MATH 231 B/C LECTURE 2: INTRODUCTION TO TAYLOR POLYNOMIALS AND SERIES

Motivating Problem. Find a practical method for calculating $e^x, \sin(x), \ldots$ Often you have an acccuracy in mind (e.g "to five decimal places").

Taylor polynomials/series give you a way to estimate the value of a function f near a real number a, if you know the derivatives of f at a.

Let f be a function, and let a be a real number. Let $n \geq 0$ be an integer.

Definition 1. The *degree n ("Taylor polynomial") approximation to f at a* is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example 2.

$$T_1(x) = f(a) + f'(a)(x-a)$$

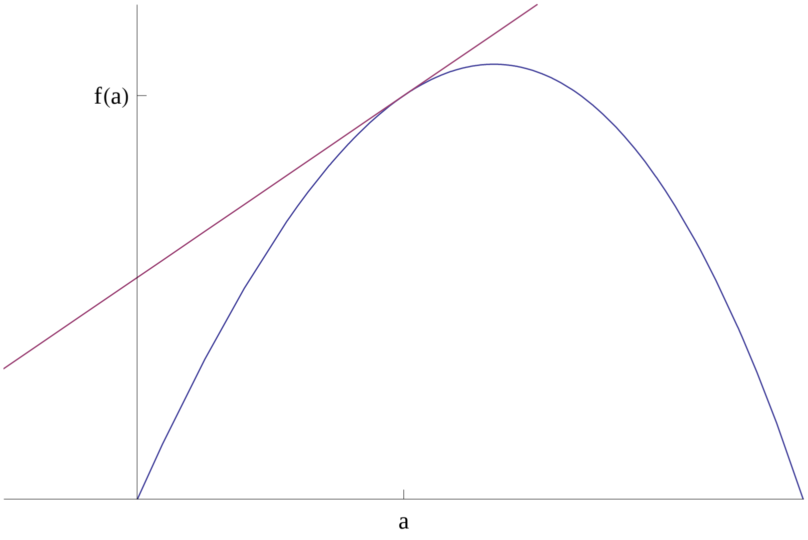
is called the "linear approximation to f at a." In the following picture, the blue curve is the graph $y = f(x)$. The red line is the tangent line at $(a, f(a))$. It has slope $f'(a)$, so by the point-slope method its equation is

$$y - f(a) = f'(a)(x-a)$$

which you could also write

$$y = f(a) + f'(a)(x-a).$$

That's T_1.



Date: August 2009.

Properties of T_n. Here are some important properties of T_n.

a) T_n is a polynomial of degree at most n
  [usually the degree is n, but it is less than n if $f^{(n)}(a)$ happens to equal 0]

b) $T_n^{(k)}(a) = f^{(k)}(a)$ if $k \le n$
  [so T_n has the same derivatives as f at $x = a$, up to order n]

c) $T_n^{(k)}(a) = 0$ if $k > n$
  [taking more than n derivatives gives zero, because T_n has degree at most n]

Let's check this last fact.

$$T_n'(x) = f'(a) + \frac{f''(a)}{2!}2(x-a) + \frac{f^{(3)}(a)}{3!}3(x-a)^2 + \text{H.O.T. ("Higher Order Terms")}.$$

"Higher order terms" means terms involving $(x-a)^k$ with $k > 2$. Substituting $x = a$ shows

$$T_n'(a) = f'(a).$$

For the second derivative,

$$T_n''(x) = f''(a) + \frac{f^{(3)}(a)}{3!}3 \cdot 2(x-a) + \text{H.O.T.}$$

where here H.O.T. means terms involving $(x-a)^k$ with $k > 1$. Substituting $x = a$,

$$T_n''(a) = f''(a).$$

What happens at the next stage...?

Example 3. $f(x) = e^x, a = 0$. Here's a table of derivatives of e^x at 0 :

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
n	e^x	1

So

$$\begin{aligned} T_n(x) &= 1 + 1(x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \cdots + \frac{1}{n!}(x-0)^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}. \end{aligned}$$

Example 4. $f(x) = \sin(x), a = 0, n = 5$. Here's a table of derivatives of $\sin(x)$ at 0 :

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1

(there's a pattern: every fourth line is the same). For example,

$$T_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Notice there are only odd powers of x : T_5 is odd, just like the sine function.

Remark 5.

$$T_2(x) = 0 + x + 0 = x$$

so we say that "$\sin x$ equals x to degree 2, or second order"

Example 6. $f(x) = \cos x, a = 0, n = 6.$

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
5	$-\sin x$	0
6	$-\cos x$	-1

Again, every fourth line is the same. We have

$$T_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6.$$

Remark 7. We say that "$\cos x = 1 - x^2/2$ to third order", since $T_3(x) = 1 - x^2/2$.

