1. M. Zaki. 8/29 and 9/3.

The Modular Group

Let R be a commutative ring.

$SL_2(R) = \{2 \times 2 \text{ matrices over } R \text{ with } \det = 1\}$

$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

$SL_2(\mathbb{R})$ acts on $\tilde{\mathbb{C}}$ by

$\gamma(z) = \frac{az+b}{cz+d}$, $\gamma(\infty) = \frac{a}{b}$.

This action preserves $\mathbb{H}$, since $\mathbb{I}m(\gamma(z)) = \frac{\mathbb{I}m(\gamma(z))}{|cz+d|^2}$

I and $-I$ act trivially.

Definition. *The full modular group is* $\Gamma = SL_2(\mathbb{Z})$.

Definition. *We call a closed set F a fundamental domain of Γ if every $z \in \mathbb{H}$ is Γ equivalent to some point of F, and no two distinct interior points of F are Γ equivalent.*

$$\text{Set } F := \{z \in \mathbb{H} : \frac{-1}{2} \leq \mathbb{R}e(z) <= \frac{1}{2} \text{and } |z| \geq 1\}$$

Theorem 1. *F is a fundamental domain for Γ and two distinct points z_1, $z_2 \in F$ are Γ equivalent iff*

(1)

$$z_1 = z_2 \pm 1$$

or

(2)

$$|z_1| = 1, \; z_2 = \frac{-1}{z_1}$$

Proof: The group $\Gamma = SL_2(\mathbb{Z})$ contains two fractional linear transformations which act as building blocks for the entire group: $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \to z+1$; $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \to \frac{-1}{z}$. To prove that every $z \in \mathbb{H}$ is Γ -equivalent to a point in F, the idea is to use translations T^j to move a point inside the strip $\frac{-1}{2} \leq \mathbb{R}ez \leq \frac{1}{2}$. If it lands outside the unit circle, it is in F. Otherwise use S to throw the point outside the unit circle, then use a translation T^k to bring it inside the strip, and continue in this way until you get a point inside the strip and outside the unit circle. For a precise proof look at Koblitz's text page 101.

Theorem 2. *We have that $\bar{\Gamma} = \Gamma/\{\pm I\}$ is generated by S and T. Every transformation is a word in S and T.*

Proof: *Let Γ' denote the subgroup of Γ generated by S and T. Let z be any point in the interior of F. Let g be an element of Γ. Consider the point $gz \in \mathbb{H}$. By the first part of the proof of proposition 1, there exists $\gamma \in \Gamma'$ such that $\gamma(gz) \in F$. But since z is in the interior of F, it follows by theorem 1 and proposition 3 that $\gamma(g) = \pm I$, i.e., $g = \pm\gamma^{-1} \in \Gamma'$. This shows that any $g \in \Gamma$ is actually (up to a sign) in Γ'. The proposition is proved.*

If $z \in \mathbb{H}$, define "isotropy subgroup" $\Gamma_z := \{\gamma \in \bar{\Gamma} : \gamma(z) = z\}$

Proposition 3. *If $z \in \mathbb{H}$ then Γ_z is trivial unless*

(1) $z = i$, $\Gamma_z = \langle S \rangle$
(2) $z = \omega$, $\Gamma_z = \langle ST \rangle$
(3) $z = -\bar{\omega}, \Gamma_z = \langle TS \rangle$

Congruence Subgroups and Cusps

Let N be a positive integer.

Definition. $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$

$\Gamma(N)$ is called the principal subgroup of level N.

Definition. *A subgroup $\Gamma' \leq \Gamma$ is a congruence subgroup of Level N if $\Gamma(N) \leq \Gamma'$.*

Note. If $\Gamma(N) \leq \Gamma'$, then $\Gamma(M) \leq \Gamma'$ for all $N \mid M$

The main groups of interest for us are the following.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N \right\}$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N), a \equiv d \equiv 1 \pmod N \right\}$$

If $\Gamma' \leq \Gamma$, define a fundamental domain F' for Γ' as before.

Proposition 4. *Suppose that $[\Gamma : \Gamma'] = n$ and $\Gamma = \bigcup_{j=1}^{n} \alpha_j \Gamma'$ is a coset decomposition. Then $F' = \bigcup_{j=1}^{n} \alpha_j^{-1} F$ is fundamental domain for Γ'.*

For proof see Koblitz, half is exercise.

Definition. *Modular triangle is a set γF for $\gamma \in \Gamma$.*

Recall that linear transformations map circles and lines onto circles and lines.

Example. Let $\Gamma' = \Gamma_0(p)$, p is a prime. Then $[\Gamma : \Gamma'(p)] = p + 1$. We have $\Gamma = \Gamma_0(p) \cup \bigcup_{j=0}^{p-1} A_j \Gamma_0(p)$ where $A_j = T^{-j} S = \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Proposition 4 implies fundamental domain for $\Gamma_0(p)$ is $F \bigcup \bigcup_{j=0}^{p-1} ST^j F$.

Definition. *The cusps are $\mathbb{Q} \bigcup \{\infty\}$. If $\Gamma' \leq \Gamma$ then the cusp of Γ' is an equivalence class of cusps under the action of Γ'.*

Example. If $\frac{a}{c} \in \mathbb{Q}$ is in lowest terms then complete to matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}$.

Example. $\Gamma_0(p)$ has two cusps: 0, and ∞ (HW 18).

Modular forms on $\Gamma = SL_2(\mathbb{Z})$.

Fourier expansions at ∞.

Suppose $f(z)$ is meromorphic on $\mathbb{H}$, and $f(z+1) = f(z)$ for all $z \in \mathbb{H}$. The map z to $q = e^{2\pi i z}$ takes $\mathbb{H}$ to the punctured open unit disk.

Definition. *F(q) by F(q) = f(z) where* $q = e^{2\pi i z}$

Well defined by the fact that $f(z+1) = f(z)$ for all $z \in \mathbb{H}$.
Note: $dq = 2\pi i e^{2\pi i z} dz$

$$F'(q) = \frac{d}{dq} f(z) = f'(z) \frac{dz}{dq} = \frac{f'(z)}{2\pi i e^{2\pi i z}}$$

This implies F(q) is meromorphic in $0 < |q| < 1$. So, $F(q) = \sum_{-\infty}^{\infty} a(n) q^n$. Write $f(z) = \sum_{-\infty}^{\infty} a(n) q^n$. Fourier expansion at ∞.

2. C. Gugg. 9/5 and 9/8

Recall: If $f(z)$ is meromorphic on $\mathfrak{H}$ and $f(z+1) = f(z)$ for all $z \in \mathfrak{H}$, then f has a Fourier expansion at infinity:

$$f(z) = \sum_{n=-\infty}^{\infty} a(n) q^n, \quad q = e^{2\pi i z}.$$

Define $f(\infty) := a(0)$. We call f

- meromorphic at infinity if the Fourier expansion of f has finitely many negative terms,
- holomorphic at infinity if f has no negative terms,
- vanishing at infinity if $a(n) = 0$ for $n \leq 0$

Definition. *f is a meromorphic (holomorphic) modular form of weight $k \in \mathbb{Z}$ on Γ if*

i) *f is meromorphic (holomorphic) on $\mathfrak{H}$,*
ii) $f(\gamma z) = (cz+d)^k f(z)\ \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, *and*
iii) *f is meromorphic (holomorphic) at infinity.*

Definition. *f is a cusp form if it is a holomorphic form of weight k on Γ vanishing at infinity.*

$M_k(\Gamma) :=$ the $\mathbb{C}$-vector space of holomorphic forms of weight k.
$S_k(\Gamma) :=$ the subspace of $M_k(\Gamma)$ consisting of cusp forms.

Remarks:

(1) $f(z) = f(-Iz) = (-1)^k f(z) \Rightarrow M_k = \{0\}$ if k is odd.
(2) (a) Serre uses $2k$ instead of k.
    (b) Koblitz calls meromorphic modular forms "modular functions."
    (c) In this class, a modular function is a meromorphic modular form of weight zero.

(3) If $k=0$, ii) $\Rightarrow f(\gamma z)=f(z)\ \forall\ \gamma \Rightarrow f(z)$ is a function on $\Gamma\backslash\mathfrak{H}$.
$\frac{d(\gamma z)}{dz}=(cz+d)^{-2} \Rightarrow$ if $k=2$, then (ii) $\Leftrightarrow\ f(z)d(\gamma z)=f(z)dz$ "differential form on $\Gamma\backslash\mathfrak{H}$".
For general k, (ii)$\Leftrightarrow f(z)(dz)^{k/2}$ is invariant under $z\to\gamma z$.
Then (group action)
$f(\gamma_1\gamma_2 z)(d\gamma_1\gamma_2 z)^{k/2}=f(\gamma_2 z)(d\gamma_2 z)^{k/2}=f(z)(dz)^{k/2}$. So, to check (ii), it is enough to check

$$(ii)'\ f(z+1)=f(z) \qquad \textit{action of } T$$
$$f(-1/z)=z^k f(z) \qquad \textit{action of } S.$$

(4) It is easy to check that

$$M_{k_1}(\Gamma)M_{k_2}(\Gamma)\subseteq M_{k_1+k_2}(\Gamma).$$

The first examples of modular forms are "Eisenstein series."
For $k\geq 4$ even, define

$$G_k(z)={\sum}'_{m,n\in\mathbf{Z}}\frac{1}{(mz+n)^k} \qquad (z\in\mathfrak{H})$$

Idea: Force γ-invariance by "averaging" over Γ.

Theorem 5. *If $k\geq 4$ is even, then $G_k(z)\in M_k(\Gamma)$. Also, $G_k(\infty)=2\zeta(k)$, where $\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}$.*

Proof.
Claim 1: ${\sum}'_{m,n\in\mathbf{Z}}\frac{1}{(mz+n)^k}$ converges absolutely if $k\geq 4$.
Claim 2: The series converges uniformly on the fundamental region F.
Claim 1 $\Rightarrow G_k(z+1)={\sum}'_{m,n}\frac{1}{(mz+(m+n))^k}={\sum}'_{m,n}\frac{1}{(mz+n)^k}$.
Similarly, $G_k(-1/z)=z^kG_k(z)$. So, $G_k(\gamma z)=(cz+d)^kG_k(z)\ \forall\gamma\in\Gamma$. (*)

Note. Claim 2 and (*) $\Rightarrow$ the series converges uniformly on any compact set. This implies that $G_k(z)$ is holomorphic on $\mathfrak{H}$.

$$G_k(\infty)=\lim_{z\to i\infty}{\sum}'_{m,n}\frac{1}{(mz+n)^k}=\sum_{n\neq 0}\frac{1}{n^k}=2\zeta(k)$$

(by uniform convergence and taking the limit inside the sum).

We now prove the claims. Recall that a lattice Λ is a set

$$\Lambda=\{m\omega_1+n\omega_2 : m,n\in\mathbf{Z}, \textit{where } \omega_1 \textit{ and } \omega_2 \textit{ are } \mathbb{R}-\textit{linearly independent}\}.$$

Lemma 6. . *If Λ is a lattice and $k>2$, then ${\sum}'_{\lambda\in\Lambda}\frac{1}{|\lambda|^k}$ converges.*

Sketch of Proof. Find a parallelogram in $\mathbf{C}$ of area A_Λ and maximum diagonal length d_Λ. Let N_R = number of lattice points λ with $|\lambda| \leq R$. Then

$$\pi R^2 \leq N_R A_\Lambda \leq \pi(R + d_\Lambda)^2$$

so that

$$N_R = \frac{\pi}{A_\Lambda} R^2 + O(R).$$

So

$$N_{R+1} - N_R = O(R).$$

Note:

$$\sum_{\lambda\in\Lambda}{}' \frac{1}{|\lambda|^k} = \sum_{|\lambda|\leq 1}{}' \frac{1}{|\lambda|^k} + \sum_{R=1}^{\infty} \sum_{R<|\lambda|\leq R+1} \frac{1}{|\lambda|^k} \leq \sum_{|\lambda|\leq 1}{}' \frac{1}{|\lambda|^k} + \sum_{R=1}^{\infty} \frac{N_{R+1} - N_R}{R^k}.$$

Since $\sum'_{|\lambda|\leq 1} \frac{1}{|\lambda|^k}$ is finite and $\sum_{R=1}^{\infty} \frac{N_{R+1}-N_R}{R^k}$ is $O(\sum_{R=1}^{\infty} \frac{1}{R^{k-1}})$, we see that $\sum'_{\lambda\in\Lambda} \frac{1}{|\lambda|^k}$ converges if $k > 2$. This proves Lemma 6. $\square$

We still need to show that $G_k(z)$ converges uniformly on F.
For $z \in$ F,

$$|mz+n|^2 = (mz+n)(m\bar{z}+n) = m^2|z|^2 + 2\mathbb{R}e(z)mn + n^2 \geq m^2 - mn + n^2 = |m\omega+n|^2.$$

Now $\sum \frac{1}{(m\omega+n)^k}$ converges absolutely $\Rightarrow$ G_k converges absolutely on F.

Fourier Expansions

We define the Bernoulli numbers B_k by:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots.$$

So $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = -\frac{1}{42}$, ….
Definition. $\sigma_k(n) := \sum_{d|n} d^k$.

Theorem 7. *If* $k \geq 4$ *is even, then*

$$G_k(z) = 2\zeta(k)E_k(z),$$

where

$$E_k(z) = 1 - \frac{2k}{B_K} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in \mathbb{Q}[[q]].$$

Proof. Fact: $-\frac{(2\pi i)^k}{2}\frac{B_k}{k} = \zeta(k)$ if $k > 0$ is even.
Start with $\sin(\pi z) = \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$.
Then

$$\begin{aligned}\pi\cot(\pi z) &= \frac{d}{dz}\log(\sin(\pi z))\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{-2z/n^2}{1-z^2/n^2}\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{-2z}{n^2-z^2}\\ &= \frac{1}{z}+\sum(\frac{1}{z+n}+\frac{1}{z-n}).\end{aligned}$$

Also,

$$\pi\cot(\pi z)=\frac{\pi\cos(\pi z)}{\sin(\pi z)}=\pi\frac{\frac{e^{\pi iz}+e^{-\pi iz}}{2}}{\frac{e^{\pi iz}-e^{-\pi iz}}{2}}=\pi i\frac{q+1}{q-1}=\pi i(1+\frac{2}{q-1})=\pi i-\frac{2\pi i}{1-q}=\pi i-2\pi i\sum_{d=0}^{\infty}q^d$$

so that

$$\boxed{\tfrac{1}{z}+\textstyle\sum_{n=1}^{\infty}(\frac{1}{z+n}+\frac{1}{z-n})=\pi i-2\pi i\sum_{d=0}^{\infty}q^d}.$$

Differentiate $k-1$ times to get

$$(-1)^{k-1}(k-1)!\sum_{n\in\mathbf{Z}}\frac{1}{(z+n)^k}=-(2\pi i)^k\sum_{d=1}^{\infty}d^{k-1}q^d. \qquad (*)$$

Recall:

$$\frac{1}{2\pi i}\frac{d}{dz}=q\frac{d}{dq}.$$

$$(q=e^{2\pi iz}\Rightarrow\frac{dq}{dz}=2\pi ie^{2\pi iz})$$

Note:

$$\frac{d}{dz}q^d=\frac{d}{dz}e^{2\pi idz}=2\pi ide^{2\pi idz}=2\pi idq^d$$

$$\Rightarrow\sum_{n\in\mathbf{Z}}\frac{1}{(z+n)^k}=\frac{(2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty}d^{k-1}q^d.$$

So,

$$
\begin{aligned}
(*) \Rightarrow G_k(z) &= \sum_{m,n}{}' \frac{1}{(mz+n)^k} \\
&= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbf{Z}} \frac{1}{(mz+n)^k} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}q^{md} \\
&= 2\zeta(k)(1 - \frac{2k}{B_k}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}q^{md}) \qquad\qquad (n = md) \\
&= 2\zeta(k)(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}(\sum_{d|n} d^{k-1})q^n). \quad \square
\end{aligned}
$$

We have:

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$$

Recall:

$$E_4^3, E_6^3 \in M_{12}(\Gamma).$$

$$\text{(Important fact: } M_k M_{k'} \subseteq M_{k+k'})$$

$$\Rightarrow E_4^3 - E_6^2 \in M_{12}.$$

$$
\begin{aligned}
E_4^3 - E_6^2 &= (1 + 3\cdot 240q + \ldots) - (1 - 2\cdot 504q + \ldots) \\
&= 720q + 1008q + \ldots \\
&= 1728q + \ldots
\end{aligned}
$$

so that

$$E_4^3 - E_6^2 \in S_{12}.$$

Define. $\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728} = q + \ldots$
Then $\Delta(z) \in S_{12}(\Gamma)$.

Note: Koblitz's Δ is $(2\pi)^{12}$ times our delta.
When $k = 2$, define

$$E_2(z) = \frac{1}{2\zeta(z)}\sum_{m=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2}.$$

Difference: This double sum does not converge absolutely.
One can show:

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n.$$

So,

$$E_2(z+1) = E_2(z).$$

However,

$$E_2(-1/z) = z^2E_2(z) + \frac{12z}{2\pi i}.$$

We say that $E_2(Z)$ is "quasi-modular." See Koblitz for a proof of the last equation.
We have in fact:

Theorem 8. *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, *then*

$$E_2(\gamma z) = (cz+d)^2E_2(z) + \frac{12c}{2\pi i}(cz+d).$$

3. R. Santhanam. 9/10 and 9/12

Definition. *Suppose* f *is a meromorphic modular form of weight* k *on* Γ. *If* $\tau \in \mathbb{H}$, *define* $v_\tau(f) =$ *order of* f *at* τ. *So, if* $f(z) = \sum_{n\geq v}\alpha_n(z-\tau)^n$ *is such that* $\alpha_v \neq 0$ *then,* $v_\tau(f) = v$.
Similarly, $v_\infty(f) =$ *order of* f *at* ∞ *i.e., if* $f(z) = \sum_{n\geq v}a(n)q^n$ *with* $a(v) \neq 0$ *then* $v_\infty(f) = v$.

Theorem 9 (Valance Formula). *If* f *is a nonzero meromorphic modular form of weight* k *on* Γ *then,*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{P\in\Gamma/\mathbb{H}, P\neq i,\omega} v_P(f) = \frac{k}{12}$$

To be proved later

Remark. *If* $\gamma \in \Gamma$ *then,* $v_\tau(f) = v_{\gamma\tau}(f)$. *To see this, set* $v_\tau(f) = v$. *Then* $\lim_{z\to\tau}(z-\tau)^{-v}f(z) \neq 0, \infty$. *We can write* $z = \gamma w$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Hence,

$$\begin{aligned}\lim_{z\to\gamma\tau}(z-\gamma\tau)^{-v}f(z) &= \lim_{\gamma w\to\gamma\tau}(\gamma w-\gamma\tau)^{-v}f(\gamma w) \\ &= \lim_{w\to\tau}\frac{(w-\tau)^{-v}}{(wc+d)(c\tau+d)^{-v}}(cw+d)^kf(w) \neq 0,\infty\end{aligned}$$

Consequences of Valance Formula
Write $M_k = M_k(\Gamma)$ and $S_k = S_k(\Gamma)$. Note that if $f \in M_k$ then, all terms on the left of valance formula are non negative.

Theorem 10. *Suppose* k *is an even integer. Then,*

(a) $M_0 = \mathbb{C}$
(b) $M_k = \{0\}$ *if* $k < 0$ *or* $k = 2$.
(c) $M_k = \mathbb{C}E_k$ *if* $k = 4, 6, 8, 10, 14$.
(d) $S_k = \Delta M_{k-12}$ *for all* k *(that implies* $S_k = \{0\}$ *if* $k = 4, 6, 8, 10, 14$.*)*
(e) $M_k = S_k \bigoplus \mathbb{C}E_k$ *if* $k \geq 4$.
(f) $\dim M_k = \begin{cases} 1 + [\frac{k}{12}] \textit{if } k \not\equiv 2 \pmod{12}. \\ [\frac{k}{12}] \textit{if } k \equiv 2 \pmod{12}. \end{cases}$

Proof:

(a) Suppose $f(z) \in M_0$. Let c be a value of f. Then $f(z) - c \in M_0$ and has a zero. By Valance formula $f(z) - c \equiv 0$.
(b) Obvious from the valance formula as RHS would be negative or $\frac{1}{6}$ which cannot be obtained for any integer values of v_P.
(c) Suppose $f \in M_k$. Apply the valance formula to see that
   - If $k = 4$ then $v_\omega(f) = 1$ and all others are zero.
   - If $k = 6$ then $v_i(f) = 1$ and all others are zero.
   - If $k = 8$ then $v_\omega(f) = 2$ and all others are zero
   - If $k = 10$ then $v_i(f) = v_\omega(f) = 1$ and all others are zero.
   - If $k = 14$ then $v_i(f) = 1, v_\omega(f) = 2$ and all others are zero.

   This implies that if nonzero $f_1, f_2 \in M_k$ for $k = 4, 6, 8, 10, 14$ then their zeros are the same and so $\frac{f_1}{f_2} \in M_0$. By (a) then $f_1 = cf_2$ for some $c \in \mathbb{C}$. But, $E_k \in M_k$ implies that $M_k = \mathbb{C}E_k$.
(d) Recall that $\Delta(z) \in S_{12}$ implies that $v_\infty(\Delta) = 1$ and that Δ has no other zeros. If $f \in S_k$ then $v_\infty(f) \geq 1$. Therefore, $\frac{f(z)}{\Delta(z)} \in M_{k-12} \implies f \in \Delta M_{k-12}$.
(e) Suppose $f \in M_k$ where $k \geq 4$. Write $f(z) = a(0) + a(1)q + \cdots$. Then $f(z) - a(0)E_k(z)$ vanishes at ∞ and hence is in S_k.
(f) (d) and (e) imply that $\dim M_k = \dim M_{k-12} + 1$ for all $k \geq 4$. Now the statement follows from induction. ☐

Other Consequences of Valance formula:

- We get various identities for the Eisenstein series. For example, $E_8 = E_4^2, E_{10} = E_6E_4, E_{14} = E_6E_8$ (since dim $M_k = 1$).
- If $f \in M_k$ and $v_\infty(f) > \frac{k}{12}$, then f is identically zero.
- Construction of bases. Assume $k \geq 4$. Define $\tilde{E_k} = \begin{cases} 1 \text{ if } k \equiv 0 \pmod{12}. \\ E_{14} \text{ if } k \equiv 2 \pmod{12}. \\ E_4 \text{ if } k \equiv 4 \pmod{12}. \\ E_6 \text{ if } k \equiv 6 \pmod{12}. \\ E_8 \text{ if } k \equiv 8 \pmod{12}. \\ E_{10} \text{ if } k \equiv 10 \pmod{12} \end{cases}$

  Define $\tilde{k}$ = Weight of $\tilde{E_k}$. Note $k = \tilde{k} + 12n$ where $\dim M_k = n + 1$.

