

Math 595 Minicourse

Geometry and analysis in sub-Riemannian spaces

Spring 2009

Instructor: Prof. Jeremy Tyson

Dates: 3/16/2009 – 5/6/2009

This minicourse will cover the foundations of (first-order) analysis, geometric measure theory and differential geometry in the Heisenberg group and more general sub-Riemannian manifolds.

The Heisenberg group \mathbb{H} is the simplest noncommutative Carnot (nilpotent stratified Lie) group. Its analytic relevance stems largely from its representation as the group of biholomorphic automorphisms of the unit ball in \mathbb{C}^2 . Under the name *nilgeometry*, it features as one of the eight three-dimensional Thurston geometries. Sub-Riemannian manifolds locally modelled on the Heisenberg group arise in the differential geometry of solution spaces for partial differential equations (jet spaces), control theory (path planning algorithms for wheeled motion) and neurobiology (structure and function of the first layer of the mammalian visual cortex).

Geometric analysis synthesizes tools and methods from geometric function theory, differential and metric geometry, harmonic analysis and geometric measure theory to study the structure of spaces and functions defined thereon, especially in nonsmooth contexts such as Sobolev spaces, bi-Lipschitz and quasiconformal mappings, rectifiable sets and currents. Recent years have witnessed a strong trend towards the development of these and related subjects in increasingly abstract contexts (*analysis in metric measure spaces*). The sub-Riemannian case served as an important motivation and impetus for these developments and continues to inspire and generate new avenues for research.

In this course, we will describe this circle of ideas in the sub-Riemannian Heisenberg group with an eye towards more general Carnot groups and sub-Riemannian manifolds. Following Korányi and Reimann, we will discuss quasiconformal and (bi-)Lipschitz mappings on \mathbb{H} in the context of Mostow's rigidity theorem and the Pansu–Rademacher differentiability theorem. We will also briefly introduce Pansu's celebrated isoperimetry conjecture in \mathbb{H} (1982) and discuss its history and present status. Finally, we will give a short introduction to sub-Riemannian geometric measure theory and fractal geometry. This course should be of interest to anyone working in analysis and/or geometry.

Suggested texts/reference materials:

- R. Montgomery, *A tour of sub-Riemannian geometries, their geodesics and applications*, AMS Mathematical Surveys and Monographs, vol. 91, 2002.
- E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press, 1993: Chapters XII and XIII.
- O. Calin, D.-C. Chang and P. Greiner, *Geometric analysis on the Heisenberg group and its generalizations*, AMS/IP Studies in Advanced Mathematics, vol. 40, 2007.
- L. Capogna, D. Danielli, S. D. Pauls and J. T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Birkhäuser, 2007.

Prerequisites: Solid preparation in analysis (at the level of Math 540 and 542) and differential geometry (at the level of Math 521) are essential. The topics in the course will intersect with harmonic analysis, PDE, several complex variables, geometric group theory and Gromov hyperbolic geometry, and Lie groups; background or interest in any of these areas would be helpful but not necessary.