Contents

Section 1.3: Linear Inequalities in the Plane

Definition: *linear equality*

Linear describes a mathematical expression in which all variables appear to the first power

Ex: $-2x + 5y = 12$.

Equality means that the expression contains an equality sign.

Graphically, a linear equality represents a line in the xy-plane, cutting the plane into two pieces.

Definition: *linear inequality*

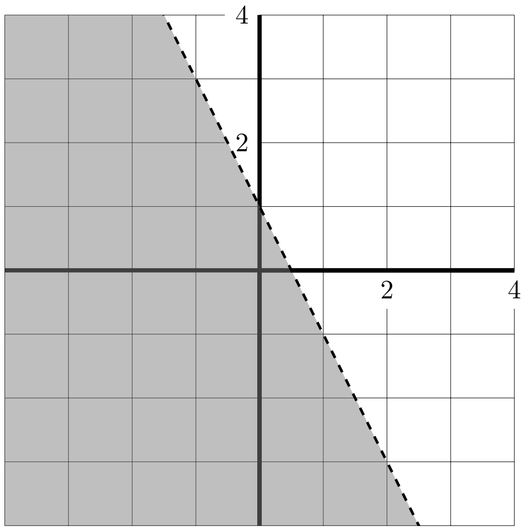
This is a linear expression containing an inequality sign rather than an equality sign.

Graphically, a linear inequality defines one half of the xy-plane.

Definition: *half-plane*

A region defined by a linear inequality, i.e. those points (x, y) for which the inequality is true.

Example 1: Sketch the half-plane defined by $2x + y < 1$.



Step 1: graph $2x + y = 1$ using intercepts $(0, 1)$ and $(1/2, 0)$ OR using the slope-intercept form $y = mx + b$; i.e. $y = -2x + 1$.

Step 2: choose a test point. A test point is any point not on the line. We can use $(0, 0)$, and since $2(0) + 0 = 0 < 1$, (0, 0) is in our region.

Step 3: shade the region. Use a dashed line for $<$ and $>$ and a solid line for $\leq$ and $\geq$.

Definition: *system of linear inequalities*

A collection of more than one linear inequality.

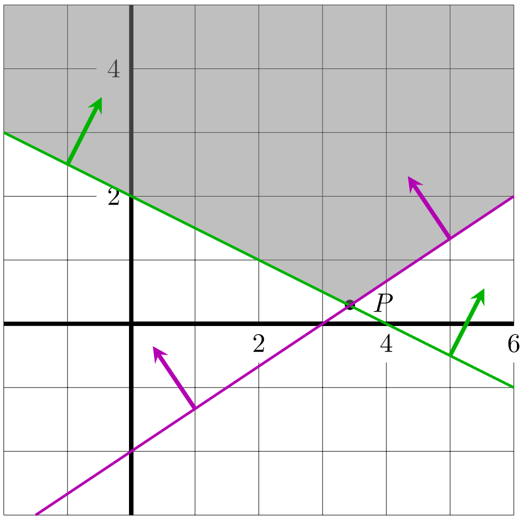
Definition: *the feasibility region*

The region determined simultaneously by all the inequalities in a system of linear inequalities.

To find the feasible region, sketch all the half-planes and find their intersection; i.e. the points common to all the half-planes.

Example 2: Graph the region defined by the system of inequalities and find the corner points.

$$\begin{aligned} x + 2y &\geq 4 \\ -2x + 3y &\geq -6 \end{aligned}$$



Step 1: graph $x + 2y = 4$ using intercepts $(0, 2)$ and $(4, 0)$, solid.

Step 2: test point $(0, 0)$: $0 + 2(0) \not\geq 4$, so $(0, 0)$ is not in our region.

Step 3: graph $-2x + 3y \geq -6$ using intercepts $(0, -2)$ and $(3, 0)$, solid.