Proposition 11. *If* $k \geq 4$, *then following gives a basis for* M_k.

$$\{\tilde{E_k}{E_4}^{3n}, \tilde{E_k}\Delta{E_4}^{3(n-1)}, \tilde{E_k}\Delta^2{E_4}^{3(n-2)}, \cdots, \tilde{E_k}\Delta^n\}$$

Proof: The given set is has as many elements as the dimension of M_k. So we need to check only linear independence among these elements. But, their Fourier series expansions start with $1, q, q^2$ and so on. Therefore, this is indeed a basis for M_k.

Note $\Delta = \frac{{E_4}^3 - {E_6}^2}{1728}$. This with Proposition 11 says that every element of M_k can be written as a linear combination of $\{{E_4}^a {E_6}^b\}_{4a+6b=k}.(*)$ $\square$

Proposition 12. *If $k \geq 4$, $(*)$ is a basis for M_k.*

Proof: Just need to check that $\dim M_k =$ No. of solutions to $4a + 6b = k$.

Remark. *Recall*

$$\frac{f'(z)}{f(z)} = \frac{v_\tau(f)}{z-\tau} + g(z)$$

$g(z)$ is analytic at $z = \tau$ implies that

$$\frac{1}{2\pi\tau}\int_C \frac{f'(z)}{f(z)}dz = v_\tau(f)$$

Arc $\tilde{P}$ (a part of C) has radius $\epsilon \to 0$ and angle $\to \theta$. Then

$$\frac{1}{2\pi i}\int_{\tilde{P}} \frac{f'(z)}{f(z)}dz \to \frac{\theta}{2\pi} v_\tau(f).$$

Proof of Theorem 9. Consider the given contour

- There are arcs of radius $\epsilon \to 0$ at ω, $\overline{\omega}$ and i.
- There is a detour around P on the boundary with $v_P(f) = 0$.
- HA at height T is large enough so that it has no zeros or poles above it in the fundamental domain.

Residue Theorem says that $\frac{1}{2\pi i}\int_{textContour}\frac{f'(z)}{f(z)}dz = \sum_{\tau\in\Gamma/\mathbb{H},\tau\neq i,\omega} v_\tau(f)$ Note that since $f(z+1) = f(z)$ we have, $\int_{AB} + \int_{GH} = 0$. As $\epsilon \to 0$ we have,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{BC}\frac{f'(z)}{f(z)}dz &\to \frac{\pi/3}{2\pi}v_\omega(f) = -\frac{v_\omega(f)}{6} \\
\frac{1}{2\pi i}\int_{FG}\frac{f'(z)}{f(z)}dz &\to -\frac{v_{-\overline{\omega}}}{6}(f) = -\frac{v_\omega(f)}{6} \\
\frac{1}{2\pi i}\int_{DE}\frac{f'(z)}{f(z)}dz &\to -\frac{v_i(f)}{2} \text{ as } \epsilon \to 0
\end{aligned}$$

On HA, the contour is given by $x + iT$ and $q = e^{2\pi i(x+iT)} = e^{2\pi T}e^{2\pi ix}$. So HA is a circle of radius Q : circle of radius $e^{-2\pi iT}$ over $\mathbb{C}$ oriented clockwise. We have,

$$\begin{aligned}
F(q) &= f(z) \\
f'(z) &= \frac{d}{dz}F(q) = F'(q)\frac{dq}{dz} \\
\frac{1}{2\pi i}\int_{HA}\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_Q \frac{F'(q)}{F(q)}dq \\
&= -v_\infty(f)
\end{aligned}$$

Claim : $\frac{1}{2\pi i}(\int_{CD}+\int_{EF}) \to \frac{k}{12}$ *as* $\epsilon \to 0$.

Reason : Let $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S(z) = -\frac{1}{z}$. S takes EF to DC and vice versa.

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{EF}\frac{f'(z)}{f(z)}dz \\ &= -\frac{1}{2\pi i}\int_{CD}\frac{f'(S(z))}{f(S(z))}dS(z) \end{aligned}$$

Recall that

$$f(S(z)) = f(-\frac{1}{z}) = z^k f(z) \implies F'(S(z))\frac{dS(z)}{dz} = z^k f'(z) + kz^{k-1}f(z)$$

So,

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{CD}\frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}k\int_{CD}\frac{1}{z} \\ \implies \frac{1}{2\pi i}\int_{CD}\frac{f'(z)}{f(z)}dz + \int_{EF}\frac{f'(z)}{f(z)}dz &= -\frac{1}{2\pi i}\int_{CD}\frac{k}{z}dz \\ &\to -\frac{1}{2\pi i}\int_{e^{\frac{2\pi i}{3}}}^{e^{\frac{2\pi i}{4}}} k\frac{1}{z}dz \text{ as } \epsilon \to 0 \\ &= -\frac{k}{2\pi i}2\pi i(\frac{1}{4}-\frac{1}{3}) \\ &= \frac{k}{12} \end{aligned}$$

Now putting all the integrals together we get the valance formula. □

The j function

Define $j(z) := \frac{E_4^3(z)}{\Delta(z)} = q^{-1}+744+1968884q+\cdots$. where $E_4(z) = 1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n$ and $\Delta(z) = \frac{E_4^3-E_6^2}{1728}$. Then $j(z)$ defines a modular form on Γ. Note $j(z) = \frac{1728E_4^3}{E_4^3-E_6^2} \implies j(z) - 1728 = \frac{E_6^2}{\Delta(z)}$.

Facts:

$j(z)$ is holomorphic on $\mathbb{H}$. It has a simple pole at ∞ and zero of multiplicity 3 at ω.

E_6 has a simple zero at i. Therefore $j(z) - 1728$ has a double zero at i.

Theorem 13. (a) *$j(z)$ gives a bijection between $\Gamma\backslash\overline{\mathbb{H}} = \Gamma\backslash(\mathbb{H}\cup\infty)$ and $\overline{\mathbb{C}} = \mathbb{C}\cup\infty$.*

(b) *Let $\mathcal{M}$= Set of meromorphic modular functions of weight k on Γ. Then $\mathcal{M} = \mathbb{C}(j(z))$ (rational functions in j.)*

Proof:

(a) Note $j(p) = \infty$ if and only if $p = \infty$. Suppose $c \in \mathbb{C}$ and $h(z) = j(z) - c$. Then h has a simple pole at ∞. Valance formula says h has exactly 1 zero.

(b) Suppose $f \in \mathcal{M}$. Note that Valance formula says that $v_i(f) \equiv 0 \pmod 2$, $v_\omega(f) \equiv 0 \pmod 3$ Let $\tau_1, \cdots, \tau_n$ be all the zeros and poles in $\Gamma/\mathbb{H}$ (without multiplicity) distinct from i, ω.

Let $g(z) = j(z)^{\frac{v_\omega(f)}{3}}(j(z)-1728)^{\frac{v_i(f)}{2}}\prod_{i=1}^n (j(z)-j(\tau_i))^{v_{\tau_i}(f)}$

Then $h(z) = \frac{f(z)}{g(z)}$ has no zeros or poles on $\mathcal{H}$. Valance formula gives that $v_\infty(h) = 0$, i.e., h has no zeros or poles at ∞, $h \in \mathcal{M}_0$. Therefore, h is a constant function. Hence f is a rational function in j.

$\square$

4. O.Y. Chan. 9/15 and 9/17

The Dedekind Eta Function.

Define the Dedekind eta function by:

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n); \quad q = e^{2\pi iz}; \quad z \in \mathbb{H}$$

This product converges absolutely on $\mathbb{H}$. (i.e., the sum $\sum |q|^n$ converges).

This implies that $\eta(z)$ is non-vanishing on $\mathbb{H}$. It converges uniformly on compact subsets, and is holomorphic on $\mathbb{H}$.

Theorem 14. *Let $\sqrt{\cdot}$ represent the branch with non-negative real part ($\sqrt{1} = 1$). Then we have:*

$$\eta\left(\frac{-1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z) \tag{1}$$

Idea of proof. The logarithmic derivative of (1) is equivalent to
$E_2(-1/z) = z^2E_2(z) + \frac{12z}{2\pi i}$.

Definition. $F(z) := \eta^{24}(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$

Notes:

1. $F(z)$ is holomorphic on $\mathbb{H}$, and vanishes at ∞.
2. $F(z+1) = F(z)$.
3. $F(-1/z) = z^{12}F(z)$. Thus F is a cusp form of weight 12.
   But since $S_{12} = \mathbb{C}\Delta$, we have:
4. $F(z) = \Delta$.

Theorem 15.

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728} = F(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

Modular Forms on Congruence Subgroups.

Recall:

$$\Gamma(N) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\right\}$$

$$\Gamma_0(N) = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

are all subgroups of Γ.

Definition. *Call a subgroup Γ' of Γ a a congruence subgroup of level N if $\Gamma(N) \leq \Gamma'$.*

Example. *Let $f(z) = \Delta(2z)$. Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$. Set $\gamma' = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$. Then*

$$\begin{aligned} f(\gamma z) = \Delta(2\gamma z) = \Delta\left(\frac{2az+2b}{cz+d}\right) &= \Delta\left(\frac{a(2z)+2b}{\frac{c}{2}(2z)+d}\right) \\ = \Delta\left(\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}(2z)\right) &= \Delta(\gamma'(2z)) \\ = (\frac{c}{2}(2z)+d)^{12}\Delta(2z) &= (cz+d)^{12} f(z) \end{aligned}$$

since $\gamma' \in \Gamma$.

Notation. *The Slash Operator:*

Suppose $f(z)$ is meromorphic on $\mathbb{H}$, $k \in \mathbb{Z}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$. Define

$$f(z)|[\gamma]_k = f(z)|_k\gamma := \det(\gamma)^{k/2}(cz+d)^{-k} f(\gamma z)$$

Note: f is a meromorphic modular form of weight k for $\Gamma \iff f|_k\gamma = f$ for all $\gamma \in \Gamma$.

Note: $\left(\frac{d}{dz}(\gamma z)\right)^{k/2} = \det(\gamma)^{k/2}(cz+d)^{-k}$. Thus, the chain rule gives us:

$$f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2$$

Definition. *Suppose $\Gamma' < \Gamma$ is a congruence subgroup of level N. Suppose $k \in \mathbb{Z}$. Then $f(z)$ is a meromorphic modular form of weight k for Γ' if:*

1. *f is meromorphic on $\mathbb{H}$.*
2. *$f|_k\gamma = f$, $\forall\, \gamma \in \Gamma'$*
3. *$\forall\, \gamma_0 \in \Gamma, f|_k\gamma_0$ has the form*

$$\sum_{n=-\infty}^{\infty} a(n) q_N^n$$

with $a(n) = 0$ for all sufficiently small n, and $q_N := q^{1/N} = e^{2\pi i/N}$.

Definition. *f is a modular form if f is a meromorphic modular form that is holomorphic on $\mathbb{H}$ and at the cusps. (i.e., $a(n) = 0$ for all $n < 0$ in (3)).*

Definition. *f is a cusp form if f is a modular form and $a(0) = 0$ for all γ_0 in (3).*

We denote the spaces of modular forms and cusp forms by $M_k(\Gamma'), S_k(\Gamma')$ respectively.

Note: $\Gamma'' < \Gamma' \Rightarrow M_k(\Gamma') \subseteq M_k(\Gamma'')$.

Interpretation of (3) as a condition "at the cusps".

Recall that if $g(z+N) = g(z)$, then g has a Fourier expansion

$$g(z) = \sum a(n) q_N^n$$

Note: Since $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$, if $g|_k\gamma_0 = g$, $\forall \gamma_0 \in \Gamma(N)$, then $g(z+N) = g(z)$.

Suppose $\gamma_0 \in \Gamma$, and set $g(z) = f|_k\gamma_0$. Suppose further that $f|_k\gamma' = f$ for all $\gamma' \in \Gamma'$.

Note: If $\gamma_0^{-1}\gamma\gamma_0 \in \gamma_0^{-1}\Gamma'\gamma_0$, then we have:

$$\begin{aligned} g|_k\gamma_0^{-1}\gamma\gamma_0 &= f|_k\gamma_0\gamma_0^{-1}\gamma\gamma_0 = f|_k\gamma\gamma_0 \\ &= f|_k\gamma_0 = g \end{aligned}$$

Recall: $\Gamma(N) < \Gamma'$ and $\Gamma(N)$ is normal in Γ. Thus

$$\begin{aligned} &\Gamma(N) < \gamma_0^{-1}\Gamma'\gamma_0 \\ &\Rightarrow g(z+N) = g(z) \\ &\Rightarrow \text{we can write } g(z) = \sum a(n) q_N^n \end{aligned}$$

Condition (3) refers to these expansions.

Recall: "cusps" are $\mathbb{Q} \cup \{\infty\}$ and "cusp of Γ'" is an equivalence class under the action of Γ'.

Our next result shows that condition 3 depends only on the Γ' equivalence class of the cusp in question.

Proposition 16. *Suppose Γ' is a congruence subgroup of level N.*

Suppose $f|_k\gamma = f$ for all $\gamma \in \Gamma'$.

Suppose $\gamma_1\infty, \gamma_2\infty$ are cusps, $\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$, and that $\gamma_1\infty = \gamma'\gamma_2\infty$ for some $\gamma' \in \Gamma'$.

Write $f|_k\gamma_1 = \sum a(n)q_N^n$, $f|_k\gamma_2 = \sum b(n)q_N^n$.

Then $\exists j \in \mathbb{Z}$ such that

$$b(n) = (\pm 1)^k e^{\frac{2\pi i n j}{N}} a(n), \quad \forall n$$

In particular, $a(n) = 0 \iff b(n) = 0$.

Proof of Proposition 16.

Given that $\gamma_1^{-1}\gamma'\gamma_2\infty = \infty$, this implies
$\gamma_1^{-1}\gamma'\gamma_2 = \pm T^j$ for some $j \in \mathbb{Z}$.
$\Rightarrow \gamma_2 = \pm(\gamma')^{-1}\gamma_1 T^j$
$\Rightarrow f|_k\gamma_2 = f|_k(\pm I(\gamma')^{-1}\gamma_1 T^j)$.

Note that $f|_k(-I) = (-1)^k f$, so that
$f|_k\gamma_2 = (\pm 1)^k f|_k(\gamma')^{-1}\gamma_1 T^j = (\pm 1)^k f|_k\gamma_1 T^j$ since $\gamma' \in \Gamma'$.
Therefore we get

$$\begin{aligned} f|_k\gamma_2 &= \sum b(n) e^{\frac{2\pi i n z}{N}} \\ &= (\pm 1)^k \sum a(n) e^{\frac{2\pi i n z}{N}} \Big|_k \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = (\pm 1)^k \sum a(n) e^{\frac{2\pi i n (z+j)}{N}} \end{aligned}$$

The proposition follows by equating coefficients of q_N. ☐

Thus condition 3 is a finite set of conditions, one for each cusp of Γ'.

Example. (1) $\Gamma' = \Gamma$. *There is only 1 cusp:* ∞. $f(z) = \sum_{n=0}^{\infty} a(n)q^n$.
(2) $\Gamma' = \Gamma(p)$, *p prime. There are 2 cusps:* $0, \infty$. *There are then 2 conditions:*

(a) $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ *(at* ∞*)*

(b) $f\Big|_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = z^{-k}f(-1/z) = \sum_{n=0}^{\infty} b(n)q_p^n$. *(at 0)*

Dirichlet Characters.

Definition. *Given a homomorphism* $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \longrightarrow \mathbb{C}$, *define a Dirichlet Character* $\chi' : \mathbb{Z} \longrightarrow \mathbb{C}$ *by:*

$$\chi'(n) = \begin{cases} \chi(\bar{n}) & \text{if} \quad (n, N) = 1 \\ 0 & \text{if} \quad (n, N) > 1 \end{cases}$$

Note that χ' has period N, is completely multiplicative, and we will write $\chi' = \chi$.

Definition. *The trivial character mod N (or principal character mod N) is defined by*

$$\chi_N^{triv}(n) = \begin{cases} 1 & \text{if} \quad (n, N) = 1 \\ 0 & \text{if} \quad (n, N) > 1 \end{cases}$$

Definition. *Suppose χ is a Dirichlet character mod N, call d an induced modulus for χ if $\chi(a) = 1$ whenever $(a, N) = 1$ and $a \equiv 1 \pmod{d}$.*

Definition. *The conductor of χ is the smallest positive induced modulus*

Note: The conductor of χ_N^{triv} is 1.

Example. *Consider a character χ mod 12 defined by:*

n	$\chi(n)$
1	*1*
5	*1*
7	*-1*
11	*-1*

The conductor of χ is 4, and we may define χ_1 mod 4 by

n	$\chi_1(n)$
1	*1*
3	*-1*

Notice that $\chi = \chi_1\chi_{12}^{triv}$

Definition. *A character χ mod N is primitive if the conductor is N.*

Theorem 17. *If χ is a Dirichlet character mod N, then $\chi = \chi_1\chi_N^{triv}$, where χ_1 is a primitive character modulo the conductor of χ.*

5. J. PAULHUS. 9/19 AND 9/22

Recall that to say $f \in M_k(\Gamma_1(N))$ means that $f|_k\gamma = f$ for all $\gamma \in \Gamma_1(N)$ plus some cusp conditions and "holomorphicness".

Now suppose we have ourselves a Dirichlet character $\chi \mod N$.

Definition. $M_k(\Gamma_0(N), \chi) := \{f \in M_k(\Gamma_1(N)) : f|_k\gamma = \chi(d)f \ \forall \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\}$.

Note that since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma_0(N)$ then $N|c$. Combining this with the fact that $\det(\gamma) = 1$ we get that $(d, N) = 1$. Additionally if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma_1(N)$, then $d \equiv 1 \mod N$ and so $\chi(d) = 1$.

Note also that $-I \in \Gamma_0(N)$. So $f \in M_k(\gamma_0(N), \chi)$ implies that $f|_k \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \chi(-1)f$. But by definition, $f|_k \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)^k f$ which tells us that if $\chi(-1) \neq (-1)^k$ then $M_k(\Gamma_0(N), \chi)$ is trivial (we call this the "parity condition").

The following is Proposition 28 in Koblitz.

Theorem 17. $M_k(\Gamma_1(N)) = \bigoplus_{\chi \mod N} M_k(\Gamma_0(N), \chi)$, *and similarly with* M_k *replaced with* S_k.

Some Representation Theory

The setup is as follows: G is a finite group and V a vector space over, say $\mathbb{C}$ for simplicity, of dimension n. A representation of G in V is a homomorphism ρ from G into $GL(V)$ where $GL(V)$ is the set of invertible linear transformations from V to V. We can identify $GL(V)$ with $GL_n(\mathbb{C})$ once we fix a basis.

We need the following to prove Theorem 17.

Fact. *If G is an abelian group then ρ decomposes as a direct sum of characters (1 dimensional representations $\chi : G \to \mathbb{C}$) i.e. there exists a basis of V such that*

$$\rho(g) \sim \begin{pmatrix} \chi_1(g) & & & \\ & \chi_2(g) & & \\ & & \ddots & \\ & & & \chi_n(g) \end{pmatrix}$$

with zeros off the diagonal.

Read the first 10 pages of an algebra book with representation theory to see the proof of this!

Proof of Theorem 17. We begin with the following exact sequence:

$$0 \to \Gamma_1(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \to 0 \tag{2}$$

where the map from $\Gamma_0(N)$ to $(\mathbb{Z}/N\mathbb{Z})^*$ is defined by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\overline{d}$. Since $\Gamma_1(N)$ is the kernel of this map, it is a normal subgroup of $\Gamma_0(N)$ with $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. Now $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are in the same coset of $\Gamma_0(N)/\Gamma_1(N)$ $\iff d \equiv d' \mod N$. So suppose $f \in M_k(\Gamma_1(N))$ and γ, γ' are in the same coset. Then $\gamma = \gamma_N \gamma'$ where $\gamma_N \in \Gamma_1(N)$ which implies $f|_k\gamma = f|_k\gamma_N\gamma' = f|_k\gamma'$. (It is important to note that we only care about cosets here).

Now we define $<>$: $(\mathbb{Z}/N\mathbb{Z})^* \to GL(M_k(\Gamma_1(N)))$ which sends $\overline{d}$ to $<\overline{d}>$ where $f|<\overline{d}>:= f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ any matrix in $\Gamma_0(N)$ with $d \equiv \overline{d} \mod N$.

$<>$ is well defined by above comments. Finally there exists an exercise in checking details to show that $<>$ is a representation. Then the remark above implies that $<>$ decomposes as a sum of characters $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$, i.e. there is a basis for $M_k(\Gamma_1(N))$ such that $<\overline{d}> \sim \begin{pmatrix} \chi_1(d) & & & \\ & \chi_2(d) & & \\ & & \ddots & \\ & & & \chi_n(d) \end{pmatrix}$ where the entries off the diagonal are zero.

Recall from the definition that $M_k(\Gamma_0(N), \chi) = \{f \in M_k(\Gamma, (N)) : f|<\overline{d}>= \chi(d)f\}$. By collecting subspaces that correspond to distinct characters χ we get the result. ☐

Other Operators on Modular Forms

Let's define some other operators on spaces of modular forms.

Definition. *For $t \in \mathbb{N}$ define $f(z)|V_t := f(tz)$. This is the V-operator.*

Definition. *If $f = \sum a(n)q^n$ and χ is a Dirichlet character, define $f \otimes \chi := \sum \chi(n)a(n)q^n$. We call this the twisting operator and say "f is twisted by χ".*

Proposition 18. *The map $f \to f|V_t$ takes $M_k(\Gamma_0(N), \chi)$ to $M_k(\Gamma_0(Nt), \chi)$.*

Proposition 19. *Suppose $f \in M_k(\Gamma_0(M), \psi)$ and suppose that χ is a Dirichlet character $\mod N$. Let $T := lcm(M, N^2, N \cdot cond(\psi))$. Then $f \otimes \chi$ is in $M_k(\Gamma_0(T), \psi\chi^2)$.*

Propositions 18 and 19 are still true with M_k replaced by S_k.

Note, we always have $f \otimes \chi \in M_k(\Gamma_0(MN^2), \psi\chi^2)$ (in a way this is the worst we can do). There seems to be a gap in Koblitz' proof. He uniformly substitutes $\chi(v)\overline{\chi}(v)$ with 1 for all v but this is only true if $(N, v) = 1$. The ideas in Koblitz' proof are correct, though.

Let's look at an example:

Example. $\Delta(z) \in S_{12}(\Gamma_0(1))$, $\Delta \otimes \chi_N^{triv} \in S_{12}(N^2, \chi_{triv}) = S_{12}(\Gamma_0(N^2))$ *where* $\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$ *and so* $\Delta \otimes \chi_N^{triv} = \sum_{(n,N)=1} \tau(n)q^n$.

This demonstrates that anything reasonable you do to the index set preserves modularity in a very precise way. We also have a silly example:

Example. $\chi(n) = \left(\frac{n}{p}\right)$ *with* p *prime. Then* $\Delta \otimes \chi = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \tau(n) q^n \in S_{12}(\Gamma_0(p^2))$.

