MATH 595: INTRODUCTION TO q-SERIES AND THETA FUNCTIONS WITH APPLICATIONS TO NUMBER THEORY 9–9:50 MWF, 447 ALTGELD HALL

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The basic theorems in q-series and theta functions will be developed. We then will apply these theorems to problems in number theory, chiefly in the theory of partitions. In particular, we shall prove Ramanujan's famous congruences for the partition function $p(n)$, namely, $p(5n+4) \equiv 0 \pmod{5}$, $p(7n+5) \equiv 0 \pmod{7}$, and $p(11+6) \equiv 0 \pmod{11}$. Next, Ramanujan's famous τ-function will be studied. It also satisfies some famous congruences, such as, $\tau(n) \equiv n\sigma_9(n) \pmod{25}$, where $\sigma_k(n) = \sum_{d|n} d^k$. Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares. Jacobi established some famous formulas for $r_{2k}(n), 1 \leq k \leq 4$, which we shall also prove (but in ways much different from those of Jacobi). Similarly, there exists beautiful formulas for $t_k(n)$, the number of representations of n as a sum of k triangular numbers.

Next, the theory of basic hypergeometric series will be developed. In particular, famous theorems of Heine, Watson, Rogers, and Ramanujan will be proved. Further applications to number theory will then be made, most notably to the Rogers–Ramanujan identities.

If time permits, we will show that the most fundamental theta function relates to the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$. After proving this theorem, we shall show how this leads to Ramanujan's theory of modular equations, which can be utilized to prove theorems about partitions and sums of squares, in particular. Ramanujan's famous continued fraction and possibly some other q-continued fractions of Ramanujan shall be examined.

Prerequisites

A course in complex analysis and a course in elementary number theory.