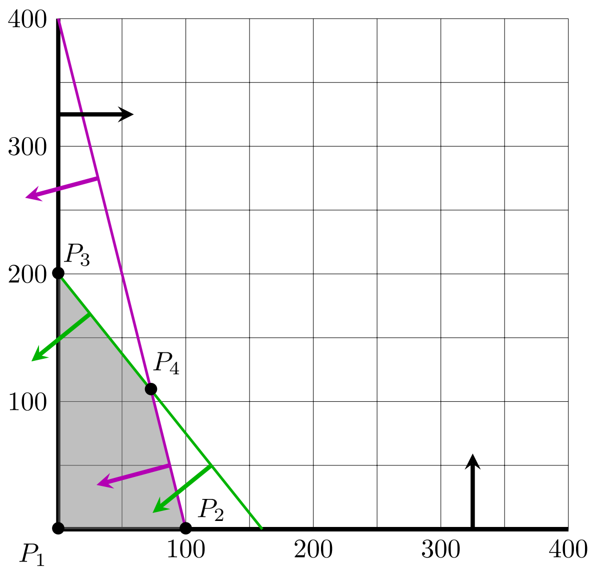
Step 4: test point $(0, 0)$: $-2(0) + 3(0) \geq -6$, so $(0, 0)$ is in our region.

Solve for x in the first equation ($x = 4 - 2y$) and substitute into the second equation:

$$-2(4 - 2y) + 3y = -6 \qquad \text{OR} \qquad -8 + 4y + 3y = -6 \qquad \text{OR} \qquad 7y = -6 + 8 = 2$$

so $y = 2/7$, and $x = 4 - 2(2/7) = 24/7$, so the corner point $P = (24/7, 2/7)$.

Example 3: A certain shingle company produces two kinds of shingles. The first kind takes 250 pounds of asphalt and 100 pounds of gravel to make. The second kind takes 200 pounds of asphalt and 25 pounds of gravel to make. The company has 40,000 pounds of asphalt and 10,000 pounds of gravel on hand. Find the linear mathematical model (i.e. the set of linear inequalities) that describes all the feasible combinations of shingle production. Graph the feasibility region and find the corner points.



(!) Clearly define the variables.

$x = \#$ of batches of the 1st type of shingle
$y = \#$ of batches of the 2nd type of shingle

(!) Interpret the information you are given. Make sure the units match on both sides of inequalities (here lbs):

Asphalt: $250x + 200y \leq 40000$
Gravel: $100x + 25y \leq 10000$

(!) You can't make a negative number of shingles, so $x \geq 0$ and $y \geq 0$

Plot Asphalt intercepts $(0, 200)$, $(160, 0)$, solid; gravel intercepts $(0, 400)$, $(100, 0)$, solid.

Corner Points: 3 are easy to see: $P_1 = (0, 0)$, $P_2 = (100, 0)$, and $P_3 = (0, 200)$. The fourth corner point is on the intersection of the two lines $250x + 200y = 40,000$ and $100x + 25y = 10,000$. Multiply the second one by -8:

$$\begin{array}{rcrcr} 250x & + & 200y & = & 40,000 \\ -800x & - & 200y & = & -80,000 \\ \hline -550x & & & = & -40,000 \end{array}$$

Hence $x = 800/11$, and so $P_4 = (800/11, 1200/11)$.