Example 8. $f(x) = 1/x = x^{-1}, a = 1, n = 4$. This time we cannot use $a = 0$. (Why not?!) Here's a table of derivatives of $1/x$ at 1 :

k	$f^{(k)}(x)$	$f^{(k)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3!x^{-4}$	$-3!$
4	$4!x^{-5}$	$4!$

We have

$$\begin{aligned} T_4(x) &= 1 - 1(x-1) + \frac{2}{2!}(x-1)^2 - \frac{3!}{3!}(x-1)^3 + \frac{4!}{4!}(x-1)^4 \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4. \end{aligned}$$

Series. For many functions f, and many choices of a and x, it turns out that $T_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$. Then we say the "Taylor series converges to $f(x)$." We will learn later how mathematicians make sense of convergence of series with infinitely many terms. For now, we just state how our examples behave.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} && \text{for all } x \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} && \text{for all } x \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} && \text{for all } x \\
\frac{1}{x} &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k (x-1)^k && \text{for } 0 < x < 2
\end{aligned}
$$

The last series for $1/x$ obviously cannot be valid at $x = 0$. It is also false at $x = 2$ (try it!).

Four things you can do with series.

1. Numerical approximation. Use a degree two (also called "second order" or "quadratic") Taylor polynomial to estimate $\cos(0.1)$.

We'll use $a = 0$ since this is near to $x = 0.1$ and we know the Taylor polynomial for cosine when $a = 0$. We find

$$\cos(0.1) \approx 1 - \frac{(0.1)^2}{2} = 1 - 0.005 = 0.995.$$

2. Find new series. Find the sixth-order Taylor polynomial for $x \sin(2x)$ at 0.

We won't calculate afresh all the derivatives of this function at 0. Instead we take the series for $\sin x$, then substitute $2x$ in place of x, and multiply the whole series by x:

$$
\begin{aligned}
x \sin(2x) &= x \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \text{H.O.T.} \right) \\
&= 2x^2 - \frac{8x^4}{3!} + \frac{32x^6}{5!} + \text{H.O.T.}
\end{aligned}
$$

Exercises.

(a) Find the series for e^{-x^2}, when $a = 0$.

(b) Find the first few terms of the series for $e^{-x^2} \cos x$, when $a = 0$.

3. Find new equations. Remember that i is the complex number with $i^2 = -1$. So $i^3 = -i$ and $i^4 = 1$. Then

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \text{H.O.T.} \\ &= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \text{H.O.T.} \end{aligned}$$

Let's compare this to the series for $\cos(x) + i\sin(x)$. We have

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \text{H.O.T.}$$

and

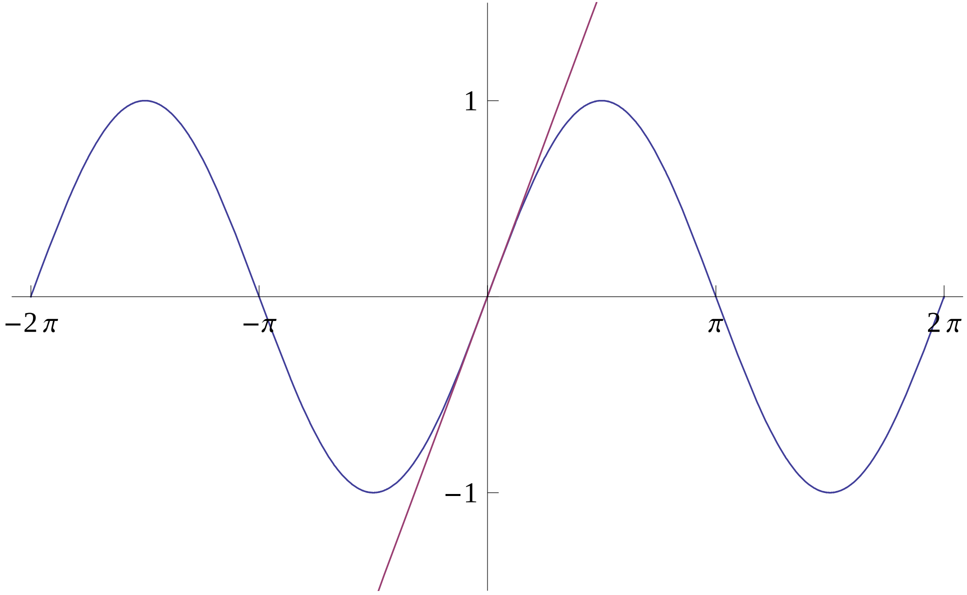
$$i \sin x = ix - i\frac{x^3}{3} + \text{H.O.T.}$$

It turns out that indeed $\boxed{e^{ix} = \cos x + i \sin x}$. This is called "Euler's Theorem."