Now recall the cusp condition: If $\gamma_0 \in \Gamma$ then

$$f|_k \gamma_0 = \sum_{n=n_0}^{\infty} a(n) q_N^n \tag{3}$$

where $n_0 = \begin{cases} 0, & \text{if modular form;} \\ 1, & \text{if cusp form.} \end{cases}$

Lemma 20. *Suppose (3) holds with either* $n_0 = 0$ *for all* $\gamma_0 \in \Gamma$ *or* $n_0 = 1$ *for all* $\gamma_0 \in \Gamma$*. Then for all* $\alpha \in GL_2^+(\mathbb{Q})$*,* $f|_k \alpha$ *has the form* $\sum_{n=an_0}^{\infty} b(n) q_{Nd}^n$ *for some positive integers* a *and* d *(which depend on* α*).*

Proof of Lemma 20. If $A > 0$ then $f|_k \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = f$ (i.e. the positive scalar matrices act trivially). So we may assume α has integer entries. We find a $\gamma_0 \in \Gamma$ such that $\gamma_0^{-1}\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where a, d are positive integers (an "exercise in linear algebra").

Then $f|_k \alpha = f|_k \gamma_0 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$$= \left(\sum_{n=n_0}^{\infty} a(n) e^{\frac{2\pi i n z}{N}} \right) |_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (ad)^{\frac{k}{2}} (d)^{-k} \left(\sum_{n=n_0}^{\infty} a(n) e^{\frac{2\pi i n}{N}\left(\frac{az+b}{d}\right)} \right).$$

We only care about the first term: $(ad)^{\frac{k}{2}} (d)^{-k} a(n_0) e^{\frac{2\pi i n_0 b}{Nd}} e^{\frac{2\pi i n_0 a z}{Nd}} + \cdots = (\text{constant}) \cdot q_{Nd}^{n_0 a} + \cdots$. $\square$

Proof of Proposition 18. Suppose $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$. Define $g(z) := f(z)|V_t = t^{-k/2} f(z)|_k \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. But then Lemma 20 implies that g satisfies the same cusp conditions as f. We check the Transformation Law: Suppose $\gamma = \begin{pmatrix} a & b \\ cNt & d \end{pmatrix} \in \Gamma_0(Nt)$. Then $g(z)|_k \gamma = t^{(-k/2)} f(z)|_k \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ cNt & d \end{pmatrix}$ which is $t^{(-k/2)} f|_k \begin{pmatrix} at & bt \\ cNt & d \end{pmatrix}$ $= t^{(-k/2)} f|_k \begin{pmatrix} a & bt \\ cN & d \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = t^{(-k/2)} \chi(d) f|_k \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \chi(d) g$. $\square$

Some Facts from Finite Fourier Analysis

We set $\zeta := e^{2\pi i/N}$. Then

$$\sum_{m=0}^{N-1} \zeta^{mn} = \begin{cases} 0, & \text{if } N \nmid n; \\ N, & \text{if } N \mid n. \end{cases}$$

Suppose that χ is any function on the integers with period N (You may think of χ as a function on $\mathbb{Z}/N\mathbb{Z}$.) We define the Fourier transform $\widehat{\chi}(v) := \frac{1}{N} \sum_{m \bmod N} \chi(m)\zeta^{-mv}$.

Fact. $\chi(n) = \sum_{v \bmod N} \widehat{\chi}(v)\zeta^{nv}$.

Proof of the fact. The right hand side of the above equation is:
$\sum_{v \bmod N} \frac{1}{N} \sum_{m \bmod N} \chi(m)\zeta^{-mv}\zeta^{nv}$
$= \frac{1}{N} \sum_{m \bmod N} \chi(m) \sum_{v \bmod N} \zeta^{v(n-m)}$ but this is just $\chi(n)$.

Proof of Proposition 19. Suppose $f \in M_k(\Gamma_0(M), \psi)$. Then: $f \otimes \chi = \sum_{n=0}^{\infty} \chi(n)a(n)q^n$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{v \bmod N} \widehat{\chi}(v)\zeta^{nv}a(n)q^n \\
&= \sum_{v \bmod N} \widehat{\chi}(v) \sum_{n=0}^{\infty} a(n)e^{\frac{2\pi inv}{N}} e^{2\pi inz} \\
&= \sum_{v \bmod N} \widehat{\chi}(v) \sum_{n=0}^{\infty} a(n)e^{2\pi in}(z + v/N) \\
&= \sum_{v \bmod N} \widehat{\chi}(v) f(z + v/N) \\
&= \sum_{v \bmod N} \widehat{\chi}(v) f|_k \begin{pmatrix} 1 & v/N \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Applying Lemma 20, we see that $f \otimes \chi$ satisfies the same cusp condition as f. Let $T = \operatorname{lcm}(M, N^2, N \cdot \operatorname{cond}(\psi))$. If we can show that $(f \otimes \chi)|_k\gamma = \psi\chi^2(d)(f \otimes \chi)$ for $\gamma \in \Gamma_0(T)$ (Transformation Law) we will be done.

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Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(T)$. It remains to show that $(f \otimes \chi)|_k\gamma = \psi\chi^2(d) f \otimes \chi$. Let $\gamma_v := \begin{pmatrix} 1 & v/N \\ 0 & 1 \end{pmatrix}$. Then

$$\gamma_v \gamma \gamma_{v'}^{-1} = \begin{pmatrix} a + \frac{vc}{N} & b + \frac{vd - av'}{N} - \frac{vv'c}{N^2} \\ c & d - \frac{v'c}{N} \end{pmatrix}.$$

Now $N^2|c$, and $(a, N) = (d, N) = 1$, so for each $v \bmod N$, there is a unique $v' \bmod N$ such that $vd \equiv av' \bmod N$. For this v', the entries in $\gamma_v\gamma\gamma_{v'}^{-1}$ are all integers, and since $M|c$, $\gamma_v\gamma\gamma_{v'}^{-1} \in \Gamma_0(M)$. We also have $\gamma \in \Gamma_0(M)$ so $\det\gamma = 1$ and hence $(d, M) = 1$.

For the same reason, $(d - \frac{v'c}{N}, M) = 1$. By assumption, $\frac{c}{N}$ is divisible by $\mathrm{cond}(\psi)$, so $\psi(d - \frac{v'c}{N}) = \psi(d)$. Then

$$f \otimes \chi|_k\gamma = \sum_{v \bmod N} \hat{\chi}(v) f|_k\gamma_v\gamma = \sum_{v \bmod N} \hat{\chi}(v)(f|_k\gamma_v\gamma\gamma_{v'}^{-1})|_k\gamma_{v'} = \psi(d) \sum_{v \bmod N} \hat{\chi}(v) f|_k\gamma_{v'}.$$

Claim. $\hat{\chi}(v) = \chi^2(d)\hat{\chi}(v')$.

The claim implies that

$$(f \otimes \chi)|_k\gamma = \psi(d) \sum_{v \bmod N} \chi^2(d)\hat{\chi}(v') f|_k\gamma_{v'} = \psi(d)\chi^2(d) f \otimes \chi.$$

Proof of Claim. By definition, and since $dv \equiv av' \mod N$,

$$\hat{\chi}(v) = \frac{1}{N} \sum_{m \bmod N} \chi(m)\zeta^{-mv} = \frac{1}{N} \sum_{m \bmod N} \chi(dm)\zeta^{-mdv} = \frac{\chi(d)}{N} \sum_{m \bmod N} \chi(m)\zeta^{-mav'}$$
$$= \frac{\chi(d)\overline{\chi}(a)}{N} \sum_{m \bmod N} \chi(am)\zeta^{-mav'} = \chi(d)\overline{\chi}(a)\hat{\chi}(v').$$

Then $\gamma \in \Gamma_0(N)$ implies $ad \equiv 1 \bmod N$ so $\overline{\chi}(a) = \chi(d)$. Hence $\chi(d)\overline{\chi}(a)\hat{\chi}(v') = \chi^2(d)\hat{\chi}(v')$. ☐

Example. Let p be prime. Let $f(z) := \sum_{(\frac{n}{p})=1} \sigma_{k-1}(n)q^n$, where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Recall that $E_k = 1 - \frac{2k}{B_k}\sum_n \sigma_{k-1}(n)q^n$. Then

$$-\frac{B_k}{2k}E_k \otimes \chi_p^{triv} = \sum_{(n,p)=1} \sigma_{k-1}(n)q^n$$

and

$$-\frac{B_k}{2k}E_k \otimes \left(\frac{\bullet}{p}\right) = \sum_n \left(\frac{n}{p}\right)\sigma_{k-1}(n)q^n,$$

so

$$f(z) = \frac{1}{2}\left(-\frac{B_k}{2k}E_k \otimes \chi_p^{triv} - \frac{B_k}{2k}E_k \otimes \left(\frac{\bullet}{p}\right)\right).$$

Some examples: theta function and eta function

Theta function:

Recall that $M_k(\Gamma_1(4)) = \bigoplus_{\chi \bmod 4} M_k(\Gamma_0(4), \chi) = M_k(\Gamma_0(4)) \oplus M_k(\Gamma_0(4), \chi_{-4})$ where

$$\chi_{-4}(d) = \left(\frac{-4}{d}\right) = \begin{cases} (-1)^{(d-1)/2} & d \text{ odd} \\ 0 & d \text{ even} \end{cases}.$$

χ_4^{triv} is an even character and χ_{-4} is odd.

Proposition 21. $M_k(\Gamma_1(4)) = \begin{cases} M_k(\Gamma_0(4)) & k \text{ even} \\ M_k(\Gamma_0(4), \chi_{-4}) & k \text{ odd} \end{cases}.$

Definition. $\Theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + \cdots$.

Theorem 22. $\Theta^2 \in M_1(\Gamma_0(4), \chi_{-4})$.

This theorem implies that $\Theta^{2k} \in M_k(\Gamma_0(4), \chi_{-4}^k)$ for $k = 1, 2, \ldots$. Note that Θ is a modular form of weight $\frac{1}{2}$. However, it takes work to define what this means. This will be done later.

Definition. $\Theta_0(t) = \sum\limits_{n=\infty}^{\infty} e^{-\pi n^2 t}$, *Re*$(t) > 0$.

Proposition 23. $\Theta_0(t) = \frac{1}{\sqrt{t}}\Theta_0\left(\frac{1}{t}\right)$, *Re*$(t) > 0$.

Proof of Proposition 23. This is a 453 result. For a proof, the reader is referred to Koblitz §II.4. □

Proof of Theorem 22. First note that $\Theta(z) = \Theta_0(-2iz)$. Then

$$\Theta(z) = \Theta_0(-2iz) = \frac{1}{\sqrt{-2iz}}\Theta_0\left(-\frac{1}{2iz}\right) = \frac{1}{\sqrt{-2iz}}\Theta_0\left(-2i\left(-\frac{1}{4z}\right)\right) = \frac{1}{\sqrt{-2iz}}\Theta\left(-\frac{1}{4z}\right).$$

This implies that $\Theta^2(z) = -\frac{1}{2iz}\Theta^2\left(-\frac{1}{4z}\right)$. Set $\alpha_4 = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$. Then

$$\Theta^2|_1\alpha_4 = 2(4z)^{-1}\Theta^2\left(-\frac{1}{4z}\right) = 2(4z)^{-1}(-2iz)\Theta^2(z) = -i\Theta^2(z).$$

By problem 13 of section III.1 in Koblitz, $\Gamma_0(4)$ is generated by $-I$, T, and ST^4S. So to verify the transformation law, it is enough to check it for these matrices. First, $\Theta^2|_1T = \Theta^2$ is clear since $\Theta^2(z+1) = \Theta^2(z)$, and $\Theta^2|_1 - I = -\Theta^2 = \chi_{-4}(-1)\Theta^2$. Now $ST^4S = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}\alpha_4 T\alpha_4 = \begin{pmatrix} -1 & 0 \\ -4 & -1 \end{pmatrix}$. Then

$$\Theta^2|_1ST^4S = \Theta^2|_1\begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}\alpha_4T\alpha_4 = \Theta^2|_1\alpha_4T\alpha_4 = -i\Theta^2|_1T\alpha_4$$
$$= -i\Theta^2|_1\alpha_4 = (-i)^2\Theta^2 = -\Theta^2 = \chi_{-4}(-1)\Theta^2.$$

Finally, $\Gamma_0(4)$ has three cusps: ∞, 0, and $\frac{1}{2}$. Since $\Theta^2 = 1 + 4q + \cdots$, Θ^2 is clearly holomorphic at ∞. Next, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\infty = 0$, and

$$\Theta^2|_1\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \Theta^2|_1\alpha_4\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} = -i\Theta^2|_1\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{i}{2} + \cdots,$$

so Θ^2 is holomorphic at 0. The cusp condition at $\frac{1}{2}$ is left as an exercise: compute the expansion of $\Theta^2|_1\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. (Hint: see III.3 #1 in Koblitz.) □

Eta function:

Next we will use the function $\eta(z) = q^{1/24}\prod\limits_{n=1}^{\infty}(1-q^n)$ to build modular forms on congruence subgroups, but first we look at some technicalities at the cusps of $\Gamma_0(N)$. Let $\Gamma = \bigcup\limits_{i=1}^{t}\Gamma_0(N)A_i$ be a coset decomposition of $\Gamma = SL_2(\mathbb{Z})$, where $t = N\prod\limits_{p|N}(1+\frac{1}{p})$.

We could also write $\Gamma = \bigcup_{A\in\Gamma_0(N)\backslash\Gamma} \Gamma_0(N)A$. Suppose $s = \gamma\infty$, $\gamma \in \Gamma$, is a cusp. Then $\gamma \in \Gamma_0(N)A_i$ for some i, so $s = \gamma\infty \overset{\Gamma_0(N)}{\sim} A_i\infty$. Therefore the cusps are contained in $\{A_i\infty\}$.

Definition. *Given a cusp $s = A\infty$, the* fan width *of s, denoted h_s, is the number of distinct cosets $\Gamma_0(n)B$ that take ∞ to the equivalence class of s.*

The name arises from the picture of the fundamental domain of $\Gamma_0(N)$. There are h_s triangles meeting at the point s which create a fan shape.

Proposition 24. *With the notation as above, suppose $s = A\infty$. Then h_s is the least positive integer h_0 such that $AT^{h_0}A^{-1} \in \Gamma_0(N)$. Furthermore, $h_s|N$.*

Proof of Theorem 24. We first show that h_s is well-defined. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then $AT^hA^{-1} = \begin{pmatrix} 1-cah & a^2h \\ -c^2h & 1+cah \end{pmatrix}$, so $AT^NA^{-1} \in \Gamma_0(N)$, and hence h_0 exists. Note that $AT^NA^{-1} \in \Gamma_0(N)$ and $AT^{h_0}A^{-1} \in \Gamma_0(N)$ imply that $AT^{(h_0,N)}A^{-1} \in \Gamma_0(N)$, so $h_0|N$ by the minimality of h_0. Now recall that

$$\begin{aligned} A\infty \overset{\Gamma_0(N)}{\sim} B\infty &\iff A\infty = \gamma B\infty \qquad \gamma \in \Gamma_0(N) \\ &\iff A^{-1}\gamma B = T^n \qquad \text{for some } n \\ &\iff AT^nB^{-1} \in \Gamma_0(N) \\ &\iff \Gamma_0(N)B = \Gamma_0(N)AT^n. \end{aligned}$$

Then since h_0 is the least positive integer with $\Gamma_0(N)A = \Gamma_0(N)AT^{h_0}$, the distinct cosets taking ∞ to s are $\Gamma_0(N)A, \Gamma_0(N)AT, \ldots, \Gamma_0(N)AT^{h_0-1}$. Thus $h_s = h_0$. □

Note. If $s = A\infty = \frac{a}{c}$ then $h_s = \frac{N}{(c^2,N)}$.

Suppose f is a meromorphic modular form of weight k on $\Gamma_0(N)$, $s = A\infty = \frac{a}{c}$ is a cusp, and h_s is the fan width. We know that $f|_kA$ has the form $\sum a(n)q_N^n$. But $AT^{h_s}A^{-1} = \gamma \in \Gamma_0(N)$, so $AT^{h_s} = \gamma A$, and hence $(f|_kA)|_kT^{h_s} = f|_k\gamma A = f|_kA$. Thus $f|_kA$ has period h_s, which implies that $f|_kA = \sum a(n)q_{h_s}^n$.

Definition. *We call q_{h_s} a* local variable *at the cusp s.*

Note that $1 + cah_s$ is the lower right entry of $AT^{h_s}A^{-1}$, so if $f \in M_k(\Gamma_0(N), \chi)$ then $f|_kA$ has the form $\sum a(n)q_{h_s}^{n+\mu}$, where $\chi(1 + cah_s) = e^{2\pi i\mu}$.

7. S. H. Chan. 9/29 and 10/1

Recall: Let $f \in M_k(\Gamma_0(N))$ and $s = \frac{a}{c}$ a cusp. If $s = A\infty$, then let h_s denote the least positive integer such that $AT^{h_s}A^{-1} \in \Gamma_0(N)$. h_s is known as the "fan width".

$$h_s = \frac{N}{(c^2, N)}, \qquad f|_kA = \sum_{n\geq n_0} a(n)q_{h_s}^n, \qquad (a(n_0) \neq 0).$$

We define $\mathrm{ord}_s f := n_0$, the "order of vanishing with respect to local variable h_s".

8. Eta function

We define the eta function by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Then

$$\Delta(q) = \eta^{24}(z) \text{ and } \Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z) \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

This implies that

$$\Delta\left(\frac{az+b}{cz+d}\right) = \varepsilon\sqrt{cz+d}\eta(z),$$

where ε is some 24^{th} root of unity. ε can be determined.

For $h, k \in \mathbb{Z}$, we defind the Dedekind Sum as

$$S(h,k) := \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

Theorem 25 (Dedekind). *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* $c > 0$*, then*

$$\eta\left(\frac{az+b}{cz+d}\right) = exp\left(\pi i \left(\frac{a+d}{12c} + S(-d,c)\right)\right)(-i)^{1/2}(cz+d)^{1/2}\eta(z),$$

where $exp\left(\pi i\left(\frac{a+d}{12c} + S(-d,c)\right)\right)(-i)^{1/2}$ is a 24-th root of unity.

Proof. See Apostol Chapter 3. ☐

Note that sums of $S(h,k)$ satisfy certain "reciprocity laws". This implies that we can deduce transformation laws for eta-quotients.

Definition. *An eta-quotient is a function of the form*

$$f(z) = \eta^{r_1}(\delta_1 z)\eta^{r_2}(\delta_2 z)\dots\eta^{r_t}(\delta_t z),$$

where $\delta_i \in \mathbb{N}, r_i \in \mathbb{Z}$.

It is clear that we can express

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z), \tag{4}$$

where N is any multiple of $\operatorname{lcm}(\delta_1, \dots, \delta_t)$.

Example.

$$\begin{aligned} f_1(z) =& \eta^2(z)\eta^2(11z), \\ f_2(z) =& \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}, \end{aligned}$$

and

$$f_3(z) = \frac{\eta^l(lz)}{\eta(z)} \text{ for any prime } l \geq 5.$$

Theorem 26 (Gordon, Huges, Newman)**.** *Suppose $f(z)$ is as in (4). Suppose also that*

$$
\begin{aligned}
&1)\ k := \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z},\\
&2)\ \sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24},\\
&3)\ N \cdot \sum_{\delta|N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}.
\end{aligned}
$$

Then $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, *we have* $f|_k\gamma = \chi(d)f$, *where* χ *is the Dirichlet character* mod N *defined by* $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$, *where* $s = \prod_{\delta|N} \delta^{r_\delta}$ *. (Thus only the parity of* r_δ *matters for* χ*.)*

To determine the cusp conditions, we often use the following theorem.

Theorem 27 (Ligozat)**.** *Suppose that $s = a/c$ is in the lowest terms. Let f be as in* (4)*. Then the order of f at s is*

$$\frac{N}{24(c^2, N)} \sum_{\delta|N} \frac{(\delta, c)^2}{\delta} r_\delta \text{ (computed with respect to } q_{h_s}\text{).}$$

Fact: A complete set of representatives for cusps of $\Gamma_0(N)$ is

$$c_N = \left\{ \frac{a_c}{c} \in \mathbb{Q} : c|N, 1 \le a_c \le N, (a_c, N) = 1, a_c \equiv a'_c \pmod{(c, N/c)} \iff a_c = a'_c \right\}.$$

Corollary 28. *If $f(z)$ is as in (4), then f is holomorphic at cusps if and only if* $\sum_{\delta|N} \frac{(\delta,c)^2}{\delta} r_\delta \ge 0\ \forall c|N$.

Thus f vanishes at cusps $\iff$ all sums ≥ 0.

Example. $f_1(z) = \eta^2(z)\eta^2(11z)$. *Then*

$$
\begin{aligned}
k =& \frac{1}{2}(2+2) = 2,\\
\sum \delta r_\delta =& 2\cdot 1 + 2 \cdot 11 = 24,\\
\sum \frac{r_\delta}{\delta} =& \frac{2}{1} + \frac{2}{11} = \frac{24}{11}.
\end{aligned}
$$

So we may take $N = 11$.

$$\chi(d) = \left(\frac{(-1)^2 \cdot 11^2}{d}\right) \text{ is the trivial character.}$$

f_1 clearly vanishes at all cusps. Therefore $f_1 \in S_2(\Gamma_0(11))$.

Example. $f_2(z) = \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}$. *Check that* $f_2(z) \in M_1(\Gamma_0(4), (\frac{-4}{\cdot}))$. *In fact* $f_2 = \theta^2(z)$.

Example. *$f(z) = \frac{\eta^l(lz)}{\eta(z)}$ where $l \geq 5$ prime. Then*

$$\begin{aligned} k =& \frac{l-1}{2}, \\ \sum \delta r_\delta =& l^2 - 1 \equiv 0 \pmod{24}, \\ \sum \frac{r_\delta}{\delta} =& \frac{l}{l} - 1 = 0. \end{aligned}$$

So we may choose $N = l$.

$$\chi(d) = \left(\frac{(-1)^{\frac{l-1}{2}} \cdot l^l}{d}\right)\left(\frac{(-1)^{\frac{l-1}{2}} \cdot l}{d}\right) = \left(\frac{d}{l}\right) \text{ by quadratic reciprocity.}$$

We have the following cusp conditions.

$$\begin{aligned} c = 1: \ & \frac{(l,1)^2}{l} \cdot l - \frac{(1,1)^2}{1} = 0. \\ c = l: \ & \frac{(l,l)^2 \cdot l}{l} \cdot l - \frac{(l,1)^2}{1} = l^2 - 1. \end{aligned}$$

Thus $f \in M_{\frac{l-1}{2}}\left(\Gamma_0(l), \left(\frac{\cdot}{l}\right)\right)$.