Section 1.4: Linear Programming in the Plane

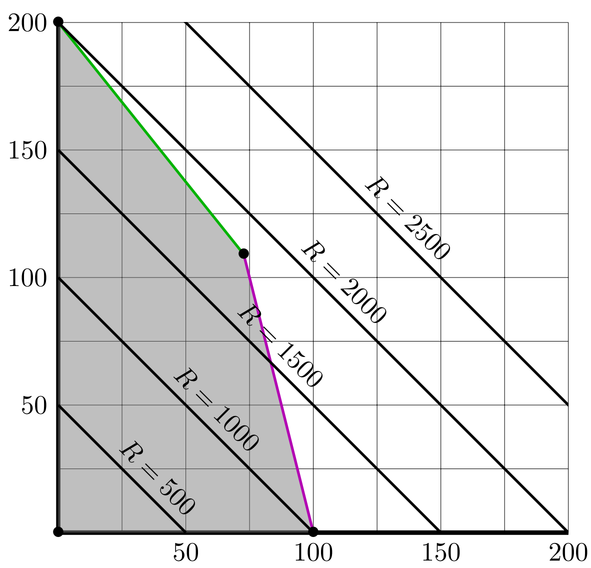
Definition: *a linear function*

A function, where every variable in the expression defining the function is raised to the first power (at most). Example: $f(x, y) = 2x - (5/7)y$.

Example 1: Consider the feasibility region for the shingle company from Lesson 1.3, Example 3. Now suppose each bundle of shingles sells for \$10. Then the revenue R can be written as a linear function of x and y. Specifically, $R = 10x + 10y$ where x is the number of the first kind of shingles sold and y is the number of the second kind of shingles sold. (Note that different values of R yield different lines with the exact same slope.)

Consider $R = 10x + 10y$: solve for y to get the equation of a line $y = -x + R/10$. Thus, for any value of R, the line has the same slope $m = -1$.

Choosing different values of R, graph some of the lines for $R = 10x + 10y$ over the feasibility region from the shingle company example. Can you draw any conclusions about what type of shingles to produce?



$R = 500$, $y = -x + 500/10 = -x + 50$
$R = 1000$, $y = -x + 100$
$R = 1500$, $y = -x + 150$
$R = 2000$, $y = -x + 200$
$R = 2500$, $y = -x + 250$

These are lines of constant revenue.

We pick the line with the largest revenue that still intersects the feasible region.

Here, the line for $R = 2000$ has one point in common with the feasible region, namely $(0, 200)$. Thus, the maximum revenue is $10(0) + 10(200) = \$2000$.

Definition: *lines of constancy*

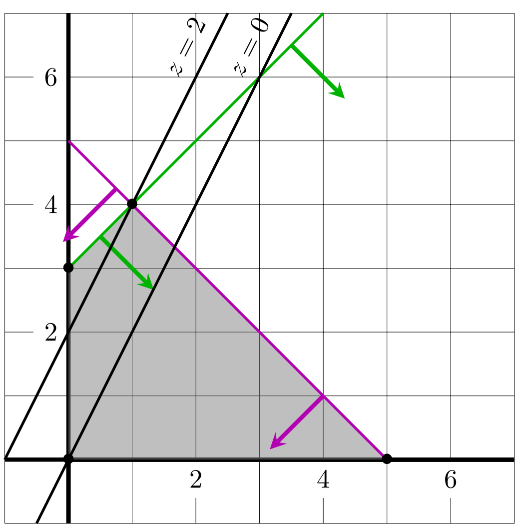
Assume that we have a linear function $f(x, y) = ax + by$. ***Lines of constancy*** are lines along which $f(x, y)$ has a constant value. A line is the set of points (x, y) for which $f(x, y) = c$ for some constant c. (!) $f(x, y) = 1$ means the same thing as $ax + by = 1$.

Definition: *a linear program*

Problem of maximizing or minimizing a linear function over a set of constraints (generally linear inequalities).

Example 2: Minimize $z = -2x + y$ subject to the following restraints:

$$\begin{aligned} x - y &\geq -3 \\ x + y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



Step 1: Graph the feasibility region and determine the corner points.

$x - y = 3$: test point $(0, 0)$ gives $0 - 0 \geq -3$ YES!

$x + y = 5$: test point $(0, 0)$ gives $0 + 0 \leq 5$ YES!

Find the intersection of the lines
$x - y = -3$
$x + y = 5$
$2x = 2$ and so $x = 1$, $y = 4$

The corner points are $(0, 0)$, $(0, 3)$, $(1, 4)$, and $(5, 0)$.