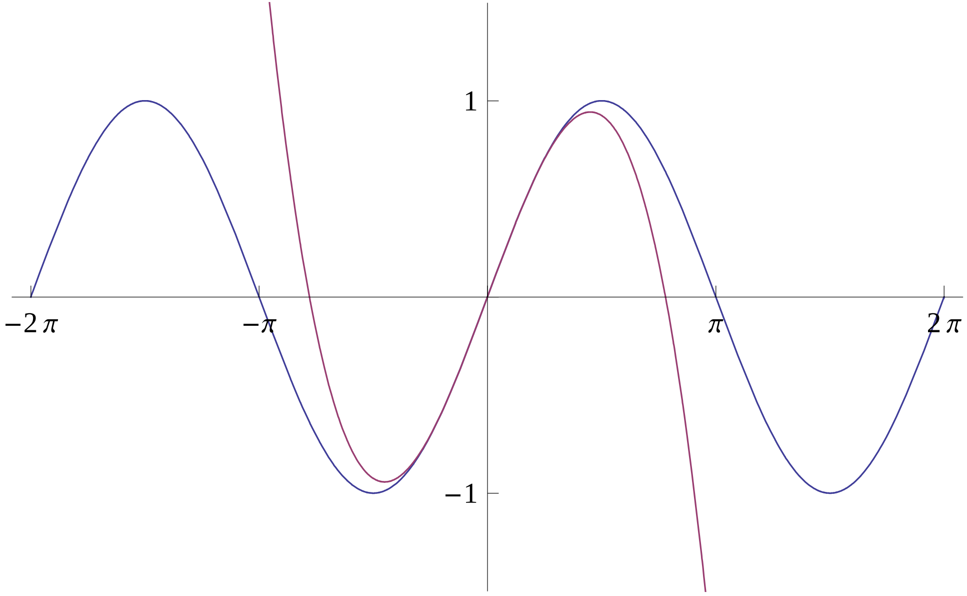
4. Graphical approximation. We conclude with some illustrative graphs of polynomial approximations to $\sin x$ at $a = 0$. (You might want to try the same thing yourself, with the cosine, or with e^x, or your favorite function.)

$$\begin{aligned} T_1(x) &= x \\ T_3(x) &= x - \frac{1}{3!}x^3 \\ T_5(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \\ T_7(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \end{aligned}$$

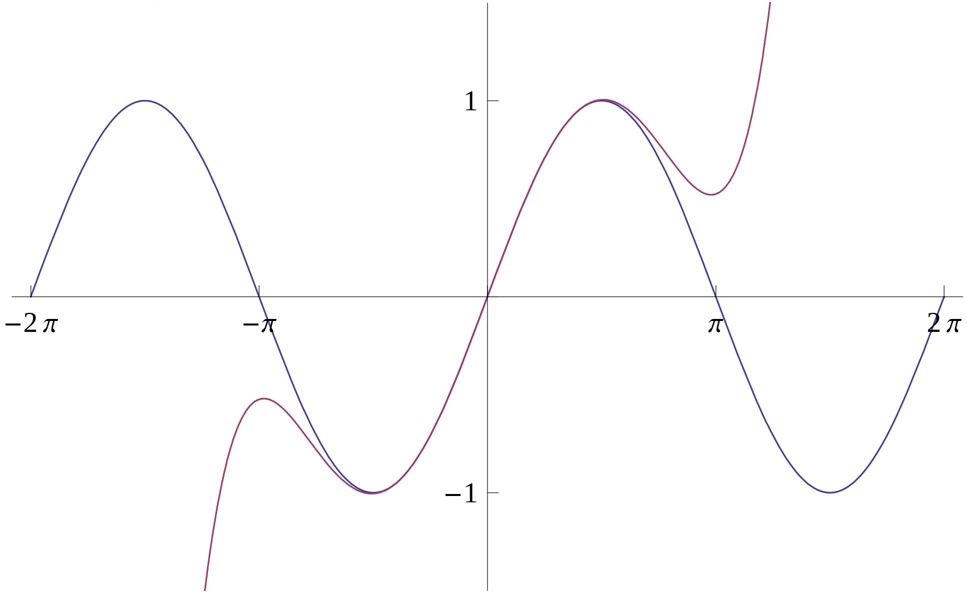
This is the linear approximation, T_1. The graph of sin is blue, the linear approximation is red. Graphically, $\sin x \simeq T_1(x) = x$ when x is near $a = 0$ (say, when $|x| \lesssim \pi/6$).