Example. *Let $l \geq 5$ be a prime. Then $\frac{\eta^l(z)}{\eta(lz)} \in M_{\frac{l-1}{2}}\left(\Gamma_0(l), \left(\frac{\cdot}{l}\right)\right)$.*

$$\frac{\eta^l(z)}{\eta(lz)} = \frac{q^{l/24}\prod(1-q^n)^l}{q^{l/24}\prod(1-q^{ln})} \equiv 1 \pmod{l}$$

.

9. Valence formula on $\Gamma_0(N)$

For $\tau \in \mathbb{H}$, let $\Gamma_\tau := \{\gamma \in \Gamma : \gamma\tau = \tau\}$ be the "isotropy subgroup".

Recall that $\Gamma_i = \{\pm I, \pm S\}, \Gamma_\omega := \pm\{I, ST, (ST)^2\}$, and $\Gamma_\tau = \{\pm I\}$ if $\tau \in F$ and $\tau \neq i, \omega, -\bar{\omega}$, where F is the fundamental domain. $\Gamma_{A\tau} = A\Gamma_\tau A^{-1}$ for any $A \in \Gamma$. Thus for $\tau \in \mathbb{H}$, we have

$$|\Gamma_\tau| = \begin{cases} 4 & \text{if } \tau \overset{\Gamma}{\sim} i, \\ 6 & \text{if } \tau \overset{\Gamma}{\sim} \omega, \\ 2 & \text{otherwise.} \end{cases}$$

Set $\Gamma_0(N)_\tau = \Gamma_0(N) \cap \Gamma_\tau$.

Define $l_\tau := \frac{1}{2}|\Gamma_0(N)_\tau|$.

Note: For all $\tau \in \mathbb{H}, l_\tau = 1, 2, 3$. ($l_\tau = 2, 3 \iff \tau$ "elliptic fixed points".)

Recall that if f is meromorphic of weight k on $\Gamma_0(N)$ with s a cusp of $\Gamma_0(N)$, (and hence fan width h_s,) then $\text{ord}_s f = n_0$ where f has expansion at s,

$$\sum_{n \geq n_0} a(n) q_{h_s}^n, \qquad (a(n) \neq 0).$$

Theorem 28. *Valence formula on $\Gamma_0(N)$.*

Suppose f is a non-zero meromorphic modular form of weight k on $\Gamma_0(N)$. Then

$$\sum_{\text{cusps } s \text{ of } \Gamma_0(N)} ord_s f + \sum_{\tau \in \Gamma_0(N)\backslash\mathbb{H}} \frac{1}{l_\tau} ord_\tau f = \frac{k}{12}[\Gamma : \Gamma_0(N)].$$

Note that we recover our old valence formula when $N = 1$.

Note also that Thoerem 28 says that # zeros −# poles = $\frac{k}{12}[\Gamma : \Gamma_0(N)]$ when the number of zeros and poles are counted correctly.

A similar statement holds for any subgroup of Γ. (See Schoeneberg's book.)

Corollary 29. *A holomorphic modular form of weight k on $\Gamma_0(N)$ which vanishes to order $> \frac{k}{12}[\Gamma : \Gamma_0(N)]$ at any cusp must be identically zero .*

Note: Suppose $f \in M_k(\Gamma_0(N), \chi)$ vanishes to order $> \frac{k}{12}[\Gamma : \Gamma_0(N)]$ at some cusp. Then $f^N \in M_{kN}(\Gamma_0(N))$ vanishes to order $> \frac{Nk}{12}[\Gamma : \Gamma_0(N)]$. So we have $f^N \equiv 0$ and consequently, $f \equiv 0$.

Example.

$$\begin{aligned} f_1(z) =& \theta^2(z) = \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^2, \\ f_2(z) =& \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}. \end{aligned}$$

Then $f_1, f_2 \in M_1(\Gamma_0(4), \chi_{-4})$.

$$\frac{k}{12}[\Gamma : \Gamma_0(4)] = \frac{1}{12} \cdot 4 \cdot \left(1 + \frac{1}{2}\right) = \frac{1}{2}.$$

Therefore it suffices to check the linear terms to see that $f_1 = f_2$, and so

$$\theta(z) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}.$$

Example.

$$\begin{aligned} E_4 =& 1 + 240 \sum \sigma_3(n) q^n, \\ E_6 =& 1 - 504 \sum \sigma_5(n) q^n, \\ f_1(z) =& \frac{\eta^{16}(z)}{\eta^8(2z)} + 2^8 \frac{\eta^{16}(2z)}{\eta^8(z)}, \\ f_2(z) =& \frac{\eta^{24}(z)}{\eta^{12}(2z)} - 2^5 \cdot 3 \cdot 5 \cdot \eta^{12}(2z) - 2^9 \cdot 3 \cdot 11 \cdot \frac{\eta^{12}(2z)\eta^8(4z)}{\eta^8(z)} + 2^{13} \frac{\eta^{24}(4z)}{\eta^{12}(2z)}. \end{aligned}$$

Check that $f_1(z) \in M_4(\Gamma_0(4))$ and $f_2(z) \in M_6(\Gamma_0(4))$.

After comparing a few coefficients, we see that

$$f_1 = E_4, \qquad f_2 = E_6.$$

(Note that $M_4(\Gamma) \subseteq M_4(\Gamma_0(4))$).

Proposition 30. *Every modular form on $SL_2(\mathbb{Z})$ is a rational function in $\eta(z), \eta(2z)$, and $\eta(4z)$.*

Proof. Every modular form on $SL_2(\mathbb{Z})$ is a polynomial in E_4, E_6. ☐

The following theorem is an important consequence of the valence formula.

Theorem 31.

$$\dim_{\mathbb{C}} M_k(\Gamma_0(N)) \leq 1 + \frac{k}{12}[\Gamma : \Gamma_0(N)].$$

Proof. Suppose that $f_1, f_2, \ldots, f_h$ are linearly independent elements of $M_k(\Gamma_0(N)$.

Claim: By taking linear combinations of these, it is easy to produce a non-zero form f which vanishes to order $\geq h-1$ at infinity.

Proof of claim. Induction on h. ☐

By Corollary 29, we have $h-1 \leq \frac{k}{12}[\Gamma : \Gamma_0(N)]$. This completes the proof of Theorem 31. ☐

10. S. K. Park. 10/3 and 10/6

Proof of Theorem 28 Valance Formula

Proof. Suppose f is meromorphic of weight k on $\Gamma_0(N)$. Let $t = [\Gamma : \Gamma_0(N)]$. Write $\Gamma = \bigcup_{j=1}^{t} \Gamma_0(N)A_j$. Define

$$F(z) = \Pi_{j=1}^{t} f|_k A_j.$$

Note: If $\gamma \in \Gamma$, then $\{A_j\gamma\}$ runs through a complete set of coset representatives of $\Gamma_0(N)\backslash\Gamma$ as $\{A_j\}$ does.
$\Longrightarrow F(z)|_{tk}\gamma = \Pi_{j=1}^{t} f(z)|_k A_j\gamma = F(z)$
$\Longrightarrow F(z)$ is meromorphic of weight tk on Γ.
Valance formula on Γ.

$$ord_\infty F + \frac{1}{2} ord_i F + \frac{1}{3} ord_\omega F + \sum_{\substack{\tau \in \Gamma\backslash\mathbb{H} \\ \tau \neq i,\omega}} ord_\tau F = \frac{tk}{12} = \frac{k}{12}[\Gamma : \Gamma_0(N)]$$

Cusps : show

$$ord_\infty F = \sum_{\substack{\text{cusps} \\ s \text{ of } \Gamma_0(N)}} ord_s f$$

At each cusp s of $\Gamma_0(N)$, f has an expansion of the form :

$$a(n_0) q_{h_s}^{n_0} \ + \ \cdots ,$$

where $a(n_0) \neq 0, \ n_0 = ord_s f$.

- There are h_s distinct cosets $\Gamma_0(N)A_j$ which take ∞ to s.
- Multiplying the corresponding $f|_k A_j$.

We obtain :

$$(\text{constant}) \left(q_{h_s}^{n_0}\right)^{h_s} + \cdots = (\text{constant}) \cdot q^{h_0} + \cdots$$

$$\Longrightarrow \ ord_\infty F = \sum_{\substack{cusps \ s \\ \text{of } \Gamma_0(N)}} ord_s f.$$

Points $\tau \in \Gamma_0(N)\backslash\mathbb{H}$ $\quad(\ \Gamma_0(N)\backslash\mathbb{H}, \quad \Gamma\backslash\mathbb{H} \subseteq \mathbb{C}\ )$.
If $\tau \in \Gamma_0(N)\backslash\mathbb{H}$, then $\exists!\ \tau_0 \ \in\ \Gamma\backslash\mathbb{H}$ such that for some $\gamma \in \Gamma,\ \tau = \gamma\tau_0$.

$$\begin{aligned}
&\Gamma = \bigcup_{j=1}^{t} \Gamma_0(N)A_j \implies \gamma \in \Gamma_0(N)A_j \text{ for some } j \\
&\implies \tau \in \Gamma_0(N)A_j\tau_0 \\
&\implies \tau = \gamma_0 A_j \tau_0 \ \text{ for some } \gamma_0 \in \Gamma_0(N)
\end{aligned}$$

Q : How do the points $\{A_j\tau_0\}$ split into $\Gamma_0(N)$-equivalence classes?
Case I. $\tau_0 \neq i,\ \omega$. Then

$$\Gamma_{\tau_0} = \{\pm I\}.$$

If $\tau = \gamma_0 A_j \tau_0$ then only $\pm\ \gamma_0 A_j \in \Gamma_0(N)A_j$ takes τ_0 to τ.
$\implies \{A_j\tau_0\} = \{\tau \in \Gamma_0(N)\backslash\mathbb{H}\ :\ \tau \underset{\Gamma}{\sim} \tau_0\}$.
We have

$$ord_{\tau_0}F = \sum_{j=1}^{t} ord_{\tau_0} f|_k A_j = \sum_{j=1}^{t} ord_{A_j\tau_0} f = \sum_{\substack{\tau \in \Gamma_0(N)\backslash\mathbb{H} \\ \tau \underset{\Gamma}{\sim} \tau_0}} ord_\tau f.$$

Case II. $\tau_0 = i,\ \Gamma_i = \{\pm I,\ \pm S\}$
If $\tau = \gamma_0 A_j i$ then only $\pm\gamma_0 A_j,\ \pm\gamma_0 A_j S$ take τ to i.
Q : Are $\gamma_0 A_j S,\ \gamma_0 A_j$ in different cosets of $\Gamma_0(N)\backslash\Gamma$?
(i) Let $\gamma_0 A_j S,\ \gamma_0 A_j$ be in different cosets. Then

$$\gamma_0 A_j S \cdot (\gamma_0 A_j)^{-1} \ \notin\ \Gamma_0(N).$$

<u>Note.</u> $\gamma_0 A_j S\,(\gamma_0 A_j)^{-1}\,\tau = \tau$

$$\implies \Gamma_\tau \ = \ \{\pm I,\ \pm\gamma_0 A_j S(\gamma_0 A_j)^{-1}\}.$$

Since $\pm\gamma_0 A_j S(\gamma_0 A_j)^{-1} \ \notin\ \Gamma_0(N),\ \Gamma_0(N)_\tau = \{\pm I\}$.
$\implies\ l_\tau = 1$.
(ii) Let $\gamma_0 A_j S,\ \gamma_0 A_j$ be in the same coset. Then

$$\gamma_0 A_0 S \cdot (\gamma_0 A_j)^{-1} \ \in\ \Gamma_0(N).$$

So $\Gamma_\tau = \Gamma_0(N)_\tau \implies\ l_\tau = 2$.
We have :

$$\begin{aligned}
ord_i F \quad &= \sum_{j=1}^{t} ord_i f|_k A_j = \sum_{j=1}^{t} ord_{A_j i} f \\
&= \sum_{\substack{\tau \in \Gamma_0(N)\backslash\mathbb{H} \\ \tau \underset{\Gamma}{\sim} i,\ l_\tau = 2}} ord_\tau f \ + \ 2 \sum_{\substack{\tau \in \Gamma_0(N)\backslash\mathbb{H} \\ \tau \underset{\Gamma}{\sim} i,\ l_\tau = 1}} ord_\tau f.
\end{aligned}$$

$$\implies \frac{1}{2} ord_i F = \sum_{\substack{\tau \in \Gamma_0(N)\backslash\mathbb{H} \\ \tau \underset{\Gamma}{\sim} i}} \frac{1}{l_\tau} ord_\tau f.$$

Similarly, $\dfrac{1}{3} ord_\omega F = \sum\limits_{\substack{\tau \in \Gamma_0(N)\backslash\mathbb{H} \\ \tau \underset{\Gamma}{\sim} \omega}} \dfrac{1}{l_\tau} ord_\tau f.$

$\square$

Congruences between modular forms

Suppose $f, g \in M_k(\Gamma_0(N), \chi) \bigcap \mathbb{Z}[[q]],\ m \in \mathbb{N}$. Write

$$f(z) = \sum_{n=0}^{\infty} a(n) g^n,$$

$$g(z) = \sum_{n=0}^{\infty} b(n) q^n.$$

Definition. $f \equiv g\ (mod\ m) \iff \forall n,\ a(n) \equiv b(n)\ (mod\ m)$

Define $ord_m f = \min\{n\ :\ a(n) \not\equiv 0 \text{ (mod } m)\}$.
(If $a(n) \equiv 0$ (mod m), $\forall n$, then $ord_m f = +\infty,\ f \equiv 0$ (mod m).)

Theorem 32. *(Sturm's Theorem) Suppose* $f \in M_k(\Gamma_0(N), \chi) \bigcap \mathbb{Z}[[q]],\ m \in \mathbb{N}$. *If* $ord_m f > \frac{k}{12}[\Gamma : \Gamma_0(N)]$ *then* $ord_m f = +\infty$.

Point : 1. To verify a congruence (mod m) between 2 forms it suffices to verify that the first few coefficients agree (mod m).
2. $ord_m(f-g) > \frac{k}{12}[\Gamma : \Gamma_0(N)] \Longrightarrow f \equiv g$ (mod m)
3. If $\mathcal{O}_K$ = ring of integers of a number field K, and $\mathcal{M}$ is an ideal, then the same result holds with

$$\mathbb{Z} \longrightarrow \mathcal{O}_K$$
$$m \longmapsto \mathcal{M}.$$

Proof of Theorem 32.
In M_k, let $t = \dim\ M_k - 1$. Then $t \leq \left\lfloor \frac{k}{12} \right\rfloor \leq \frac{k}{12}$.
There exists a basis for M_k of the form

$$\{f_0, f_1, \cdots, f_t\},\ \text{each } f_i \in \mathbb{Z}[[q]]$$

and

$$\begin{aligned} f_0 &= 1 + O(q) \\ f_1 &= \qquad q + O(q^2) \\ &\qquad\quad \vdots \\ f_t &= \qquad\qquad\qquad q^t + O(q^{t+1}). \end{aligned}$$

Example. $k \equiv 4\ (mod\ 12) \implies k = 12j + 4,\ t = dim\ M_k - 1 = j$
Take $f_0 = E_4^{3j+1},\ f_1 = E_4^{3j+2}\Delta, \cdots, f_i = E_4^{3j+1-3i}\Delta^i, \cdots, f_j = E_4 \Delta^j$.

Suppose $f(z) \in M_k \bigcap \mathbb{Z}[[q]]$ has $ord_m f = s > \frac{k}{12} \geq t$. Write

$$\begin{aligned} f(z) &= \alpha_0 f_0 + \alpha_1 f_1 + \cdots + \alpha_t f_t \\ &= \alpha_0(1+\cdots) + \alpha_1(q+\cdots) + \cdots + \alpha_\tau(q^t+\cdots) \\ &\equiv (\text{constant})q^s + \cdots\cdots \pmod{m}. \end{aligned}$$

$$\begin{aligned} s > t \qquad &\Longrightarrow \alpha_0 \equiv \cdots \equiv \alpha_t \equiv 0 \pmod{m} \\ &\Longrightarrow f(z) \equiv 0 \pmod{m}. \end{aligned}$$

To prove general case, we use similar method to that used to prove valance formula on $\Gamma_0(N)$. However, also need additional knowledge of " integrality " properties. $\square$

U-operator

Let

$$f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi), \ m \in \mathbb{N}.$$

Define

$$f|U_m := \sum_{n=0}^{\infty} a(mn)q^n = \sum_{\substack{n=0 \\ m|n}}^{\infty} a(n)q^{\frac{n}{m}}.$$

Theorem 33. *If* $m|N$ *then* U_m: $M_k(\Gamma_0(N),\chi) \longrightarrow M_k(\Gamma_0(N),\chi)$
: $S_k\ (\Gamma_0(N),\chi) \longrightarrow S_k\ (\Gamma_0(N),\chi)$

Remark. If $m \nmid N$, note that $M_k(\Gamma_0(N),\chi) \subset M_k(\Gamma_0(\mathrm{lcm}(m,N)),\chi)$.
Theorem 33 $\Longrightarrow U_m\ :\ M_k(\Gamma_0(N),\chi) \longrightarrow M_k(\Gamma_0(\mathrm{lcm}(m,N)),\chi)$

Lemma 34. *With these hypotheses, we have*

$$f|U_m = m^{\frac{k}{2}-1} \sum_{v=0}^{m-1} f|_k \begin{pmatrix} 1 & v \\ 0 & m \end{pmatrix}.$$

Proof of Lemma 34.

$$\begin{aligned} \text{RHS} &= m^{\frac{k}{2}-1} m^{\frac{k}{2}} m^{-k} \sum_{v=0}^{m-1} f\left(\frac{z+v}{m}\right) \\ &= \frac{1}{m} \sum_{n=0}^{\infty} \sum_{v=0}^{m-1} a(n) e^{2\pi i n\left(\frac{z+v}{m}\right)} \\ &= \sum_{n=0}^{\infty} a(n) e^{\frac{2\pi i n z}{m}} \left(\frac{1}{m} \sum_{v=0}^{m-1} e^{\frac{2\pi i n v}{m}} \right) \\ &= \sum_{\substack{n=0 \\ m|n}}^{\infty} a(n) q^{\frac{n}{m}} \ = \ f|U_m. \end{aligned}$$

$\square$

Lemma 34, Lemma 20 $\Longrightarrow$ $f|U_m$ satisfies same cusp conditions as f.
To prove Theorem 33, check transformation law. Like twisting theorem, but easier. Suppose

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ m|N,\ m|c,\ \text{and } \gcd(a,m) = 1.$$

Show : $f|U_m|_k\gamma = \chi(d)f|U_m$. Set

$$\gamma_v = \begin{pmatrix} 1 & v \\ 0 & m \end{pmatrix}.$$

Then

$$\gamma_v\gamma\gamma_{v'}^{-1} = \begin{pmatrix} a+vc & \frac{b+vd-v'a}{m} - \frac{vv'c}{m} \\ mc & d-v'c \end{pmatrix}.$$

$\forall v \pmod m$, $\exists!\ v' \pmod m$ such that $av' \equiv b+vd \pmod m$, since $\gcd(a,m)=1$. As v runs through $\{1,2,\cdots,m-1\}$, so does v'. For such a v', $\gamma_v\gamma\gamma_{v'}^{-1} \in \Gamma_0(N)$. Now

$$f|U_m = m^{\frac{k}{2}-1}\sum_{v=0}^{m-1} f|_k\gamma_v.$$

So

$$\begin{aligned} f|U_m|_k\gamma &= m^{\frac{k}{2}-1}\sum_{v=0}^{m-1}\left(f|_k\gamma_v\gamma\gamma_{v'}^{-1}\right)|_k\gamma_{v'} \\ &= \chi(d)m^{\frac{k}{2}-1}\sum_{v'=0}^{m-1} f|_k\gamma_{v'} = \chi(d)f|U_m. \end{aligned}$$

$\square$

Note. : In general, $(fg)|U_m \neq (f|U_m)(g|U_m)$.
However, if

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a(n)q^{nm} \\ g(z) &= \sum_{n=0}^{\infty} b(n)q^n \end{aligned}$$

then

$$(fg)|U_m = (f|U_m)(g|U_m).$$

Application : Ramanujan's Congruences for the Partition Function

Definition. $p(n)$=*number of partitions of* n
=*number of non-increasing sequences of positive integers whose sum is* n

Example.

$$\begin{aligned}
p(n) &= 0 \qquad \textit{if } n < 0 \\
p(0) &= 1 \\
p(1) &= 1 \\
p(2) &= 2 \ (2, 1+1) \\
p(3) &= 3 \ (3, 2+1, 1+1+1) \\
p(4) &= 5 \ (4, 3+1, 2+2, 2+1+1, 1+1+1+1) \\
p(n) &\sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}, \quad \textit{as } n \to \infty \qquad (\textit{Hardy,Ramanujan})
\end{aligned}$$

Euler : $$\frac{1}{\Pi_{n=1}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} p(n)q^n.$$

This is true since

$$\begin{aligned}
\Pi_{n=1}^{\infty}\frac{1}{1-q^n} &= \frac{1}{1-q}\cdot\left(\frac{1}{1-q^2}\right)\cdot\left(\frac{1}{1-q^3}\right)\cdot\left(\frac{1}{(1-q^4)}\right)\cdots \\
&= \left(\sum_{m_1=0}^{\infty} q^{m_1}\right)\left(\sum_{m_2=0}^{\infty}\left(q^2\right)^{m_2}\right)\left(\sum_{m_3=0}^{\infty}\left(q^3\right)^{m_3}\right)\cdots \\
&= \sum_{n=0}^{\infty} p(n)q^n.
\end{aligned}$$

Ramanujan's Congruences :

$$\begin{aligned}
p(5n+4) &\equiv 0 \pmod{5}, \quad \forall n \\
p(7n+5) &\equiv 0 \pmod{7}, \quad \forall n \\
p(11n+6) &\equiv 0 \pmod{11}, \quad \forall n.
\end{aligned}$$

<u>Note.</u>

$$\begin{aligned}
4 &\equiv \frac{1}{24} \pmod{5} \\
5 &\equiv \frac{1}{24} \pmod{7} \\
6 &\equiv \frac{1}{24} \pmod{11}
\end{aligned}$$

For $l \geq 5$, prime. Set $\delta_l = \frac{l^2-1}{24} \in \mathbb{Z}$. Then

$$-\delta_l = -\left(\frac{l^2-1}{24}\right) = \frac{1}{24} \pmod{l}.$$

Restatement of Ramanujan's Congruences.
If $l = 5, 7, 11$ then $\forall n,\ p(l_n - \delta_l) \equiv \pmod{l}$.
(**Proof**) For $l \geq 5$, prime
set

$$\begin{aligned} f_l(z) &= \frac{\eta^l(lz)}{\eta(z)} \in M_{\frac{l-1}{2}}\left(\Gamma_0(l), \left(\frac{\cdot}{l}\right)\right) \\ &= q^{\delta_l}\Pi_{n=1}^{\infty}\frac{\left(1-q^{ln}\right)^l}{1-q^n} \end{aligned}$$

$$\begin{aligned} \text{Euler} \quad \Longrightarrow f_l(z) &= \Pi_{m=1}^{\infty}(1-q^{lm})^l\sum_{n=0}^{\infty}p(n)q^{n+\delta_l} \\ n \to n-\delta_l, \qquad &= \Pi_{m=1}^{\infty}(1-q^{lm})^l\sum_{n=0}^{\infty}p(n-\delta_l)q^n \end{aligned}$$

$$\text{Conclusion :} \quad f_l(z)|U_l = \Pi_{m=1}^{\infty}(1-q^m)^l\sum_{n=0}^{\infty}p(ln-\delta_l)q^n.$$