Example 2 (Continued):

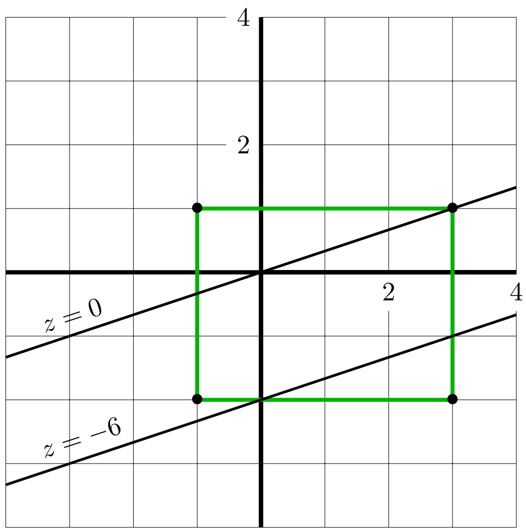
Step 2: Look for the lines of constancy of $z = -2x + y$, i.e. the line $y = 2x + z$.

$z = 0$ gives the line $y = 2x$.

$z = 2$ gives the line $y = 2x + 2$.

We conclude that as z increases, the lines move upward. Therefore, the smallest z is at $(5, 0)$: $z_{\text{min}} = -2(5) + 0 = -10$.

Example 3: Find the maximum and minimum values of the linear function $z = -x + 3y$ over the following rectangular region.



Try different values of z to create some lines of constancy.

$z = 0$ gives line $0 = -x + 3y$ or $y = x/3$

$z = -6$ gives line $y = x/3 - 2$

Graphically, this suggests that the maximum value of z arises from the upper left corner and the minimum value of z comes from the lower right hand corner.

The max of z is $z_{\text{max}} = -(-1) + 3(1) = 4$ and is located at $(-1, 1)$.

The min of z is $z_{\text{min}} = -(3) + 3(-2) = -9$ and is located at $(3, -2)$.

Definition: *polygon*

A ***polygon*** is a bounded closed geometric region in the plane formed by three or more straight line segments intersecting only at corners.

Theorem: Let $z = ax + by$ be a linear function, and let P be a polygon in the plane. Then the maximum and minimum values of z are attained at corner points of P.

Remarks:

1. Since the feasibility regions for the last couple of problems were polygons, we could simply have checked the value of the linear function at each of the corners and then picked the max and min of these values.
2. If the feasibility region of a linear program is finite, we can always use this theorem.

Procedure for solving a linear program with a polygonal feasibility region:

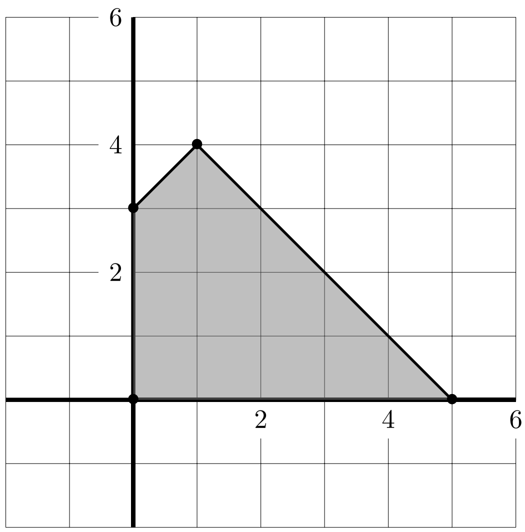
Step 1: Graph the constraints (the feasibility region).

Step 2: Find the corner points.

Step 3: Evaluate the z function at the corner points: give the location and values of the max/min of z

Example 4: Minimize $z = -2x + y$ subject to the following constraints:

$$\begin{aligned} x - y &\geq -3 \\ x + y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



See example 2 for steps 1 and 2.