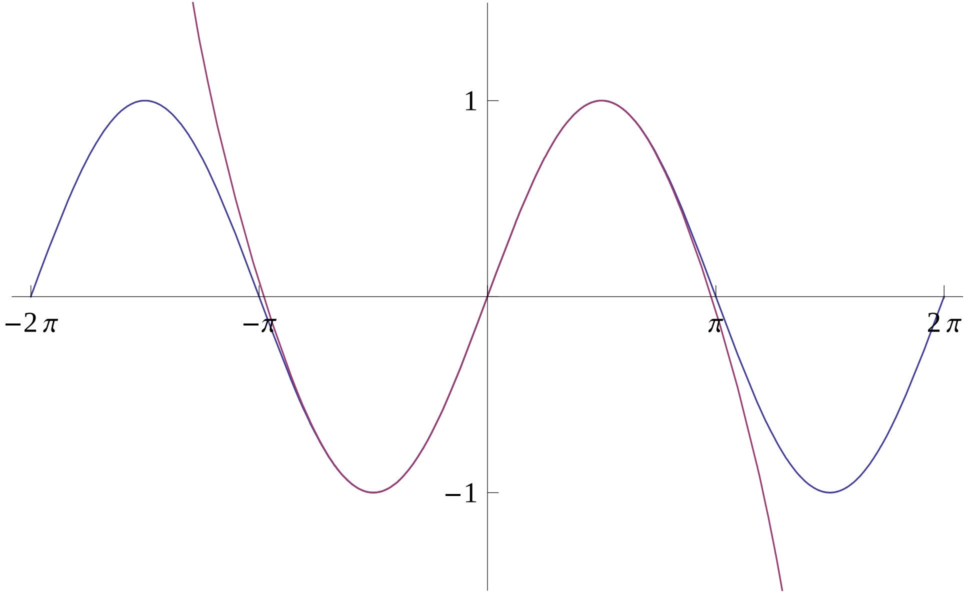
This is the third-degree approximation, T_3. Again the graph of sin is blue, the approximation is red. Graphically, $\sin x \simeq T_3(x) = x - x^3/3!$ when x is near $a = 0$ (say, when $|x| \lesssim \pi/3$).



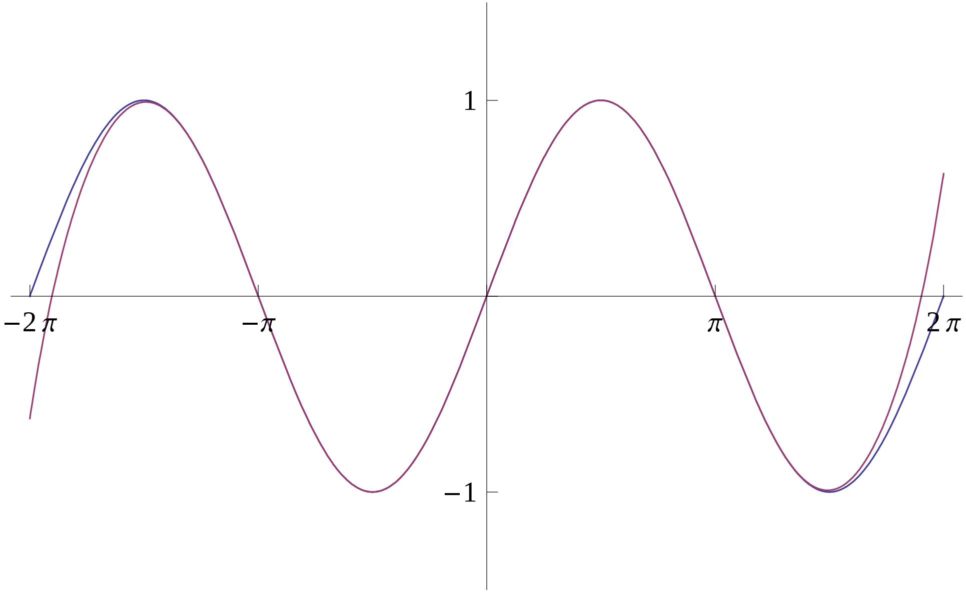
Fifth degree.



Seventh degree.



Just for fun, here's the degree-13 approximation. The red plot is the approximation. Notice that we finally have 5 roots, but after that the approximation blows up.



What you cannot do with Taylor series. The trigonometric meaning of sine and cosine is hidden by the Taylor series. For example, putting $x = \pi/2$ into the series for $\cos x$ must give $\cos \pi/2 = 0$, but one cannot see that result just from looking at the series!