11. R. Rojas. 10/8 and 10/10.

NOTES FROM THE EIGHTH AND TENTH OF OCTOBER

Recall the U - operator: if

$$f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi) \bigcap \mathbb{Z}[[q]]$$

and if $0 < m \in \mathbb{Z}$, then

$$f|U_m = \sum_{n=0}^{\infty} a(nm)q^n = \sum_{n=0 \ \ m|n}^{\infty} a(n)q^{n/m}$$

Note that if $m|N$, then

$$f|U_m \in M_k(\Gamma_0(N), \chi) \bigcap \mathbb{Z}[[q]]$$

If also, $g = \sum_{n=0}^{\infty} b(n)q^{nm}$, then

$$(fg)|U_m = (f|U_m)(g|U_m)$$

Finally, if $j \in \mathbb{Z}$, then

$$\prod_{n=1}^{\infty}(1-q^{mn})^j|U_m = \prod_{n=1}^{\infty}(1-q^n)^j$$

Now we consider Ramanujan Congruences. For all $n \in \mathbb{Z}$, we have

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

These three congruences are equivalent to the statement that if $l \in \{5, 7, 11\}$, then for all $n \in \mathbb{Z}$,

$$p(ln - \delta_l) \equiv 0 \pmod{l} \qquad \delta_l := \frac{l^2 - 1}{24}$$

Proof. Set

$$f_l(z) = \frac{\eta^l(lz)}{\eta(z)} = q^{\delta_l} \prod_{n=1}^{\infty} \frac{(1 - q^{ln})^l}{1 - q^n} \in M_{\frac{l-1}{2}}\left(\Gamma_0(l), \left(\frac{\cdot}{l}\right)\right)$$

By Euler,

$$f_l(z) = \prod_{m=1}^{\infty} (1 - q^{lm})^l \sum_{n=0}^{\infty} p(n) q^{n+\delta_l} = \prod_{m=1}^{\infty} (1 - q^{lm})^l \sum_{n=0}^{\infty} p(n - \delta_l) q^n$$

Therefore,

$$f_l|U_l = \left(\prod_{m=1}^{\infty} (1 - q^{lm})^l \sum_{n=0}^{\infty} p(n - \delta_l) q^n\right) |U_l = \left(\prod_{m=1}^{\infty} (1 - q^{ml})^l |U_l\right) \left(\sum_{n=0}^{\infty} p(n - \delta_l) q^n) |U_l\right)$$

$$= \prod_{m=1}^{\infty} (1 - q^m)^l \sum_{n=0}^{\infty} p(ln - \delta_l) q^n = (1 + ...) \sum_{n=0}^{\infty} p(ln - \delta_l) q^n$$

Hence, $f_l|U_l \equiv 0 \pmod{l} \Leftrightarrow$ for all $n \in \mathbb{Z}, p(ln - \delta_l) \equiv 0 \pmod{l}$. Let $l = 11$ ($l = 5, 7$ have essentially identical proofs). We know that

$$f_{11} \in M_5\left(\Gamma_0(11), \left(\frac{\cdot}{11}\right)\right) \bigcap \mathbb{Z}[[q]]$$

so it follows that

$$f_{11}|U_{11} \in M_5\left(\Gamma_0(11), \left(\frac{\cdot}{11}\right)\right) \bigcap \mathbb{Z}[[q]]$$

By Sturm, $f_{11}|U_{11} \equiv 0 \pmod{11} \Leftrightarrow f_{11}$ vanishes mod11 to order

$$> \frac{5}{12}[\Gamma_0(1) : \Gamma_0(11)] = \frac{5}{12}(11)\left(1 + \frac{1}{11}\right) = 5$$

Thus, to prove that $p(11n + 6) \equiv 0 \pmod{11}$, it suffices to verify that the first six coefficients of $f_{11}|U_{11}$ vanish mod11. As a final note, to prove that

$$p(5n + 4) \equiv 0 \pmod{5}$$

for all $n \in \mathbb{Z}$, check two coefficients mod5 of $f_5|U_5$ because

$$2 > \frac{2}{12}(5)\left(1 + \frac{1}{5}\right) = 1$$

□

NOTE: We can compute $f_l|U_l$ explicitly.
DEFINITION:

$$\sigma_{k,\chi}(n) = \sum_{d|n} \chi(d) d^k$$

NOTE: $\dim(M_2(\Gamma_0(5), (\frac{\cdot}{5}))) = 2$. One basis element is

$$\frac{\eta^5(z)}{\eta(5z)} = 1 - 5\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{d}{5}\right) d\right) q^n = 1 - \frac{2 * 2}{B_{2,(\frac{\cdot}{5})}} \sum_{n=1}^{\infty} \sigma_{1,(\frac{\cdot}{5})}(n) q^n = 1 - ...$$

The other basis element is

$$f_5(z) = \frac{\eta^5(5z)}{\eta(z)} = \sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{d}{5}\right)\right)q^n = q + ...$$

We find that

$$\frac{\eta^5(5z)}{\eta(z)}|U_5 = 5q + ... = 5\frac{\eta^5(5z)}{\eta(z)}$$

NOTE: $dim(M_3(\Gamma_0(7), (\frac{\cdot}{7}))) = 3$. A basis is shown below:

$$\{\frac{\eta^7(z)}{\eta(7z)} = 1 - ..., \frac{\eta^7(7z)}{\eta(z)} = q^2 + ..., \eta^3(z)\eta^3(7z) = q + ...\}$$

We find that

$$f_7|U_7 = 7\eta^3(7z)\eta^3(z) + 7^2\frac{\eta^7(7z)}{\eta(z)}$$

Transformation Law for Θ-Function:

$$\Theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{\eta^5(2z)}{\eta^2(4z)\eta^2(z)} \in M_{\frac{1}{2}}(\Gamma_0(4))$$

It should be noted that $M_{\frac{1}{2}}(\Gamma_0(4))$ has not yet been rigorously defined. Let $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$. We know that

$$\frac{\Theta(\gamma z)}{\Theta(z)} = \frac{\Theta(\frac{-1}{4z})}{\Theta(z)} = \sqrt{-2iz}$$

QUESTION: If $\gamma \in \Gamma_0(4)$, then what is $\frac{\Theta(\gamma z)}{\Theta(z)}$?
DEFINITION:

$$\varepsilon_d := \sqrt{\left(\frac{-1}{d}\right)} = i^{\frac{d-1}{2}} = \begin{cases} 1 & : \quad d \equiv 1 \pmod 4 \\ i & : \quad d \equiv 3 \pmod 4 \end{cases}$$

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4) \Rightarrow j(\gamma, z) := \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d}$$

By convention, $\text{Arg}(\sqrt{z}) \in (\frac{-\pi}{2}, \frac{\pi}{2}]$. Also, $\Theta(z)|_{\frac{1}{2}}\gamma = j(\gamma, z)^{-1}\Theta(\gamma z)$.

Theorem 35. *If* $\gamma \in \Gamma_0(4)$, *then* $\Theta(z)|_{\frac{1}{2}}\gamma = \Theta(z)$.

NOTE: Generally, if $4|N$, $k \equiv 1 \pmod 2$, and $\gamma \in \Gamma_0(N) \subseteq \Gamma_0(4)$, then

$$f|_{\frac{k}{2}}\gamma = j(\gamma, z)^{-k}f(\gamma z) = \left(\frac{\Theta(z)}{\Theta(\gamma z)}\right)^{-k} f(\gamma z)$$

Let χ be a Dirichlet character modN. Then $f \in M_{\frac{k}{2}}(\Gamma_0(N), \chi)$ if: 1. For all $\gamma \in \Gamma_0(N)$, $f|_{\frac{k}{2}} = \chi(d)f$. 2. f is holomorphic in the upper half-plane and at cusps of $\Gamma_0(N)$.
Proof. Define $\phi(z) := \Theta(\frac{z}{2}) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$. Properties of ϕ:

$$\phi(T^2 z) = \phi(z+2) = \phi(z) \qquad \phi(Sz) = \phi\left(\frac{-1}{z}\right) = \sqrt{-iz}\phi(z)$$

Let $G(2) := \pm < T^2, S > \subseteq SL_2(\mathbb{Z})$. Note that $\Gamma(2) \subseteq G(2)$. Let $\alpha = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that $\phi(z)$ transforms with respect to $G(2)$. Also, $\Theta(z) = \phi(2z) = \phi(\alpha z)$ transforms with respect to $\alpha G(2)\alpha^{-1}$ and $\alpha G(2)\alpha^{-1} \nsubseteq SL_2(\mathbb{Z})$. We call $G(2)$ the Hecke Group. Below are other properties of ϕ:

$$\phi(z) = e^{\frac{-2\pi i}{24}} \frac{\eta^2(\frac{z+1}{2})}{\eta(z)} \qquad\qquad \phi^8(z) \in M_4(G(2))$$

Finally, if p is an odd prime and $\alpha = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $\alpha(G(2) \bigcap \Gamma_0(p))\alpha^{-1} \subseteq G(2)$.
Lemma 1: If $\gamma \in \Gamma$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \pmod 2$, $d \neq 0$, and

$$\phi(\gamma z) = i^{\frac{1-c}{2}} \left(\frac{d}{c}\right) \sqrt{-i(cz+d)}\phi(z)$$

then Theorem 35 holds.
Proof of Lemma 1. If $\alpha = \left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right)$, then $\Theta(\alpha z) = \Theta(z+b) = \Theta(z)$. Also,

$$j(\alpha, z) = \left\{ \begin{array}{rcl} (\frac{0}{1})\varepsilon_1^{-1}\sqrt{1} & : & Use + sign \\ (\frac{0}{-1})\varepsilon_{-1}^{-1}\sqrt{-1} & : & Use - sign \end{array} \right. = 1$$

So Theorem 35 holds for these α. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, $c \neq 0$. Then $\Theta(\gamma z) = \phi(2\gamma z)$ and

$$2\gamma z = \frac{2az+2b}{cz+d} = \frac{-az-b}{\frac{-cz}{2} - \frac{d}{2}} = \frac{-a + 2b(\frac{-1}{2z})}{\frac{-c}{2} + d\frac{-1}{2z}} = \left(\begin{smallmatrix} 2b & -a \\ d & \frac{-c}{2} \end{smallmatrix}\right) \left(\frac{-1}{2z}\right)$$

Let $\gamma' = \left(\begin{smallmatrix} 2b & -a \\ d & \frac{-c}{2} \end{smallmatrix}\right)$. Note that $\frac{c}{2} \neq 0$ and that $4|c \Rightarrow \gamma' \equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \pmod 2$. We have that

$$\Theta(\gamma z) = \phi(2\gamma z) = \phi\left(\gamma'\left(\frac{-1}{2z}\right)\right) = i^{\frac{1-d}{2}} \left(\frac{(\frac{-c}{2})}{d}\right) \sqrt{-i(d(\frac{-1}{2z})) - \frac{c}{2}}\varphi(\frac{-1}{2z})$$

$$= i^{\frac{1-d}{2}}(\frac{-2}{d})(\frac{c}{d})\sqrt{-i\left(d\left(\frac{-1}{2z}\right)\right) - \frac{c}{2}}\sqrt{-2iz}\Theta(z)$$

For all of the different cases of $d \pmod 8$, show that $i^{\frac{1-d}{2}}(\frac{-2}{d}) = \varepsilon_d^{-1}$. It follows that

$$\Theta(\gamma z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{cz+d}\Theta(z)$$

$\square$

Lemma 5: If n is an odd, positive integer, and if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(2) \bigcap \Gamma_0(n)$, then

$$\frac{\phi(nz)}{\phi^n(z)}|_{\frac{1-n}{2}}\gamma = \left(\frac{d}{n}\right) \frac{\phi(nz)}{\phi^n(z)}$$

NOTATION: (Koblitz) If p is an odd prime, then

$$\psi(z) = \frac{\eta^p(z)}{\eta(pz)} \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$$

Lemma 2:

$$\frac{\phi(pz)}{\phi^p(z)} = \frac{\psi(z)}{\psi^2(\frac{z+1}{2})}$$

Lemma 3: If p is an odd prime, and if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(2) \bigcap \Gamma_0(p)$, then

$$\psi^3\left(\frac{z+1}{2}\right)|_{\frac{3(p-1)}{2}}\gamma = \left(\frac{d}{p}\right)\psi^3\left(\frac{z+1}{2}\right)$$

See Koblitz for proofs of Lemma 2 and Lemma 3.
Lemma 4: If p is an odd prime, and if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(2) \bigcap \Gamma_0(p)$, then

$$\frac{\phi(pz)}{\phi^p(z)}|_{\frac{1-p}{2}}\gamma = \left(\frac{d}{p}\right)\frac{\phi(pz)}{\phi^p(z)}$$

Proof of Lemma 4. Let $g(z) = \frac{\phi(pz)}{\phi^p(z)}$. First, show that $g^8|_{4(1-p)}\gamma = g^8$. We know that

$$g^8|_{4(1-p)}\gamma = (cz+d)^{4(p-1)}\frac{\phi^8(p(\gamma z))}{\phi^{8p}(\gamma z)}$$

Let $\alpha = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so $p\gamma z = \alpha\gamma z$. Let $\gamma' = \alpha\gamma\alpha^{-1} = \left(\begin{smallmatrix} a & bp \\ \frac{c}{p} & d \end{smallmatrix}\right)$. By Property 5: $\gamma' \in G(2)$. Thus:

$$g^8|_{4(1-p)}\gamma = \frac{(cz+d)^{-4}}{(cz+d)^{-4p}}\frac{\phi^8(\gamma'\alpha z)}{\phi^{8p}(\gamma z)} = \frac{(\frac{c}{p}\alpha z+d)^{-4}}{(cz+d)^{-4p}}\frac{\phi^8(\gamma'\alpha z)}{\phi^{8p}(\gamma z)} = \frac{\phi^8(2z)|_4\gamma'}{\phi^{8p}(z)|_{4p}\gamma}$$

By Property 4:

$$\frac{\phi^8(2z)|_4\gamma'}{\phi^{8p}(z)|_{4p}\gamma} = \frac{\phi^8(pz)}{\phi^{8p}(z)} = g^8(z)$$

Thus, the first part is done. Second, show that

$$g^9|_{\frac{9(1-p)}{2}}\gamma = \left(\frac{d}{p}\right)g^9$$

We know that

$$g^9 = \frac{\phi^9(pz)}{\phi^{9p}(z)}$$

By Lemma 2:

$$\frac{\phi^9(pz)}{\phi^{9p}(z)} = \frac{\psi^9}{(\psi^2(\frac{z+1}{2}))^9} = \frac{\psi^9}{(\psi^3(\frac{z+1}{2}))^6}$$

It follows that

$$g^9|_{\frac{9(1-p)}{2}}\gamma = \frac{\psi^9|_{\frac{9(1-p)}{2}}\gamma}{(\psi^3(\frac{z+1}{2})|_{\frac{3(1-p)}{2}}\gamma)^6}$$

By Lemma 3:

$$\frac{\psi^9|_{\frac{9(1-p)}{2}}\gamma}{(\psi^3(\frac{z+1}{2})|_{\frac{3(1-p)}{2}}\gamma)^6} = \frac{(\frac{d}{p})\psi^9}{((\frac{d}{p})\psi^3(\frac{z+1}{2}))^6} = \frac{(\frac{d}{p})\psi^9}{(\psi^2(\frac{z+1}{2}))^9} = \left(\frac{d}{p}\right)g^9$$

Divide the second equation by the first to obtain the desired result. □

12. T. Huber. 10/13 and 10/15

Recall

$$\theta(z) = \sum_{n=1}^{\infty} q^{n^2}$$

and

$$\gamma \in \Gamma_0(4) \implies j(\gamma, z) := \left(\frac{c}{d}\right)\epsilon_d^{-1}\sqrt{cz+d}.$$

(Note that $\phi(z) = \phi(z/2)$ and $G(2) := \pm\langle S, T^2\rangle \supseteq \Gamma(2)$)

Theorem 35. $\forall \gamma \in \Gamma_0(4)$,

$$\frac{\theta(\gamma z)}{\theta(z)} = j(\gamma, z). \tag{5}$$

Proof of Theorem 35.

It is easy to check that (5) holds for $\gamma = -I, T, ST^{-4}S$, the generators for $\Gamma_0(4)$. To prove the theorem, we would need to show

$$j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z) \qquad \forall \gamma, \gamma' \in \Gamma_0(4) \tag{6}$$

(in fact, Theorem 35 implies (6)).

However, we will not take this approach.

We will need the following facts:

1. $\phi(T^2 z) = \phi(z)$
2. $\phi(Sz) = \sqrt{-iz}\phi(z)$
3. $\phi^8 \in M_4(G(2))$
4. If p is an odd prime, $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, then

$$\alpha(G(2) \cap \Gamma_0(np))\alpha^{-1} \subseteq \Gamma_0(n) \cap G(2).$$

It suffices to prove

Lemma 1. If n is an odd positive integer, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pmod 2$, and $d \neq 0$, then

$$\frac{\phi(\gamma z)}{\phi(z)} = i^{\frac{1-c}{2}}\left(\frac{d}{c}\right)\sqrt{-i(cz+d)}.$$

Key step:

Lemma 5. If n is odd, positive, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(2) \cap \Gamma_0(n)$, then

$$\left.\frac{\phi(nz)}{\phi^n(z)}\right|_{\frac{1-n}{2}} \gamma = \left(\frac{d}{n}\right)\frac{\phi(nz)}{\phi^n(z)}.$$

Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (*mod* 2). To see that $\gamma \in G(2) \cap \Gamma_0(c)$, note that $-S\gamma \in \Gamma(2) \subseteq G(2)$. Note that both sides of the statement in Lemma 1 are invariant under $\gamma \mapsto -\gamma$. Thus, without loss of generality, $c > 0$. Applying Lemma 5 with $n = c$, we obtain

$$\frac{\phi(c\gamma z)}{\phi^c(\gamma z)} = \left(\frac{d}{c}\right)(cz+d)^{\frac{1-c}{2}}\frac{\phi(cz)}{\phi^c(z)},$$

which implies

$$\left(\frac{\phi(\gamma z)}{\phi(z)}\right)^c = \left(\frac{d}{c}\right)(cz+d)^{\frac{c-1}{2}}\frac{\phi(c\gamma z)}{\phi(cz)}.$$

Next note that

$$\begin{aligned} c\gamma z &= \frac{caz+cb}{cz+d} \\ &= \frac{caz+ad-1}{cz+d} \\ &= \frac{a(cz+d)-1}{cz+d} \\ &= a-\frac{1}{cz+d}, \end{aligned}$$

which implies

$$\begin{aligned} \phi(c\gamma z) &= \phi\left(a-\frac{1}{cz+d}\right) \\ &= \phi\left(-\frac{1}{cz+d}\right) && \text{(since } a \text{ is even)} \\ &= \sqrt{-i(cz+d)}\phi(cz+d) && \text{(by fact 2)} \\ &= \sqrt{-i(cz+d)}\phi(z) && \text{(since } d \text{ is even).} \end{aligned}$$

Therefore,

$$\left(\frac{\phi(\gamma z)}{\phi(z)}\right)^c = \left(\frac{d}{c}\right)(cz+d)^{\frac{c-1}{2}}\sqrt{-i(cz+d)}.$$

Fact (3) implies

$$\left(\frac{\phi(\gamma z)}{\phi(z)}\right)^{8k} = (cz+d)^{4k} \qquad \forall k\in\mathbb{Z}.$$

The c^{th} power of (5)

$$\begin{aligned} &= \left(\frac{\phi(\gamma z)}{\phi(z)}\right)^{c^2} \\ &= \left(\frac{d}{c}\right)(cz+d)^{\frac{c^2-c}{2}}\left(\sqrt{-i(cz+d)}\right)^c \\ &= \left(\frac{d}{c}\right)\sqrt{-i(cz+d)}^{\frac{c^2-c}{2}}(-i)^{\frac{c-1}{2}}(cz+d)^{\frac{c-1}{2}} \\ &= \left(\frac{d}{c}\right)i^{\frac{1-c}{2}}\sqrt{-i(cz+d)}(cz+d)^{\frac{c^2-1}{2}}. \end{aligned}$$

Choose k with $c^2=8k+1$, so $\frac{c^2-1}{2}=4k$. Divide to obtain

$$\frac{\phi(\gamma z)}{\phi(z)} = \left(\frac{d}{c}\right)i^{\frac{1-c}{2}}\sqrt{-i(cz+d)},$$

which completes the proof of the lemma.

To prove the theorem, it remains for us to show Lemma 5. We show this by induction on the total number of prime factors of n. The base case ($n=p$, for some prime p) is

Lemma 34 with $n = p$. For the inductive step, assume the result in the lemma above is true for n. Let $n' = np$ for some prime p. We need to show that

$$\frac{\phi(n'z)}{\phi^{n'}(z)}\Big|_{\frac{1-n'}{2}}\gamma = \left(\frac{d}{n'}\right)\frac{\phi(n'z)}{\phi^{n'}(z)}$$

$$\forall\gamma \in G(2)\cap\Gamma_0(n') \subseteq G(2)\cap\Gamma_0(n)\cap\Gamma_0(p).$$

Since

$$\begin{aligned}\frac{\phi(n'z)}{\phi^{n'}(z)} &= \frac{\phi(npz)}{\phi^{np}(z)}\cdot\frac{\phi^n(pz)}{\phi^n(pz)}\\ &= \frac{\phi(npz)}{\phi^n(pz)}\left(\frac{\phi(pz)}{\phi^p(z)}\right)^n,\end{aligned}$$

we have

$$\begin{aligned}\frac{\phi(n'z)}{\phi^{n'}(z)}\Big|_{\frac{1-np}{2}}\gamma &= \left(\left(\frac{\phi(npz)}{\phi^n(pz)}\right)\Big|_{\frac{1-n}{2}}\gamma\right)\left(\frac{\phi(pz)}{\phi^p(z)}\right)^n\Big|_{\frac{n(1-p)}{2}}\gamma\\ &= \left(\frac{d}{n}\right)\frac{\phi(npz)}{\phi^n(pz)}\cdot\left(\frac{d}{p}\right)^n\left(\frac{\phi(pz)}{\phi^p(z)}\right)^n\\ &= \left(\frac{d}{np}\right)\frac{\phi(n'z)}{\phi^{n'}(z)}\\ &= \left(\frac{d}{n'}\right)\frac{\phi(n'z)}{\phi^{n'}(z)}\end{aligned}$$

Thus, Lemma 5 is shown, and the proof of the theorem follows. ☐

Hecke Operators:

The prototypical operator is

$$\Delta(z) = \sum_{n=1}^{\infty}\tau(n)q^n \in S_{12}(\Gamma_0(1)).$$

Multiplicative properties:

- $\tau(mn) = \tau(m)\tau(n)$ if $(m,n) = 1$
- $\tau(p^k) = \tau(p^{k-1})\tau(p) - p^{11}\tau(p^{k-2})$

Why?