Step 3:

$z(0,0) = -2(0) + 0 = 0$

$z(0,3) = -2(0) + 3 = 3$

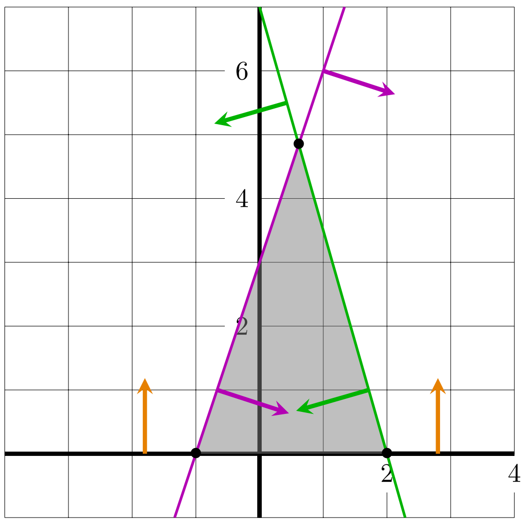
$z(1,4) = -2(1) + 4 = 2$

$z(5,0) = -2(5) + 0 = -10$ (min)

Step 4: The minimum value of z is -10 and it occurs at the point $(5,0)$.

Example 5: Maximize $z = x - 3y$ subject to the following constraints:

$$\begin{aligned} 7x + 2y &\leq 14 \\ -3x + y &\leq 3 \\ y &\geq 0 \end{aligned}$$



Step 1: Graph the constraints.

First, $7x + 2y = 14$ or $y = -(7/2)x + 7$.
test point $(0, 0)$: $0 + 0 = 0 = 14$ YES!!

Second, $-3x + y = 3$ or $y = 3x + 3$.
test point $(0, 0)$: $0 + 0 = 0 = 3$ YES!!

$y \geq 0$.

Step 2: Find the corner points. Two of these, $(-1, 0)$ and $(2, 0)$, are readily apparent. The third comes from solving the system

$$\begin{aligned} 7x + 2y &= 14 \\ -3x + y &= 3 \qquad \text{(multiply by } -2) \end{aligned}$$

$$\begin{aligned} 7x + 2y &= 14 \\ 6x - 2y &= -6 \qquad \text{(subtract)} \\ \hline 13x \qquad &= 8 \end{aligned}$$

so $x = 8/13$ and $y = 3(8/13) + 3 = 63/13$. Thus, the corners are $(8/13, 63/13)$, $(2, 0)$, and $(-1, 0)$.

Step 3: Plug corner points into z:

$z(8/13, 63/13) = 8/13 - 3(63/13) = -181/13$

$z(2, 0) = 2 - 3(0) = 2$ (max)

$z(-1, 0) = -1 - 3(0) = -1$

Step 4: Report the result: the max of z is 2 and is achieved at $(2, 0)$.

Section 1.5: Applications of Linear Programming

Outline for solving a linear program word problem:

1. Assign variables (detail and with units).
2. Define the objective function (what one is trying to optimize).
3. Define the constraints (the system of inequalities).
4. Solve the linear program.

Example 1: A company mines rubies and emeralds at two different locations. Mine A produces 15 ounces of rubies and 5 ounces of emeralds per day at a cost of \$2,200 per day. Mine B produces 10 ounces of rubies and 10 ounces of emeralds per day at a cost of \$2,600 per day. The company must fill an order for 150 ounces of rubies and 100 ounces of emeralds in 25 days with minimum cost.

$$\begin{aligned} x - y &\geq -3 \\ x + y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Step 1: Define variables.

$$\begin{aligned} x &= \text{the number of days running mine } A \\ y &= \text{the number of days running mine } B \end{aligned}$$

Step 2: Minimize cost, $C = 2200x + 2600y$.

