MATH 595: ELLIPTIC FUNCTIONS WITH APPLICATIONS TO NUMBER THEORY
9–9:50 MWF, 445 ALTGELD HALL

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This course will consist of two parts. The first 8 or 9 weeks of the course will be devoted to Ramanujan's approach to number theory through the theories of q-series and theta functions. Thus, first we will develop the basic theorems in q-series and theta functions. We then will apply these theorems to the theory of partitions. In particular, we shall prove Ramanujan's famous congruences for the partition function $p(n)$, namely, $p(5n+4) \equiv 0 \pmod{5}$, $p(7n+5) \equiv 0 \pmod{7}$, and $p(11+6) \equiv 0 \pmod{11}$. Next, Ramanujan's famous τ-function will be studied. It also satisfies some famous congruences, such as, $\tau(n) \equiv n\sigma_9(n) \pmod{25}$, where $\sigma_k(n) = \sum_{d|n} d^k$. Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares. Jacobi established some famous formulas for $r_{2k}(n), 1 \leq k \leq 4$, which we shall also prove (but in ways much different from those of Jacobi). Similarly, there exists beautiful formulas for $t_k(n)$, the number of representations of n as a sum of k triangular numbers. One of the most important theorems in the theory of elliptic functions relates the most fundamental theta function with the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$. After proving this theorem, we shall show how this leads to Ramanujan's theory of modular equations, which can be utilized to prove theorems about partitions and sums of squares, in particular. Ramanujan's famous continued fraction and possibly some other q-continued fractions of Ramanujan shall be examined.

The remaining portion of the course will be devoted to the classical theory of elliptic functions, including basic facts arising from their double periodicity. Properties of the famous elliptic functions of Weierstrass, Jacobi, and others will be established. We also shall examine in detail elliptic integrals, establish a inversion formula for elliptic integrals, and discuss the famous and eminently useful arithmetic-geometric mean.

Prerequisites

A course in complex analysis and a course in elementary number theory.

Typed Lecture Notes by the Instructor will be used during the first portion of the course.

For the second portion of the course, K. Chandrasekharan's *Elliptic Functions* is suggested for supplementary, but not required, reading.