(1) There exists a family of "nice" operators on $S_{12}(\Gamma_0(1))$, and
(2) Δ is an eigenform.

In general, on any $M_k(\Gamma_0(N),\chi)$ we will construct Hecke operators $\{T_n\}_{n=1}^{\infty}$ with the following "nice" properties:

(1) The operators commute.
(2) We have "nice" multiplicative properties.
(3) The operators are "Hermitian" with respect to an inner product; i.e., $\langle T_n f, g\rangle = \overline{\chi(n)}^{1/2}\langle f, T_n g\rangle$. This leads to "nice" bases for $M_k(\Gamma_0(N),\chi)$.

Remark: Koblitz looks at modular points (in terms of lattices), but our approach involves double cosets (which is also in Koblitz).

Recall that

$$M_k(\Gamma_1(N)) = \oplus_\chi M_k(\Gamma_0(N), \chi).$$

Our idea: given $f \in M_k(\Gamma_1(N))$, build $T_n f \in M_k(\Gamma_1(N))$. To do this, we take an "average" over groups, and this requires the use of double cosets.

Definition. *If G is a group, $\Gamma_1, \Gamma_2 \leq G$, we say Γ_1, Γ_2 are commensurable if*

$$[\Gamma_1 : \Gamma_1 \cap \Gamma_2] < \infty \qquad \textit{and} \qquad [\Gamma_2 : \Gamma_1 \cap \Gamma_2] < \infty.$$

The important case for us involves the following claim:

Claim:

If Γ' is a congruence subgroup and $\alpha \in GL_2^+(\mathbb{Q})$, then

$$\Gamma', \alpha^{-1}\Gamma'\alpha$$

are commensurable.

To prove the claim, we need Lemma 1 of III.3 of Koblitz:

Lemma 1. If $\alpha \in GL_2^+(\mathbb{Q})$ has integer entries and $\Gamma(N) < \Gamma'$, then $\Gamma(ND) < \alpha^{-1}\Gamma'\alpha$, where $D = \det(\alpha)$.

To prove the claim from the lemma, note that $\alpha^{-1}\Gamma'\alpha$ is unchanged under $\alpha \mapsto d\alpha$, so without loss of generality, α has integer entries. Lemma 1 implies the following two results:

- There exists D such that $\Gamma(ND) \subset \alpha^{-1}\Gamma'\alpha \cap \Gamma' \implies$
$$[\Gamma' : \Gamma' \cap \alpha^{-1}\Gamma'\alpha] \leq [\Gamma : \Gamma(ND)] < \infty.$$
- There exists D such that $\Gamma(ND) \subset \alpha\Gamma'\alpha^{-1} \cap \Gamma' \implies$
$$[\alpha^{-1}\Gamma'\alpha : \Gamma' \cap \alpha^{-1}\Gamma'\alpha] = [\Gamma' : \alpha\Gamma'\alpha^{-1} \cap \Gamma'] \leq [\Gamma : \Gamma(ND)] < \infty.$$

Definition. *If $\Gamma_1, \Gamma_2 \leq G$, $\alpha \in G$, define the double coset*

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}.$$

Remarks:

- $\Gamma_1\alpha \subseteq \Gamma_1\alpha\Gamma_2$
- $\Gamma_1\alpha\Gamma_2$ is the union of right cosets of Γ_1.
- For $\alpha, \beta \in, G$, $\Gamma_1\beta\Gamma_2$ are either disjoint or equal.

Proposition 36. *Suppose G is a group and $\Gamma' \leq G$, $\alpha \in G$, and $\Gamma', \alpha^{-1}\Gamma'\alpha$ are commensurable. Set $\Gamma'' := \Gamma' \cap \alpha^{-1}\Gamma'\alpha$, and write*

$$\Gamma' = \bigcup_{j=1}^{d} \Gamma''\gamma_j', \qquad \textit{where } d = [\Gamma' : \Gamma'']. \tag{7}$$

Then

$$\Gamma'\alpha\Gamma' = \bigcup_{j=1}^{d} \Gamma'\alpha\gamma_j' \tag{8}$$

is a disjoint union of right cosets. Conversely, (8) $\implies$ (7).

Remark: Coset representatives of Γ'' in Γ' are in one-to-one correspondence with representatives in a coset decomposition of $\Gamma'\alpha\Gamma'$.

Proof of Proposition 36.

We only prove one direction. Assume (7). Given $\gamma_1\alpha\gamma_2 \in \Gamma'\alpha\Gamma'$, write $\gamma_2 = \gamma''\gamma'_j$, for some $1 \leq j \leq d$ and $\gamma'' \in \Gamma'' = \Gamma' \cap \alpha^{-1}\Gamma'\alpha$. Write $\gamma'' = \alpha^{-1}\gamma'\alpha$ for some $\gamma' \in \Gamma'$. Then

$$\begin{aligned} \gamma_1\alpha\gamma_2 &= \gamma_1\alpha\gamma''\gamma'_j = \gamma_1\alpha(\alpha^{-1}\gamma'\alpha)\gamma'_j \\ &= (\gamma_1\gamma')\alpha\gamma'_j \in \Gamma'\alpha\gamma'_j. \end{aligned}$$

To show that the cosets in (8) are distinct, note that if $\Gamma'\alpha\gamma'_j = \Gamma'\alpha\gamma'_k$, then $\alpha\gamma'_j\gamma'^{-1}_k\alpha^{-1} \in \Gamma'$, which implies $\gamma'_j\gamma'^{-1}_k \in \alpha^{-1}\Gamma'\alpha \cap \Gamma' = \Gamma''$. This, in turn, implies $\Gamma''\gamma'_j = \Gamma''\gamma'_k$, so that $j = k$. □

Definition. *Suppose* Γ' *is a congruence subgroup and* $\alpha \in GL_2^+(\mathbb{Q})$. *Set* $\Gamma'' := \Gamma' \cap \alpha^{-1}\Gamma'\alpha$, *and write* $\Gamma' = \bigcup_{j=1}^{d} \Gamma''\gamma'_j$, *a coset decomposition. Suppose* $f\Big|_k\gamma = f\ \forall\gamma \in \Gamma'$. *Then define*

$$f\Big|_k\Gamma'\alpha\Gamma' := \sum_{j=1}^{d} f\Big|_k\alpha\gamma'_j. \tag{9}$$

Proposition 37.
(1) *The definition in (9) is independent of the coset representatives.*
(2) *If* $\Gamma'\alpha\Gamma' = \Gamma'\beta\Gamma'$, *then* $f\Big|_k\Gamma'\alpha\Gamma' = f\Big|_k\Gamma'\beta\Gamma'$.
(3) *If* $f \in M_k(\Gamma')$, *then* $f\Big|_k\Gamma'\alpha\Gamma' \in M_k(\Gamma')$. *Moreover, the same holds for cusp forms (as the cusp conditions are preserved.)*

13. N. Masri. 10/17 and 10/20.

Recall from last Wednesday that, if Γ' is a congruence subgroup, $\alpha \in GL_2^+(\mathbb{Q})$, then $\Gamma'' = \Gamma' \cap \alpha^{-1}\Gamma'\alpha$, $\Gamma' = \bigcup_{j=1}^d \Gamma''\gamma'_j$, $f|_k\gamma' = f$, $\forall\gamma' \in \Gamma'$. Then define

$$f|_k\Gamma'\alpha\Gamma' := \sum_{j=1}^{d} f|_k\alpha\gamma'_j \qquad (\Delta)$$

Proposition 37.
(1) *The definition* (Δ) *is independent of the coset representatives* γ'_j.
(2) *If* $\Gamma'\alpha\Gamma' = \Gamma'\beta\Gamma'$ *then* $f|_k\Gamma'\alpha\Gamma' = f|_k\Gamma'\beta\Gamma'$.
(3) *If* $f \in M_k(\Gamma')$ *then* $f|_k\Gamma'\alpha\Gamma' \in M_k(\Gamma')$. *(The same is also true for cusp forms.)*

Proof of Proposition 37:

Proof. (1) Suppose γ'_j changed to $\gamma''_j\gamma'_j$ where $\gamma''_j \in \Gamma''$. Write γ''_j as $\alpha^{-1}\gamma'\alpha$, $\gamma' \in \Gamma'$. Then $f|_k\alpha\gamma''_j\gamma'_j = f|_k\alpha(\alpha^{-1}\gamma'\alpha)\gamma'_j = f|_k\alpha\gamma'_j$.

(2) Suppose $\Gamma'\alpha\Gamma' = \bigcup_{j=1}^d \Gamma'\alpha_j$ is a coset decomposition. Proposition 36 implies that $\Gamma' = \bigcup_{j=1}^d \Gamma''\alpha^{-1}\alpha_j$ is a coset decomposition. Then $(\Delta) \implies f|_k\Gamma'\alpha\Gamma' = \sum_{j=1}^d f|_k\alpha(\alpha^{-1}\alpha_j) = \sum_{j=1}^d f|_k\alpha_j$. (Call this (Δ')). So the definition depends only on $\Gamma'\alpha\Gamma'$, and not on α.

(3) Lemma 20 $\implies f|_k\Gamma'\alpha\Gamma'$ satisfies the same cusp conditions as f.

Transformation law:

Suppose $\gamma' \in \Gamma'$. Then we claim that $\Gamma'\alpha\Gamma' = \bigcup_{j=1}^d \Gamma'\alpha_j \implies \Gamma'\alpha\Gamma' = \bigcup_{j=1}^d \Gamma'\alpha_j\gamma'$. So $f|_k\Gamma'\alpha\Gamma' = \sum f|_k\alpha_j\gamma' = \sum(f|_k\alpha_j)|_k\gamma' = (f|_k\Gamma'\alpha\Gamma')|_k\gamma'$.

□

We next define Hecke operators for a large class of congruence subgroups. Some notation:

S^+ is a nonzero additive subgroup of $\mathbb{Z}$. i.e., $S^+ = M\mathbb{Z}$ for some $M \in \mathbb{N}$.

$S^\times$ is a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$.

For $n \in \mathbb{N}$ define $\Delta^n(N, S^\times, S^+) = \{$ integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$: $ad - bc = n$, $N|c$, $a \in S^\times$, $b \in S^+\}$.

(Note that the Δ^n are not groups: multiplying two elements of determinant n yields an element of determinant n^2.)

Remarks:

(1) $\Delta^1(N, S^\times, S^+)$ is a group.

(2) If $N' = \mathrm{lcm}(N, M)$, where M is such that $S^+ = M\mathbb{Z}$, then $\Gamma(N') \subset \Delta^1(N, S^\times, S^+)$.

(3) $\Gamma_1(N) = \Delta^1(N, \{1\}, \mathbb{Z})$,
$\Gamma_0(N) = \Delta^1(N, (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Z})$ and
$\Gamma(N) = \Delta^1(N, \{1\}, N\mathbb{Z})$.

Definition of Hecke Operators:

Set $\Gamma' = \Delta^1(N, S^\times, S^+)$. Let $n \in \mathbb{N}$, and suppose $f \in M_k(\Gamma')$. Then define the Hecke operator T_n by:

$$T_n(f) = n^{\frac{k}{2}-1} \sum f|_k\Gamma'\alpha\Gamma'$$

(Where the sum is over the double cosets of Γ' in $\Delta^n(N, S^\times, S^+)$.)

Equivalently, define $T_n f = n^{\frac{k}{2}-1} \sum f|_k\alpha_j$, where $\Gamma'\alpha_j$ runs over all right cosets of Γ' in $\Delta^n(N, S^\times, S^+)$.

To see the equivalence, note that in the coset decomposition $\Gamma'\alpha\Gamma' = \bigcup \Gamma'\beta_j$, every right coset $\Gamma'\alpha_j$ gets hit exactly once as $\Gamma'\alpha\Gamma'$ runs over all the double cosets.

Fact: The number of cosets is finite, so $T_n : M_k(\Gamma') \to M_k(\Gamma')$ and $S_k(\Gamma') \to S_k(\Gamma')$.

Recall: $V_d f(z) = f(dz)$, $U_d f(z) = \frac{1}{d}\sum_{j=0}^{d-1} f(\frac{z+j}{d})$. Specialize to $\Gamma_1(N) = \Delta^1(N, \{1\}, \mathbb{Z})$. (To simplify notation, we will from now on denote $\Delta^1(N, \{1\}, \mathbb{Z})$ by Δ^1).

Let T_n be the operator on $M_k(\Gamma_1(N))$.

Theorem 38. *Let T_n be the nth Hecke operator on $M_k(\Gamma_1(N))$. Then*

(1) *T_n preserves each space $M_k(\Gamma_0(N), \chi)$.*

(2) *As a map on $M_k(\Gamma_0(N), \chi)$ we have $T_n = \sum_{d|n} \chi(d) d^{k-1} V_d \circ U_{n/d}$.*

To prove theorem 38, we need an explicit decomposition of the form $\Delta^n = \bigcup \Gamma_1(N)\alpha_j$. If $(a, N) = 1$, fix $\sigma_a \in \Gamma$, $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \pmod{N}$.

Lemma 39. *We have the coset decomposition*

$$\Delta^n = \bigcup_{\substack{a|n \\ (a,w)=1}} \bigcup_{\substack{b=0 \\ d=\frac{n}{a}}}^{d-1} \Gamma_1(N)\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

.

Proof of Lemma 39: (We will do the "$\supseteq$" part. For the other direction, see Koblitz.)

Proof. Everything on the right has determinant n, and the form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \equiv \begin{pmatrix} a^{-1} & * \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod N$$

. ☐

Proof of Proposition 38:

Proof. It is easier to prove 2nd part first. Recall that $V_d f(z) = f(dz), U_d f(z) = \frac{1}{d}\sum_{b=0}^{d-1} f(\frac{z+b}{d})$. By Lemma 39, if $f \in M_k(\Gamma_0(N), \chi)$,

$$T_n f = n^{\frac{k}{2}-1} \sum_{\substack{a|n \\ (a,N)=1}} \sum_{\substack{b=0 \\ d=\frac{n}{a}}}^{d-1} f|_k \sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = n^{\frac{k}{2}-1} \sum_{a|n} \chi(a) \sum_{b=0}^{d-1} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Note that

$$\begin{aligned} \sum_{b=0}^{d-1} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} &= (ad)^{\frac{k}{2}} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right) \\ &= a^{\frac{k}{2}} d^{-\frac{k}{2}} V_a \left(\sum_{b=0}^{d-1} f\left(\frac{z+b}{d}\right) \right) = (a^{\frac{k}{2}} d^{-k/2} V_a \circ dU_d) f \\ &= (a^{\frac{k}{2}} d^{-\frac{k}{2}+1} V_a \circ U_d) f, \end{aligned}$$

so $T_n = n^{\frac{k}{2}-1} \sum_{a|n} \chi(a) a^{\frac{k}{2}} d^{-\frac{k}{2}+1} V_a \circ U_d = \sum_{a|n} \chi(a) (ad)^{\frac{k}{2}-1} a^{\frac{k}{2}} d^{-\frac{k}{2}+1} V_a \circ U_{n/d} = \sum_{a|n} \chi(a) a^{k-1} V_a \circ U_{n/a}$. (end of part 2).

We still need to prove that T_n preserves $M_k(\Gamma_0(N), \chi)$ from part 1. There are two approaches:

Approach 1: Use that $M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_0(N), \chi)$.

Recall the operator $< d >$ on $M_k(\Gamma_1(N))$, $< d > f : f \mapsto f|_k \sigma_d$, $(d, N) = 1$.

Use coset decomposition to show that $< d >$ commutes with T_n for all n. Then T_n preserves eigenspaces for operator $< d >$. These eigenspaces are the spaces $M_k(\Gamma_0(N), \chi)$.

Using coset decomposition takes a while, so instead we try approach 2: (which we will prove). The basic idea is to use the explicit description we already have:

Approach 2: Suppose $f \in M_k(\Gamma_0(N), \chi) \subseteq M_k(\Gamma_1(N))$. Then $T_n f \in M_k(\Gamma_1(N))$. $\forall (d \mid n)$, $(V_d \circ U_{n/d}) f \in M_k(\Gamma_0(Nn), \chi_n^{triv}.\chi)$ and so $T_n f \in M_k(\Gamma_1(N)) \cap M_k(\Gamma_0(Nn), \chi.\chi_n^{triv}) = M_k(\Gamma_0(N), \chi)$.

For the proof of the last equality, one direction is clear: $M_k(\Gamma_0(N), \chi) \subseteq$ each of $M_k(\Gamma_1(N))$ and $M_k(\Gamma_0(Nn), \chi.\chi_n^{triv})$. For the "$\subseteq$" direction, if f is in the intersection, then $f|_k\sigma_d$ depends only on d mod N. Given d with $(d,N)=1$ we replace d by $d+jn$ for a suitable j to get $(d,n)=1$. Then $f|_k\sigma_d = \chi(d)\chi_n^{triv}(d)f = \chi(d)f \implies f \in M_k(\Gamma_0(n),\chi)$. □

Proposition 38 results in a number of consequences:

Proposition 40. $\forall m,n \geq 1$, $T_mT_n = \sum_{d|(m,n)} \chi(d)d^{k-1}\frac{T_{mn}}{d^2}$.

Corollary 41. (i) $T_mT_n = T_nT_m$.

(ii) *We can invert proposition 40 to get:*

$$T_{mn} = \sum_{d|(m,n)} \mu(d)\chi(d)d^{k-1}T_{\frac{m}{d}}T_{\frac{n}{d}}$$

(iii) *if* $\gcd(m,n)=1$ *then* $d=1 \implies T_{mn} = \mu(1)\chi(1)1^{k-1}T_mT_n = T_mT_n$.

(iv) *If we set* $m=p^\nu$, $n=p$ *we get:*

$$T_{p^{\nu+1}} = T_{p^\nu}T_p - \chi(p)p^{k-1}T_{p^{\nu-1}}.$$

14. J. Webster. 10/22 and 10/24

October 22

Corollary 42. *Let* $\mathcal{T}$ *be the algebra generated over* $\mathbb{C}$ *by all* T_n, *the "Hecke Algebra." Then* $\mathcal{T}$ *is commutative and is generated by* T_p *for all primes* p.

Corollary 43. *The following are true:*

(1) $\displaystyle\sum_{n=1}^{\infty} T_n n^{-s} = \prod_p \sum_{v=0}^{\infty} T_{p^v}p^{-vs}$.

(2) $\displaystyle\sum_{v=0}^{\infty} T_{p^v}p^{-vs} = \frac{1}{1-T_pp^{-s}+\chi(p)p^{k-1-2s}}$.

To prove 43.2 just multiply by denominator and compare to Corollary 41.

Proof of Proposition 40 This is primarily an exercise in variable changing.

Note: U and V won't commute in general but if $(d,t)=1$ then $V_d \circ U_t = U_t \circ V_d$.

$$(U_t \circ V_d)\sum_{n=0}^{\infty} a(n)q^n = U_t\sum_{n=0}^{\infty} a(n)q^{nd} = \sum_{t|nd} a(n)q^{\frac{nd}{t}}$$

.

$$(V_d \circ U_t)\sum_{n=0}^{\infty} a(n)q^n = V_d\sum_{t|n} a(n)q^{\frac{n}{t}} = \sum_{t|n} a(n)q^{\frac{nd}{t}}$$

.

Since $(d,t)=1$ the condition that $t|n$ is the same condition as $t|dn$. We have

$$T_nT_m = \sum_{\substack{a_1d_1=m\\a_2d_2=n}} \chi(d_1d_2)(d_1d_2)^{k-1}V_{d_1}\circ U_{a_1}\circ V_{d_2}\circ U_{a_2}$$

.

Let $\delta=(a_1,d_2)$, $a_1'=\frac{a_1}{\delta}$, and $d_2'=\frac{d_2}{\delta}$. Then

$$V_{d_1} \circ U_{a_1} \circ V_{d_2} \circ U_{a_2} = V_{d_1} \circ U_{a'_1} \circ U_{\delta} \circ V_{\delta} \circ V_{d'_2} \circ U_{a_2}$$

.

Note that because $U_{\delta} \circ V_{\delta} = Id$ we can write the above as:

$$V_{d_1} \circ U_{a'_1} \circ V_{d'_2} \circ U_{a_2} = V_{d_1 d'_2} \circ U_{a'_1 a_2}$$

.

So

$$T_n T_m = \sum_{\delta|(m,n)} \sum_{\substack{a'_1 d_1 = \frac{m}{\delta} \\ a_2 d'_2 = \frac{n}{\delta}}} \chi(\delta d_1 d'_2)(\delta d_1 d'_2)^{k-1} V_{d_1 d'_2} \circ U_{a'_1 a_2}$$

.

Now set $a = a'_1 a_2$, $d = d_1 d'_2$, and $ad = \frac{mn}{\delta^2}$ to get the result. Conversely, given $ad = \frac{mn}{\delta^2}$. Set $a'_1 = \frac{m}{(m,d\delta)}$, and $d'_2 = \frac{d\delta}{(m,d\delta)}$. Then $(a'_1, d'_2) = 1$ and corresponds to a unique index on the inner sum. So,

$$T_n T_m = \sum_{\delta|(m,n)} \chi(\delta)\delta^{k-1} \sum_{ad = \frac{mn}{\delta^2}} \chi(d) d^{k-1} V_d \circ U_a = \sum_{\delta|(m,n)} \chi(d)\delta^{k-1} T_{\frac{mn}{\delta^2}}$$

. □

Effects on Fourier expansions Let $f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ and $T_p = U_p + \chi(p)p^{k-1}V_p$. If $p|n$ then $T_p = U_p$.

$$T_p(\sum a(n)q^n = \sum (a(pn) + \chi(p)p^{k-1}a(\frac{n}{p}))q^n$$

where $a(\frac{n}{p}) = 0$ if $p \nmid n$. More generally,

$$\begin{aligned}
T_m(\sum a(n)q^n) &= \sum_{d|n} \chi(d)d^{k-1} V_d \circ U_{\frac{m}{d}}(\sum a(n)q^n) \\
&= \sum_{d|m} \chi(d)d^{k-1} V_d(\sum a(\frac{mn}{d})q^n) \\
&= \sum_{d|m} \chi(d)d^{k-1} \sum_{d|n} a(\frac{mn}{d^2})q^n \\
&= \sum_{n=1}^{\infty} (\sum_{d|(m,n)} \chi(d)d^{k-1}a(\frac{mn}{d^2}))q^n.
\end{aligned}$$

Proposition 44. *If $f = \sum a(n)q^n$ then $T_m f = \sum b(n)q^n$ where $b(n) = \sum_{d|(m,n)} \chi(d)d^{k-1}a(\frac{mn}{d^2})$*

Proof of Proposition 44 The proof is given above.