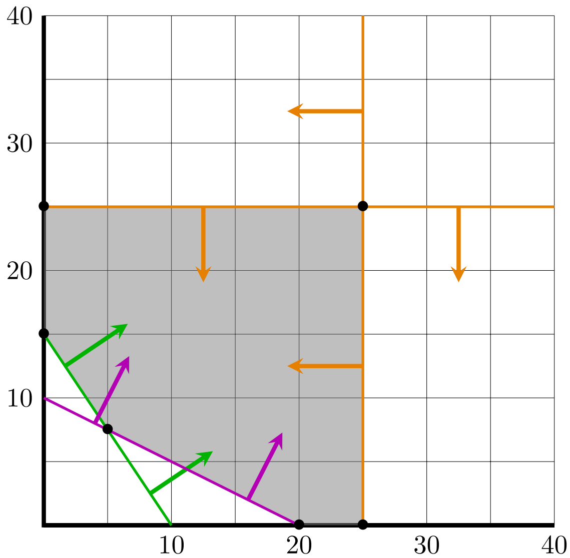
Step 3: List the constraints.

$$\begin{aligned} &\text{Rubies:} & 15x + 10y &\geq 150 \\ &\text{Emeralds:} & 5x + 10y &\geq 100 \\ &\text{Deadlines:} & x &\leq 25 \\ & & y &\leq 25 \\ & & x &\geq 0 \\ & & y &\geq 0 \end{aligned}$$

Example 1 (Continued):

The linear program: minimize $C = 2200x + 2600y$ subject to the following constraints:

$$15x + 10y \geq 150$$
$$5x + 10y \geq 100$$
$$0 \leq x, y \leq 25$$



Step 4: Graph the constraints.

Rubies: $15x + 10y = 150$
intercepts $(0, 15)$ and $(10, 0)$
test point $(0, 0)$ so $0 + 0 \not\geq 150$ NO!

Emeralds: $5x + 10y = 100$
intercepts $(0, 10)$ and $(20, 0)$
test point $(0, 0)$ so $0 + 0 \not\geq 100$ NO!

Deadlines: $0 \leq x \leq 25$, $0 \leq y \leq 25$

Step 5: Find the corner points.

$$\begin{array}{r} 15x + 10y = 150 \\ 5x + 10y = 100 \\ \hline 10x \qquad\quad = 50 \end{array} \quad \text{(subtract)}$$

so $x = 5$ and $y = 7.5$. The corners are therefore at $(0, 15)$, $(0, 25)$, $(25, 25)$, $(25, 0)$, $(20, 0)$, and $(5, 7.5)$.

Step 6: Evaluate the objective function $C = 2200x + 2600y$ at the corners.

$$C(0, 15) = 2200(0) + 2600(15) = 39000$$
$$C(0, 25) = 65000$$
$$C(25, 25) = 120000$$
$$C(25, 0) = 55000$$
$$C(20, 0) = 44000$$
$$C(5, 7.5) = 30500 \text{ (min)}$$

Step 7: Report the results. The minimum cost mining run is to operate mine A for 5 days and mine B for 7.5 days at a cost of \$30,500.

Dealing with unbounded feasibility regions:

1. Use lines of constancy to solve linear programs with unbounded feasibility regions (not the corner points method).
2. In a word problem, try to place reasonable restrictions on the variables (e.g. the number of shingle batches can't be negative).

   Warning: This won't work for a strictly mathematical problem!

Example 2: A company produces three different kinds of watches at two different plants. At the first plant, they produce 100 slim, 60 sport, and 35 petite watches per day at a cost of $3,000 per day. At the second plant, they produce 50 slim, 90 sport, and 105 petite watches per day at a cost of $3,300 per day. The company must fill an order for 5,000 slim, 5,400 sport, and 4,200 petite watches with minimum cost.

Step 1: Define variables.

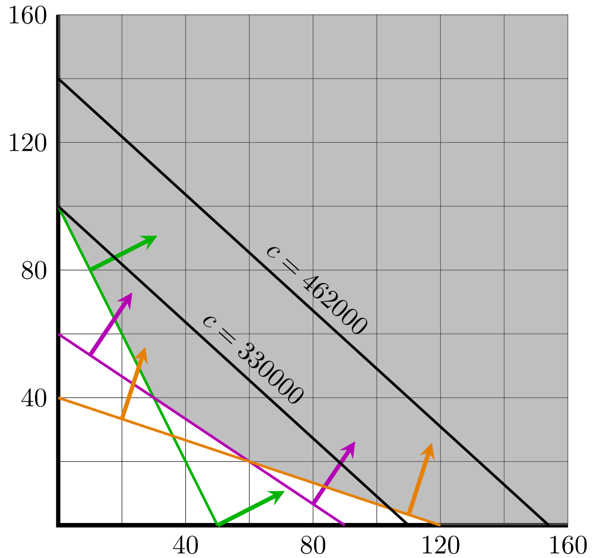
$$x = \text{number of days running plant 1}$$
$$y = \text{number of days running plant 2}$$

Step 2: Minimize cost, $c = 3000x + 3300y$.

Step 3: Determine constraints.

$$\begin{aligned} 100x + 50y &\geq 5000 \text{ (slim)} \\ 60x + 90y &\geq 5400 \text{ (sport)} \\ 35x + 105y &\geq 4200 \text{ (petite)} \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Example 2 (Continued):



Step 4: Solve the linear program.

Plot feasibility region:

slim: $100x + 50y = 5000$
Intercepts: $(0, 100)$, $(50, 0)$

sport: $60x + 90y = 5400$
Intercepts: $(0, 60)$, $(90, 0)$

petite: $35x + 105y = 4200$
Intercepts: $(0, 40)$, $(120, 0)$

How to deal with unbounded feasibility regions?

Method 1: Impose logical constraints on x and y that limit their size:

It takes 40 days to finish the petite watch order using only plant 2 ($40 = 4200/105$).

It takes 100 days to finish the slim watch order using only plant 2 ($100 = 5000/50$).

It takes 60 days to finish the sport watch order using only plant 2 ($60 = 5400/90$).

Thus, $y \leq 100$. Similarly, $x \leq 120$. Hence the feasibility region is now a polygon (and so use the corner point method).

Method 2: Use lines of constancy:

$c = 3000x + 3300y$ can be re-written as $y = -(10/11)x + c/3300$, so

$c = 330000$ gives the line $y = -(10/11)x + 100$, with intercepts at $(0, 100)$ and $(110, 0)$.

$c = 462000$ gives the line $y = -(10/11)x + 140$, with intercepts at $(0, 140)$ and $(154, 0)$.

Example 2 (Continued):

Geometrically, the minimum is located at the intersection of the sport line and slim line.

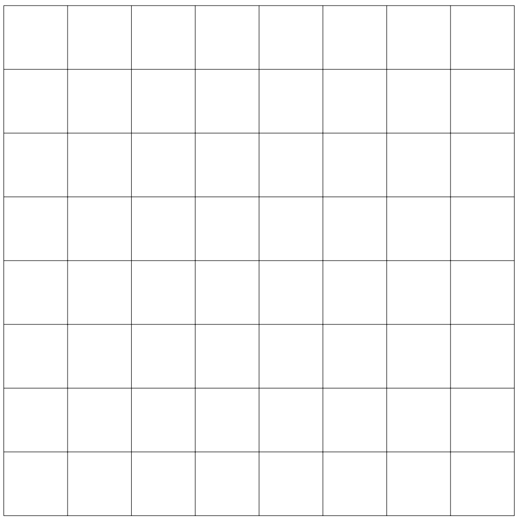
$$\begin{array}{rrrl} 100x + & 50y = & 5000 & \text{(divide by 5)} \\ 60x + & 90y = & 5400 & \text{(divide by } -3) \\ \\ 20x + & 10y = & 1000 & \\ -20x - & 30y = & -1800 & \text{(add)} \\ \hline & -20y = & -800 & \end{array}$$

so $y = 40$ and so $20x + 10(40) = 1000$, or $x = 30$. The minimum occurs at the corner $(30, 40)$ and has value $c(30, 40) = 3000(30) + 3300(40) = 222000$.