Definition. *f is an eigenform of T_m if there exists $\lambda_m \in \mathbb{C}$ such that $T_m f = \lambda_m f$.*

Proposition 45. *Suppose $f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ has $T_m f = \lambda_m f$. Then*

(1) $a(m) = \lambda_m a(1)$
(2) *if* $a(0) \neq 0$ *then* $\lambda_m = \sum_{d|m} \chi(d) d^{k-1}$.

Proof of Proposition 45 By Proposition 44,

$$\begin{aligned} T_m f &= (\sum_{d|m} \chi(d) d^{k-1}) a(0) + a(m) q + \cdots \\ &= \lambda_m f \\ &= \lambda_m a(0) + \lambda_m a(1) q + \cdots \end{aligned}$$

□

Definition. *Call f a normalized eigenform if $a(1) = 1$ and f is an eigenform of all T_m. Then $T_m f = \lambda_m f = a(m) f$ for all m.*

If f is a normalized eigenform then the Euler product for $\sum T_n n^{-s}$ becomes

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \frac{1}{1 - a(p) p^{-s} + \chi(p) p^{k-1-2s}}$$

call this $L(f, s)$ the "Modular L function."

Why are these important? Let $E/\mathbb{Q}$ be an elliptic curve and

$$L(E, S) = \prod_p \frac{1}{1 - a(p) p^{-s} + p^{1-2s}}$$

where $a(p) = p + 1 - \#$pts on $E/\mathbb{F}_p$. Wiles, et al. proved that there exists $f \in M_2(\Gamma_0(N))$ such that $L(E, S) = L(f, s)$. Further the BSD conjecture says $\mathrm{ord}_{s=1} L(E, S) = Rank(E)$.

15. October 24

Exercises from Koblitz
Problem 3.5.6 Find basis of normalized eigenforms of all T_n. $\dim(S_k(\Gamma)) = 1$ is trivial. Look at the space $S_{24}(\Gamma)$. $\dim(S_{24}(\Gamma)) = 2$. Let $f_1 = \Delta^2$ and $f_2 = E_6^2 \Delta$. Both have weight 24. One starts with q and the other with q^2.

$$f_1 = q^2 - 48q^3 + 1080q^4 + \cdots$$

$$f_2 = q - 1032q^2 + 245196q^3 + 10965568q^4 + \cdots$$

These are not normalized eigenforms because the coefficient on f_1 on q is 0. Further, f_2 is not normalized as $a_2 a_3 \neq a_6$. So we must find the basis of normalized eigenforms.

Consider T_2.

$$\begin{aligned} T_2 \sum a(n) q^n &= \sum a(2n) + 2^{23} a(\frac{n}{2})) q^n \\ &= a(2) q + (a(4) + 2^{23} a(1)) q^2 + \cdots \end{aligned}$$

$$T_2 f_1 = q + 1080q^2 + \cdots = f_2 + 2112 f_1$$

$$T_2 f_2 = -1032q + (10965568 + 2^{23}) q^2 + \cdots = -1032 f_2 + 18289152 f_1$$

The matrix of T_2 in $\{f_1, f_2\}$ is

$$\begin{pmatrix} 2112 & 18289152 \\ 1 & -1032 \end{pmatrix}$$

. Use some computer system to compute eignevectors and eigenvalues.

Eigenvalues: $540 \pm 12\sqrt{144169}$

Eigenvectors: $\{(1572 \pm 12\sqrt{144169})f_1 + f_2\} := \{F_1, F_2\}$

Since T_n commutes with T_2 for all n this implies that F_1, F_2 are normalized eigenforms of all T_n. Further F_1 and F_2 are Galois conjugates of each other. Let $\alpha = 15762 + 12\sqrt{144169}$. Then

$$f_1 = \sum a_1(n)q^n, F_1 = \sum(\alpha a_1(n) + a_2(n))q^n$$

,

$$f_2 = \sum a_2(n)q^n, F_2 = \sum(\bar{\alpha} a_1(n) + a_2(n))q^n$$

.

The n^{th} coefficient of F_1 is an eigenvalue of F_1 under T_n.

$$\begin{aligned} \sum Tr(T_n)q^n &= \sum((\alpha + \bar{\alpha})a_1(n) + 2a_2(n))q^n \\ &= \sum(3144a_1(n) + 2a_2(n))q^n \\ &= 3144f_1 + 2f_2 \in S_{24}(\Gamma). \end{aligned}$$

Problem 3.5.4 $S_8(\Gamma_0(4))$ has dimension 2.

$$\begin{aligned} f(z) = \eta^8(z)\eta^8(2z) \in S_8(\Gamma_0(2)) \subseteq S_8(\Gamma_0(4)) \\ g(z) = f(z)|V_2 \in S_8(\Gamma_0(4)) \end{aligned}$$

These are clearly linearly independent as the lowest power of q in f is 1 and the lowest power of q in g is 2. Since $\dim(S_8(\Gamma_0(2))) = 1$ then $T_n f = \lambda_n f$ for all n.

Fact: For odd n, T_n does the same thing to f considered as an element in either space.

Fact: If n is odd T_n and V_2 commute. This is because

$$T_n = \sum \chi(d)d^{k-1}V_d \circ U_{\frac{n}{d}}$$

This implies that for odd n that $T_n g = \lambda_n g$. So T_n acts by scalar multiplication on $S_4(\Gamma_0(8))$ for odd n. Look at T_2 which is U_2:

$$\begin{aligned} T_2 g = T_2 V_2 f = U_2 V_2 f = f \\ f = q - 8q^2 + 12q^3 + 64q^4 + \cdots \\ T_2 f = U_2 f = -8q + 64q^2 + \cdots = -8f \end{aligned}$$

.

The matrix is

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \end{pmatrix}$$

. The eigenvalues are $\{-8, 0\}$ and the eigenvectors are $\{f, f + 8g\}$. So the eigenvectors are normalized eigenforms of all T_n.

16. S. Kadziela. 10/27 and 10/29

Recall: f is a normalized eigenform if

1) $a(1) = 1$ and
2) f is an eigenform of all the operators T_n.

If $f = \sum a(n)q^n$ has this property, then we get an Euler product

$$\sum a(n)q^{-s} = \prod_p (1 - a(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1}$$

Dream: (almost realizable) Basis of normalized eigenforms. Need a Hermitian inner product for $S_k(\Gamma_0(N), \chi)$ (Petersson inner product).

Poincaré upper half plane $\mathbb{H}$ with $GL_2^+(\mathbb{R})$ invariant metric and measure.

Recall: $\alpha \in GL_2^+(\mathbb{R})$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then know

$$\operatorname{Im}(\alpha z) = \frac{\det(\alpha)}{|cz+d|^2} \operatorname{Im}(z) \text{ and } \frac{d(\alpha z)}{dz} = \frac{\det(\alpha)}{|cz+d|^2}$$

Fact: $ds^2 = \frac{dx^2+dy^2}{y^2}$ and $d\mu z = \frac{dxdy}{y^2}$ are invariant under $GL_2^+(\mathbb{R})$. To see the second one,

$$d(\alpha z) = \frac{d(\alpha z)}{dz} dz = \frac{\det(\alpha)}{|cz+d|^2} dx + i \frac{\det(\alpha)}{|cz+d|^2} dy$$

so under change $z \mapsto \alpha z$, the area element $dxdy$ becomes

$$\left| \frac{\det(\alpha)}{|cz+d|^2} \right|^2 dxdy = \left(\frac{\operatorname{Im}(\alpha z)}{\operatorname{Im}(z)} \right)^2 dxdy \Longrightarrow \frac{dxdy}{y^2} \text{ is invariant.}$$

Note: Geodesics in this metric are lines perpendicular to $\mathbb{R}$ and circles centered on $\mathbb{R}$.

Proposition 46. *Suppose $\Gamma' \subset \Gamma$ congruence subgroup. Define $\mu(\Gamma') = \int_{F'} \frac{dxdy}{y^2}$, where F' is a fundamental domain for Γ'. Then*

a) *Integral converges and is independent of the choice of F'.*
b) $\frac{\mu(\Gamma')}{\mu(\Gamma)} = [\overline{\Gamma} : \overline{\Gamma'}]$
c) $\alpha \in GL_2^+(\mathbb{Q})$ *and* $\alpha^{-1}\Gamma'\alpha \subset \Gamma$, *then* $[\overline{\Gamma} : \overline{\Gamma'}] = [\overline{\Gamma} : \alpha^{-1}\overline{\Gamma'}\alpha]$

Proof.

a) Write $\overline{\Gamma} = \bigcup_{j=1}^{d} \alpha_j \overline{\Gamma'}$, $d = [\overline{\Gamma}, \overline{\Gamma'}]$. Then $F' = \bigcup_{j=1}^{d} \alpha_j^{-1} F$ is a fundamental domain for Γ'. By definition,

$$\mu(\Gamma') = \sum_{j=1}^{d} \int_{\alpha_j^{-1}F} \frac{dxdy}{y^2}, \text{ so after changing } z \mapsto \alpha_j z,$$

$$= \sum_{j=1}^{d} \int_{F} \frac{dxdy}{y^2} = [\overline{\Gamma} : \overline{\Gamma'}]\mu(\Gamma)$$

provided that the integral converges.

Note: $\int\limits_F \frac{dxdy}{y^2} \leq \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} (\int\limits_{\frac{\sqrt{3}}{2}}^{\infty} \frac{dy}{y^2})dx < \infty$.

Next, show its independent of F'. Suppose G is another (nice) fundamental domain. Break G into regions R s.t. $\exists\, \alpha \in \Gamma'$ with $\alpha R \subseteq F'$. Then

$$\int\limits_{\alpha R} \frac{dxdy}{y^2} = \int\limits_{R} \frac{dxdy}{y^2} \qquad\qquad (z \mapsto \alpha^{-1} z).$$

c) If F' is a fundamental domain for Γ', then $\alpha^{-1}F'$ is a fundamental domain for $\alpha^{-1}\Gamma'\alpha$. Hence

$$\mu(\alpha^{-1}\Gamma'\alpha) = \int\limits_{\alpha^{-1}F'} \frac{dxdy}{y^2} = \int\limits_{F'} \frac{dxdy}{y^2} = \mu(\Gamma').$$

□

Definition of Petersson inner product.
Suppose $\Gamma' \subset \Gamma$ congruence subgroup. Suppose $f, g \in M_k(\Gamma')$ and at least one of f, g is a cusp form. Define

$$\langle f, g\rangle = \frac{1}{\mu(\Gamma')} \int\limits_{F'} f(z)\overline{g(z)}y^k \, \frac{dxdy}{y^2}$$

Properties:

1) $\langle f, g\rangle = \overline{\langle g, f\rangle}$
2) $\langle cf, g\rangle = c\langle f, g\rangle$
3) $\langle f, f\rangle > 0$ if $f \neq 0$.

So it is a "Hermitian" inner product.
Note: If $\alpha \in GL_2^+(\mathbb{R})$, then replacing z by αz in $f(z)\overline{g(z)}y^k$ gives

$$f(\alpha z)\overline{g(\alpha z)} \left(\frac{\det(\alpha)}{|cz+d|^2}\right)^k y^k \text{ since } \operatorname{Im}(\alpha z) = \frac{\det(\alpha)}{|cz+d|^2} \operatorname{Im}(z)$$

$$= f(z)\big|_k\alpha\, \overline{g(z)\big|_k\alpha}\, y^k \Longrightarrow \int\limits_{\alpha R} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2} = \int\limits_R f\big|_k\alpha\, \overline{g\big|_k\alpha}\, y^k \, \frac{dxdy}{y^2}.$$

Proposition 47.

a) *Integral converges absolutely.*
b) *Integral does not depend on choice of F'.*
c) *If $f, g \in M_k(\Gamma') \cap M_k(\Gamma'')$, then definition of $\langle f, g\rangle$ is the same on both spaces.*

Proof.

a) Write $\overline{\Gamma} = \bigcup\limits_{j=1}^{d} \alpha_j \overline{\Gamma'}$. Then $F' = \bigcup\limits_j \alpha_j^{-1} F$ is a fundamental domain. Then

$$\langle f, g\rangle = \frac{1}{\mu(\Gamma')} \sum_j \int\limits_{\alpha_j^{-1}F} f(z)\overline{g(z)}y^k \, \frac{dxdy}{y^2} = \frac{1}{\mu(\Gamma')} \sum_j \int\limits_F f\big|_k\alpha_j^{-1}\, \overline{g\big|_k\alpha_j^{-1}}\, y^k \, \frac{dxdy}{y^2}.$$

Show each integral converges absolutely.

To see this, suppose f is a cusp form. Then $f\big|_k\alpha_j^{-1} = \sum\limits_{n=1}^{\infty} a(n)q_N^n$.

$$\text{Note: } |q_N| = |e^{\frac{2\pi i(x+iy)}{N}}| = e^{\frac{-2\pi y}{N}} \implies |f\big|_k\alpha_j^{-1}| \le e^{\frac{-2\pi y}{N}} \sum_{n=1}^{\infty} |a(n)| e^{\frac{-2\pi y(n-1)}{N}}$$

$$\le Ae^{\frac{-2\pi y}{N}} \text{ for some A, since } y \ge \frac{\sqrt{3}}{2}, \text{ and } |g\big|_k\alpha_j^{-1}| \ll 1 \text{ as } y \to \infty$$

$\implies$ each integral converges absolutely.

b) Suppose G is another fundamental domain. Break G into regions R as before. Then

$$\int_{\alpha R} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2} = \int_R f\big|_k\alpha\, \overline{g\big|_k\alpha}\, y^k \frac{dxdy}{y^2} = \int_R f\overline{g}y^k \frac{dxdy}{y^2} \text{ since } \alpha \in \Gamma'.$$

c) done in the book.

$\square$

Goal: $\langle f, g\rangle$ is Hermitian and positive definite, i.e.

$\langle f, g\rangle = \overline{\langle g, f\rangle}$ antisymmetric

$\langle f, f\rangle > 0$ if $f \neq 0$

$\langle f, g\rangle$ linear in f, antilinear in g

Given linear map T, define adjoint T^* by $\langle Tf, g\rangle = \langle f, T^*g\rangle$. Call T Hermitian (self-adjoint) if $T = T^*$. (T is called normal if T, T^* commute).

Spectral Theorem (Lang Algebra XV section 6)

Suppose E finite dimensional vector space over $\mathbb{C}$ with positive definite Hermitian inner product. Suppose $T : E \longrightarrow E$ is self-adjoint. Then E has an orthogonal basis consisting of eigenvectors of T.

Goal: Show that T_n are "almost" self-adjoint, i.e. $\overline{\chi(n)}^{\frac{1}{2}}T_n$ is self-adjoint.

Proposition 48. *Suppose one of f, g is a cusp form. Suppose $\alpha \in GL_2^+(\mathbb{Q})$. Then*

$$\langle f\big|_k\alpha, g\big|_k\alpha\rangle = \langle f, g\rangle. \text{ (Note: } f, g \in M_k(\Gamma') \implies f\big|_k\alpha,\ g\big|_k\alpha \in M_k(\alpha^{-1}\Gamma'\alpha) \text{).}$$

Proof. done in Koblitz.

$\square$

Remark: Suppose $\alpha \in GL_2^+(\mathbb{Q})$, then $\big|_k\alpha$ is the same if α is replaced by a positive constant multiple.

WLOG, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ integral entries. Set $\alpha' = (det(\alpha))\alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $\big|_k\alpha^{-1} = \big|_k\alpha'$.

Proposition 49. *$f, g \in M_k(\Gamma'), \alpha \in GL_2^+(\mathbb{Q})$. Then*

1) $\langle f\big|_k\alpha, g\rangle = \langle f, g\big|_k\alpha'\rangle$
2) *$\langle f\big|_k\alpha, g\rangle$ depends only on the double coset $\Gamma'\alpha\Gamma'$.*

Proof.

2) Suppose change $\alpha \to \gamma_1\alpha\gamma_2$.

$$\langle f\big|_k\gamma_1\alpha\gamma_2, g\rangle = \langle f\big|_k\gamma_1\big|_k\alpha, g\big|_k\gamma_2^{-1}\rangle = \langle f\big|_k\alpha, g\rangle$$

□

Interplay with Hecke operators.

Set $\Gamma' = \Gamma_1(N), \Delta^n = \Delta^n(N, \{1\}, \mathbb{Z})$. If $d \in (\mathbb{Z}/N\mathbb{Z})^*$, fix $\delta_d \equiv \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \pmod{N}$. T_n Hecke operators on $M_k(\Gamma_1(N))$.

Theorem 50. *Suppose $f, g \in M_k(\Gamma_1(N))$, one a cusp form. Suppose $(n, N) = 1$. Then*

$$\langle T_n f, g\rangle = \langle f\big|_k\delta_n, T_n g\rangle.$$

In particular, if $f \in M_k(\Gamma_0(N), \chi)$, then

$$\langle T_n f, g\rangle = \chi(n)\langle f, T_n g\rangle.$$

Proof. later...

□

Suppose $(n, N) = 1$. Then $\overline{\chi(n)}^{\frac{1}{2}}T_n$ is self-adjoint. Why?

$$\langle\overline{\chi(n)}^{\frac{1}{2}}T_n f, g\rangle = \overline{\chi(n)}^{\frac{1}{2}}\langle T_n f, g\rangle = \overline{\chi(n)}^{\frac{1}{2}}\chi(n)\langle f, T_n g\rangle = \langle f, \overline{\chi(n)}^{\frac{1}{2}}T_n g\rangle$$

Note: $\overline{\chi(n)}^{\frac{1}{2}}T_n, T_n$ have same eigenspaces.

Theorem 51. *$S_k(\Gamma_0(N), \chi)$ has a basis consisting of eigenforms of all T_n with $(n, N) = 1$.*

Proof.

Enough to check $T_p, p \nmid N$. List these primes: $p_1, p_2, p_3 \ldots$. Spectral Theorem implies $S_k(\Gamma_0(N), \chi) = V_1 \oplus V_2 \oplus \ldots \oplus V_t$, where V_i are eigenspaces for T_{p_1}. T_{p_2} commutes with $T_{p_1} \implies T_{p_2}$ preserves all $V_i \implies$ each V_i is a direct sum of eigenspaces for T_{p_2}, by the Spectral Theorem.

Repeat with $T_{p_3}, T_{p_4}, \ldots$. Finite dimensionality implies at some point stop getting new subspaces. At this point, $S_k(\Gamma_0(N), \chi) = W_1 \oplus \ldots \oplus W_s$, each W_i joint eigenspace $\forall\, T_n$ with $(n, N) = 1$. Take any basis of forms in these spaces.

□

17. T. Kilbourn. 10/31 and 11/3

Sketch of proof of Theorem 50. $T_n f = n^{\frac{k}{2}-1}\sum f|_k\Gamma'\alpha\Gamma'$, where the sum runs over all the double cosets of Γ' in Δ^n. Recall if $\Gamma'\alpha\Gamma' = \bigcup\Gamma'\alpha_j$, then $f|_k\Gamma'\alpha\Gamma' = \sum f|_k\alpha_j$. So $\langle T_n f, g\rangle = n^{\frac{k}{2}-1}\sum\langle f|_k\Gamma'\alpha\Gamma', g\rangle$. Note that the $\langle f|_k\alpha_j, g\rangle$ are all equal. This implies that if d_α is the number of right cosets in $\Gamma'\alpha\Gamma'$, then

$$\langle T_n f, g\rangle = n^{\frac{k}{2}-1}\sum d_\alpha \cdot \langle f|_k\alpha, g\rangle. \tag{10}$$

Recall that $\alpha' = \det(\alpha)\alpha^{-1}$. We have the following two facts:

- $\Gamma'\alpha\Gamma' \to \Gamma'\sigma_n\alpha'\Gamma'$ permutes the double cosets (using the fact that $(n, N) = 1$).

- $d_{\sigma_n \alpha'} = d_\alpha$.

From (10) we see that

$$\begin{aligned} \langle T_n f, g\rangle &= n^{\frac{k}{2}-1} \sum d_{\sigma_n\alpha'} \cdot \langle f|_k \sigma_n \alpha', g\rangle \\ \text{(by Prop. 49)} \qquad &= n^{\frac{k}{2}-1} \sum d_\alpha \cdot \langle f|_k \sigma_n, g|_k \alpha\rangle \\ &= \langle f|_k \sigma_n, T_n g\rangle \\ &= \chi(n)\langle f, T_n g\rangle, \end{aligned}$$

where the last equality holds since $f \in M_k(\Gamma_0(N), \chi)$. □

Newforms on $S_k(\Gamma_0(N)) = S_k(N)$. (Atkin-Lehner, generalized to arbitrary character by Li, Miyake)

We examine the simplest case $N = 1$. Then $S_k(\Gamma)$ has a basis of eigenforms of all T_n. Suppose $f = \sum_{n=1}^{\infty} a(n)q^n$ is an eigenform. Then $a(1) \neq 0$, and we normalize such that $a(1) = 1$. Then $T_n f = \lambda_n f$ for all n, and $\lambda_n = a(n)$ for all n. This implies that each "package" of eigenvalues $\{\lambda_n\}$ determines exactly one normalized eigenform. This phenomenon is called "multiplicity 1".

Multiplicity 1 fails if $N > 1$; for example, look at $S_{12}(\Gamma_0(6))$. We have $\Delta(z)$, $\Delta(2z)$, $\Delta(3z)$, and $\Delta(6z) \in S_{12}(\Gamma_0(6))$. Also, V_d and T_n commute if $(n, d) = 1$. Then all four of these are eigenforms of T_p when $p \neq 2, 3$. However, this is not true at $p = 2$: recall $T_2 = U_2$, and $\Delta(z) = q - 24q^2 + \cdots$. Then $T_2\Delta(2z) = (U_2 \circ V_2)\Delta(z) = \Delta(z)$. Also, $\Delta(z)$ is not an eigenform of T_2, for $(U_2 + 2^{11}V_2)\Delta(z) = -24\Delta(z)$, where this operator is T_2 on $SL_2(\mathbb{Z})$. The so-called "newforms" save the day: it turns out that each package of eigenvalues $\{\lambda_n\}_{(n,N)=1}$ corresponds to a unique newform.

Definition (informal)**.** *New = not old.*

If $M|N$, then $S_k(M) \subseteq S_k(N)$. If $dM|N$, then $S_k(M)|V_d \subseteq S_k(N)$.

Definition. *S_k^{old} is the vector space spanned by $\{f|V_d : f \in S_k(M), M < N, dM|N\}$. (Think of everything that comes from a lower level.)*

Definition. *S_k^{new} is the orthogonal complement of S_k^{old} under the Petersson inner product, so $S_k(N) = S_k^{new}(N) \oplus S_k^{old}(N)$.*

Recall that T_n, V_d commute if $(n, d) = 1$. Thus $S_k^{\text{old}}(N)$ is preserved by T_n with $(n, N) = 1$. Use properties of the inner product to show $S_k^{\text{new}}(N)$ is also preserved by T_n with $(n, N) = 1$.