Step 5: Report the results: the minimum cost of production run is to run plant 1 for 30 days and plant 2 for 40 days at the cost of \$222,000.

Remark: We have produced extra petite watches. This can be seen geometrically, since the point $(30, 40)$ lies above the line for petite watches. We needed only $35x + 105y \geq 4200$, but the number made is $35(30) + 105(40) = 5250$.

Example 3: A bakery makes two different cakes with sugar and flour. The bakery's angel food cake is made with 2 cups of sugar and 1 cup of flour and sells for \$3.50. Their chiffon cake is made with 1 cup of sugar and 3 cups of flour and sells for \$3.00. The bakery wants to maximize revenue when it has 80 cups of sugar and 90 cups of flour on hand.



This is left as an exercise.

Section 2.1: Row Operations and Gaussian Elimination

Consider the following system of linear equations:

$$\begin{aligned} -3x_1 + 7x_2 - 5x_3 &= -4 \\ 5x_1 + 8x_2 + 3x_3 &= 8 \\ 4x_1 - 7x_2 + x_3 &= 2 \end{aligned}$$

Matrices are a notational way to work with systems of equations that have multiple variables. A system of linear equations has two important matrices associated with it: the ***coefficient matrix*** and the ***augmented matrix***.

Coefficient Matrix: $\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} -3 & 7 & -5 \\ 5 & 8 & 3 \\ 4 & -7 & 1 \end{bmatrix} \end{array}$ Augmented Matrix: $\left[\begin{array}{ccc|c} -3 & 7 & -5 & -4 \\ 5 & 8 & 3 & 8 \\ 4 & -7 & 1 & 2 \end{array}\right]$

Definition: *row echelon form*

A matrix is in ***row echelon form*** ("REF") if it has the following 4 properties:

1. the first non-zero entry in each row is a 1 (a leading 1);
2. each entry below a leading 1 is a 0;
3. as you move down the rows, the leading 1's move to the right; and
4. any row of all 0's is always at the bottom (or just above a row of all 0's).

Example 1: Are these matrices in row echelon form?

(a) $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Yes: every non-0 row has a leading 1, every entry below the leading 1's are zeros, the leading 1's "slope downward", and there are no rows with only zeros.

(b) $\left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right]$

No: the bottom row is not all zeros but has no leading 1. (The coefficient matrix, though, is in row echelon form.)

The Three Elementary Row Operations:

1. interchange two rows. $R_i \longleftrightarrow R_j$
2. multiply a row by a non-zero number. $R_i' = kR_i$
3. add a multiple of one row to another. $R_j' = R_j + kR_i$

Remark: The importance of the elementary row operations is that the new system of equations (corresponding to the new augmented matrix after row operations) has exactly the same solutions as the original system. In other words, all elementary row operations can be undone.

Definition: *Gaussian elimination*

This is a systematic procedure for using elementary row operations to put a matrix into row echelon form.

Remark:

Used on an augmented matrix, Gaussian elimination helps us to solve a system of linear equations.

A Procedure for Performing Gaussian Elimination:

1. Find the leftmost variable column that has a nonzero entry in it. Choose a nonzero entry in this column and interchange rows, if necessary, to put his entry in the top row.
2. Divide the top row by the nonzero entry found in step 1 so that the leftmost entry in the row is 1. This entry is called a ***leading 1*** for the row.
3. Add (or subtract) multiples of the top row to the rows beneath it to obtain 0's beneath the leading 1's.
4. Cover the top row and repeat steps 1 through 3. Stop when all rows have been covered or when 0 entries remain in the uncovered rows.

After moving an augmented matrix into row echelon form, we then write the system of equations corresponding to the row echelon form matrix and use ***the method of backward substitution*** to solve the system of equations.