Theorem 52 (Key fact)**.** *Multiplicity one holds in $S_k^{new}(N)$; that is, if $f \in S_k^{new}(N)$ is an eigenform of all T_n with $(n, N) = 1$, then it is an eigenform of all the T_n. (Or, the common eigenspaces of the T_n with $(n, N) = 1$ are 1-dimensional.)*

We know $S_k^{\text{new}}(N)$ has a basis of eigenforms of the T_n with $(n, N) = 1$. Theorem 52 implies $S_k^{\text{new}}(N)$ has a basis of eigenforms of all the T_n.

Definition. *Such an eigenform (normalized) is called a newform.*

Look at S_k^{old}: we can write $S_k^{\text{old}} = V_1 \oplus \cdots \oplus V_t$, where V_i is a joint eigenspace for T_n with $(n, N) = 1$. Let $g_1, \ldots, g_s$ be a basis for V_1. Then all the g_i have the same package $\{\lambda_n\}_{(n,N)=1}$.

Theorem 53. *1) Given $\{\lambda_n\}_{(n,N)=1}$ on $S_k(N)$, there exists a unique divisor M of N and a unique newform $g \in S_k^{new}(M)$ such that g has $\{\lambda_n\}$ as its package of eigenvalues. Or, "Every package comes from a unique newform." Moreover, 2) If $f \in S_k(N)$ has this package, then f is in the space spanned by $\{g|V_d : dM|N\}$.*

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Theorem 54. $S_k(N) = S_k^{old}(N) \oplus S_k^{new}(N) = \bigoplus_{M|N} \bigoplus_{dM|N} S_k^{new}(M)|V_d$.

This works for $S_k(\Gamma_0(N), \chi)$, but can only go down as far as χ allows. If χ is primitive, multiplicity one holds in $S_k(\Gamma_0(N), \chi)$ (Hecke).

Properties of Integrality, Rationality. We know that $M_k(\Gamma)$ and $S_k(\Gamma)$ have bases with integral coefficients. We also have the following facts:

1. $M_k(\Gamma_0(N))$, $M_k(\Gamma_1(N))$, $S_k(\Gamma_0(N))$, and $S_k(\Gamma_1(N))$ have bases with integral coefficients. ($M_k(\Gamma_0(N), \chi)$ is spanned by forms with coefficients in $\mathbb{Z}[\chi]$.)
2. If $f \in S_k(\Gamma_0(N), \chi)$ is a normalized eigenform for all T_n, then there exists a fixed number field K_f such that $a(n) \in \mathcal{O}_f$ for all n. (Here $f = \sum a(n)q^n$ and $\mathcal{O}_f$ is the ring of integers of K_f.)
3. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $f^\sigma := \sum \sigma(a(n))q^n$ is a normalized eigenform in $S_k(\Gamma_0(N), \chi^\sigma)$. Note for $S_k(\Gamma_0(N))$, the newforms come in conjugacy classes under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Corollary. *Suppose $f = \sum a(n)q^n \in S_k(\Gamma_0(N), \chi) \cap \overline{\mathbb{Q}}[[q]]$. Then 1) there exists a number field K such that $f \in K[[q]]$, and 2) f has "bounded denominators": there exists $M \in \mathbb{Z}$ such that $Mf \in \overline{\mathbb{Z}}[[q]]$.*

Modular Forms mod l. ("On l-adic representations and congruences for coefficients of modular forms", Swinnerton-Dyer, Springer LN 350.)

We will write

$$P = E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$Q = E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

and

$$R = E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

Review: If $f \in M_k(\Gamma)$, then f has a unique expression as an isobaric polynomial of weight k in Q and R. (In other words, $M_k(\Gamma)$ has a basis $\{Q^a R^b\}_{4a+6b=k}$.) Recall the "handy basis" for $M_k(\Gamma)$:

$$\{\tilde{E}_k E_4^{3n}, \tilde{E}_k E_4^{3(n-1)}\Delta, \ldots, \tilde{E}_k \Delta^n\} = \{1 + \cdots, q + \cdots, q^2 + \cdots, \ldots, q^n + \cdots\},$$

all of which have coefficients in $\mathbb{Z}$. $\tilde{E}_k$ is a product of 0 or 1 R's and 0, 1, or 2 Q's. From this it is easy to prove:

Lemma 55. *If $f = \sum a(n)q^n$ is a modular form on $SL_2(\mathbb{Z})$, and A is the additive group generated by $\{a(n)\}$, then f has a unique expression as an isobaric polynomial in $A[Q, \Delta] \oplus RA[Q, \Delta]$.*

Recall that $\Delta = \frac{Q^3-R^2}{1728}$. We also have the fundamental operators $\theta = q\frac{d}{dq}$ and $\partial f = 12\theta f - kPf$, where ∂ depends on k.

Lemma 56. *If $f \in M_k(\Gamma)$, then $\partial f \in M_{k+2}(\Gamma)$. Also, $\Delta Q = -4R$, and $\Delta R = -6Q^2$.*

Set $M(\Gamma) = \bigoplus_{k=0}^{\infty} M_k(\Gamma)$. $M(\Gamma)$ is a graded $\mathbb{C}$-algebra; that is, it is a commutative ring, a $\mathbb{C}$-module (vector space), scalars commute with everything, and $M_k(\Gamma)M_j(\Gamma) \subseteq M_{k+j}(\Gamma)$. We call an element of $M_k(\Gamma)$ homogeneous of degree k. Since M_k has a basis $\{Q^aR^b : 4a+6b=k\}$, we see that $M(\Gamma) \cong \mathbb{C}[Q,R]$.

18. F. Stan. 11/5 and 11/7

Let us denote: $P = E_2, Q = E_4, R = E_6$. Then we have :

$$M(\Gamma) := \bigoplus_{k=0}^{\infty} M_k(\Gamma) \simeq \mathbb{C}[Q,R]$$

and $\Theta = q\frac{d}{dq}$, $\Theta(\sum a(n)q^n) = \sum na(n)q^n$.
Also $\partial f = 12\Theta f - kPf, \quad \partial : M_k(\Gamma) \longrightarrow M_{k+2}(\Gamma)$.

Lemma 57. *Let $f = \sum a(n)q^n$ be a modular form on Γ, A = subgroup generated by $(a(n))_n$. Then f is isobaric element of $A[Q,\Delta] \bigoplus RA[Q,\Delta]$.*

Note: ∂ is a derivation on $M(\Gamma)$, i.e. a $\mathbb{C}$-module morphism such that

$$\partial(fg) = f\partial g + \partial fg$$

(easy to check, remember : ∂ depends on weight)

MODULAR FORMS MOD l (l=prime, usually $l \geq 5$)

$\mathbb{Q}^{(l)}$ = local ring at l :=$\{\frac{a}{b} \in \mathbb{Q}$: in lowest terms, $l \nmid b\}$

If $\frac{a}{b} \in \mathbb{Q}^{(l)}$: $\widetilde{\frac{a}{b}} := \widetilde{ab^{-1}} := ab^{-1}$ (mod l)

$\mathbb{Q}^{(l)}/l\mathbb{Q}^{(l)} \simeq \mathbb{Z}/l\mathbb{Z} = \mathbb{F}_l$

Definition:

1. M_k:=$\mathbb{Q}^{(l)}$-module of those forms $f = \sum a(n)q^n \in M_k(\Gamma)$ such that $a(n) \in \mathbb{Q}^{(l)}$, for all n
2. $\tilde{M}_k$:=$\{\sum \widetilde{a(n)}q^n : f = \sum a(n)q^n \in M_k\} \subseteq \mathbb{F}_l$[[q]], a(n) $\in \mathbb{Q}^{(l)}$, $\widetilde{a(n)} \in \mathbb{F}_l$
$\tilde{M}_k$ = a $\mathbb{F}_l$ -vector space (easy to check: $\dim_{\mathbb{F}_l} \tilde{M}_k$=$\dim_{\mathbb{C}} M_k(\Gamma)$ (Use handy basis)

Notation: If $f = \sum a(n)q^n \in \mathbb{Q}^{(l)}[[q]]$, then $\tilde{f} = \sum \widetilde{a(n)}q^n \in \mathbb{F}_l[[q]]$
If $\Phi(X,Y) \in \mathbb{Q}^{(l)}[X,Y]$, then $\tilde{\Phi}(X,Y) \in \mathbb{F}_l[[X,Y]]$ (reduce each coefficient)
Natural arguments for Φ : Q,R. View them as independent transcendentals.
So $\tilde{\Phi}(Q,R)$ polynomial. $\tilde{\Phi}(\tilde{Q},\tilde{R}) \in \mathbb{F}_l[[q]]$; substitute $\tilde{Q}$,$\tilde{R}$ for Q,R

If $f \in M_k$, then exists a polynomial Φ such that $\Phi(Q,R) = f$
Suppose $l \geq 5$ and coefficients of f lie in $\mathbb{Q}^{(l)}$.
Then f can be written as a polynomial in Q, R, Δ with coefficients in $\mathbb{Q}^{(l)}$ (see Lemma 55)
Recall: $\Delta = \frac{Q^3-R^2}{1728}$. So coefficients of Φ lie in $\mathbb{Q}^{(l)}$ (recall: Φ=unique) and $\tilde{\Phi}(\tilde{Q},\tilde{R}) = \tilde{f}$

Define:

$$\tilde{M} := \bigoplus_{k=0}^{\infty} \tilde{M}_k$$

Problem: Determine structure of $\tilde{M}$ ($l \geq 5$)
We have a (surjective) ring homomorphism $\mathbb{F}_l[Q,R] \to \tilde{M}$, sending $Q \to \tilde{Q}, R \to \tilde{R}$.
Need to find the kernel.
Denote by B_k the k^{th} Bernoulli number.

Lemma 58. *(Von Staudt-Kummer congruences) (see, e.g. Ireland and Rosen,Chap. 15.2) Let l be a prime*

1. If $k \equiv 0 \pmod{l-1}$, then $lB_k \equiv -1 \pmod{l}$
2. If $k \not\equiv 0 \pmod{l-1}$, then $\frac{B_k}{k}$ is l-integral, and $\frac{B_k}{k} \pmod{l}$ depends only on $k \pmod{l-1}$

Corollary. If $l \geq 5$ prime then: 1) $E_{l-1} \equiv 1 \pmod{l}$; 2) $E_{l+1} \equiv P \pmod{l}$
E_2 is a modular form mod l , for all l

Proof. 1) $E_{l-1} = 1 - \frac{2(l-1)}{B_{l-1}} \sum \sigma_{l-2}(n) q^n \equiv 1 \bmod l$, by part 1
2) $E_{l+1} = 1 - \frac{2(l+1)}{B_{l+1}} \sum \sigma_l(n) q^n$
But $\sigma_l(n) \equiv \sigma(n) \bmod l$, because $d^l \equiv d \bmod l$, for all l, and $\frac{l+1}{B_{l+1}} \equiv \frac{2}{B_2} \equiv -12 \pmod{l}$
So, $E_{l+1} \equiv 1 + 24 \sum \sigma(n) q^n \equiv P \pmod{l}$ □

Notation: Let A,B be polynomials such that $A(Q,R) = E_{l-1}$, $B(Q,R) = E_{l+1}$, where $A, B \in \mathbb{Q}^{(l)}[Q,R]$
Note: ∂ induces a derivation on $\mathbb{F}_l[Q,R]$ and $\partial Q = -R$, $\partial R = -6Q^2$
Θ extends to $\mathbb{F}_l[[q]]$.

Theorem 59. *Let $l \geq 5$ be a prime. Then*
1) $\tilde{A}(\tilde{Q},\tilde{R}) = 1$, $\tilde{B}(\tilde{Q},\tilde{R}) = \tilde{P}$
2) $\partial \tilde{A}(Q,R) = \tilde{B}(Q,R)$, and $\partial \tilde{B}(Q,R) = -Q\tilde{A}(Q,R)$
3) $\tilde{A}(Q,R)$ has no repeated factors and is coprime to $\tilde{B}(Q,R)$
4) $\tilde{M} \simeq \mathbb{F}_l[Q,R]/(\tilde{A}-1)$ and $\tilde{M}$ has a natural grading with values in $\mathbb{Z}/(l-1)\mathbb{Z}$, i.e.

can write

$$\tilde{M} = \bigoplus_{v=0}^{l-1} \tilde{M}_v$$

where $\tilde{M}_v = \{\tilde{f} \in \tilde{M}_k : k \equiv v \pmod{l-1}\}$

Proof. 1) done
2) Note that $\Theta\tilde{A}(\tilde{Q},\tilde{R}) = 0$. $(\partial 1 = 0)$.
So, $\partial\tilde{A}(\tilde{Q},\tilde{R}) = -(l-1)\tilde{P} = \tilde{B}(\tilde{Q},\tilde{R})$, hence $\partial A(Q,R) - B(Q,R) = \sum a(n)q^n$, where $a(n) \in l\mathbb{Q}^{(l)}$, for all n
So, $\sum \frac{a(n)}{l}q^n \in \mathbb{Q}^{(l)}[[q]]$, hence $\sum \frac{a(n)}{l}q^n \in \mathbb{Q}^{(l)}[Q,R]$
3) Fact: If $f(X,Y) \in k[X,Y]$, (k field), homogeneous of degree d, then we have:

$$f(X,Y) = \gamma X^a Y^b \prod (X - \alpha_j Y)$$

Suppose $F(Q,R)$ isobaric of weight k. Write $k = \tilde{k} + 12n$, $\tilde{k}$ defined in the 1^{st} section.
Then $F(Q,R) = E_{\tilde{k}}G(Q,R)$, with G isobaric of weight $12m$
Typical monomial in G: $Q^aR^b = (Q^3)^A(R^2)^B$, $4a + 6b = 12m$, so $a = 3A$, $b = 2B$, and $A + B = m$.
By fact: $F(Q,R) = yQ^cR^d \prod (Q^3 - \tilde{C}_j R^2)$, where $\tilde{C}_j \in \overline{\mathbb{F}}_l - \{0\}$
Back to 3)
Suppose $\tilde{A}(Q,R)$ exactly divisible by $(Q^3 - \tilde{C}R^2)^n$, $n > 0$.
Let $G = Q^3 - \tilde{C}R^2$. Then: $\tilde{A} = G^nF$, $\quad \partial\tilde{A} = G^n\partial F + nG^{n-1}\partial GF \qquad (*)$ and

$$\partial G = 3Q^2\partial Q - 2\tilde{C}R\partial R = -12Q^2R + 12\tilde{C}Q^2R = 12(\tilde{C} - 1)Q^2R$$

Note: $\tilde{C} \neq 1$, since $\tilde{A}(\tilde{Q},\tilde{R}) = 1$ ($\tilde{E}_{l-1} = 1$ has a constant term, so it is not divisible by q).So $G, \partial G$=coprime
From (*) we get $\partial\tilde{A} = \tilde{B}$ is divisible exactly by G^{n-1}
Suppose $n - 1 > 0$. By repeating this argument, we get $\partial\tilde{B} = -Q\tilde{A}$ exactly divisible by G^{n-2}, contradiction. So $n = 1$, so 3).
Note: the same argument for Q^c, R^d
4) Consider map $\mathbb{F}_l[Q,R] \to M \subset \mathbb{F}_l[[q]]$, obtained by sending $Q \to \tilde{Q}$, $R \to \tilde{R}$
Let ker be a. Show $a = (\tilde{A} - 1)$
(*) a prime ideal (image an integral domain)
(**) a not maximal (image not a field: $\tilde{Q} - \tilde{R} = \widetilde{2^3 \cdot 3 \cdot 31}q + \widetilde{2^3 \cdot 3^4 \cdot 29}q^2 \neq 0$, not invertible.

Krull dimension of a commutative ring R: the longest chain $P_1 \not\subseteq P_2 \not\subseteq P_3 ...$ of prime ideals.
Fact: If k is a field, then $k[X,Y]$ has dimension 2.
Clear that $(\tilde{A}-1) \subset a$. Only need to show that $(\tilde{A}-1)$ is a prime, i.e. $\tilde{A}-1$ irreducible over $\mathbb{F}_l$
If not, let $\Phi(Q,R) \in \mathbb{F}_l[Q,R]$ be a non-isobaric proper irreducible factor.
Fact: If $F(Q,R)$ is isobaric of weight n, and $c \in \mathbb{F}_l - \{0\}$, then $F(c^4Q, c^6R) = c^nF(Q,R)$. So $\tilde{A}$ has weight $l-1$. So $c^{l-1} = 1$.

So $\Phi(c^4Q, c^6R)$ is also a factor of $\tilde{A}-1$
Write $\Phi(Q,R) = \Phi_n(Q,R) + \Phi_{n-1}(Q,R) + ... + \alpha$, $\alpha \neq 0$, where Φ_v isobaric of weight v.
So $\Phi(c^4Q, c^6R) = c^n\Phi_n(Q,R) + c^{n-1}\Phi_{n-1}(Q,R) + ... + \alpha$ is not a scalar multiple of $\Phi(Q,R)$. So it is coprime to $\Phi(Q,R)$. □

19. E. Owiesnay. 11/10 and 11/12

We are currently considering $A(Q,R) = E_{l-1}$ and wish to show $\tilde{A}(Q,R) - 1$ irreducible in $\mathbb{F}_l(Q,R)$.

If not, let $\phi(Q,R)$ be a proper irreducible factor. Then
$\phi(Q,R) = \phi_n(Q,R) + \phi_{n-1}(Q,R) + \ldots + \alpha$.

Also, ϕ_v is isobaric of weight v, $n < l-1$, $\alpha \neq 0$.
Note that $\alpha \neq 0$ since $\tilde{A}(Q,R) - 1$ has a nonzero constant term. Now note that if C in $\mathbb{F}_l$ is a primitive $(l-1)^{st}$ root of unity then $\phi(\tilde{C}^4Q, \tilde{C}^6R) | (\tilde{A}-1)$.

Note $\phi(\tilde{C}^4Q, \tilde{C}^6R) = \tilde{C}^n\phi_n(Q,R) + \ldots + \alpha$.

$\Rightarrow \phi(\tilde{C}^4Q, \tilde{C}^6R)$ coprime to $\phi(Q,R)$

$\Rightarrow \phi_n^2 | \tilde{A}$, contradiction.

Filtrations
$\tilde{M} = \bigoplus_k \tilde{M}_k$. Suppose $\tilde{f} \in \tilde{M}$ is a graded element, i.e., $\tilde{f}^k$ = sum of elements of various $\tilde{M}_k$, where the k's are congruent $\pmod{(l-1)}$.

After multiplying the summands by powers of $\tilde{E}_{l-1}$, assume $\tilde{f} \in \tilde{M}_k$ for some k.

For such an $\tilde{f}$, define filtration $w(\tilde{f}) := \inf(k : \tilde{f} \in \tilde{M}_k)$.

Example. *Let* $E_{l-1} \in M_{l-1}$, $\tilde{E}_{l-1} \in \tilde{M}_{l-1}$
Then $w(\tilde{E}_{l-1}) = 0$.

Remark. *If* $f \in \tilde{M}_k$, *then* $w(\tilde{f}) \equiv k \pmod{(l-1)}$.

Lemma 60. *Let* $l \geq 5$ *be prime. Suppose* $f \in M_k$, *has* $f = \phi(Q,R)$, $\phi \in \mathbb{Q}^{(l)}[Q,R]$, $\tilde{f} \neq 0$. *Then* $w(\tilde{f}) < k \Leftrightarrow \tilde{A} | \tilde{\phi}$.

Proof. ($\Rightarrow$) If $w(\tilde{f}) < k$, then $\tilde{f} \in \tilde{M}_{k-(l-1)}$. This implies $\exists F(Q,R) \in \mathbb{F}_l[Q,R]$ that is isobaric and of weight $k-(l-1)$ such that $F(\tilde{Q},\tilde{R}) = \tilde{f} = \tilde{\phi}(\tilde{Q},\tilde{R})$.
Theorem 58 $\Rightarrow \tilde{\phi} - F \in (\tilde{A}-1)$

$\Rightarrow \exists G \in F_l[Q,R]$ such that $\tilde{\phi} - F = G(\tilde{A}-1)$. Write $G = G_n + G_{n-1} + \ldots$ with G_j isobaric $\forall j$. Then $\tilde{\phi} = G_n\tilde{A}$. So $\tilde{A}|\tilde{\phi}$.

($\Leftarrow$) Left as exercise. □

Recall :
$12\theta = \partial + kP$ where $P = E_2$.
So if $\tilde{f} \in \tilde{M}_k$, then $12\theta\tilde{f} = \partial\tilde{f} + k\tilde{P}\tilde{f}$. So $12\theta\tilde{f} = \tilde{E}_{l-1}\partial\tilde{f} + k\tilde{E}_{l+1}\tilde{f}$.
Therefore since $\tilde{E}_{l-1}\partial\tilde{f}$ and $k\tilde{E}_{l+1}\tilde{f}$ are both of weight $k+l+1$, we have that $12\theta\tilde{f} \in \tilde{M}_{k+l+1}$.

Theorem 61. *Let $l \geq 5$ be prime. Suppose $\tilde{f} \in \tilde{M}_k$, $\tilde{f} \neq 0$. Then $\theta\tilde{f} \in \tilde{M}_{k+l+1}$. Further, we have $w(\theta\tilde{f}) \leq w(\tilde{f}) + l + 1$ with equality iff the filtration $w(\tilde{f}) \neq 0 \pmod{l}$.*

Remark. $w(\theta\tilde{f}) \equiv w(\tilde{f}) + l + 1 \pmod{(l-1)}$

Proof. Suppose $w(\tilde{f}) = k$. Let $f = \theta(Q, R)$ be an element of M_k which reduces to $\tilde{f}$.
Then $12\theta\tilde{f} = \partial\tilde{\phi}(\tilde{Q}, \tilde{R})\tilde{A}(\tilde{Q}, \tilde{R}) + k\tilde{B}(\tilde{Q}, \tilde{R})\tilde{\phi}(\tilde{Q}, \tilde{R})$.
From here Lemma 59 completes the proof
$w(\theta\tilde{f}) < k + l + 1 \Leftrightarrow \partial\tilde{\phi}(Q, R)A(Q, R) + k\tilde{B}(Q, R)\tilde{\phi}(Q, R)$ is divisible by $\tilde{A}(Q, R)$

$\Leftrightarrow \tilde{A} | k\tilde{B}\tilde{\phi}$

$\Leftrightarrow \tilde{A} | k\tilde{\phi}$ (note $\gcd(\tilde{A}, \tilde{B}) = 1$).

Then $w(\tilde{f}) = k \Rightarrow \tilde{A}$ does not divide $\tilde{Q}$. So $\tilde{A} | k\tilde{\phi}$ iff $k = 0 \pmod{l}$. ☐

Application to partitions: Recall: $p(n) =$ number of partitions of n, and

$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$.

Ramanujan's congruences:

$\forall n$,
$p(5n+4) \equiv 0 \pmod{5}$,
$p(7n+5) \equiv 0 \pmod{7}$, and
$p(11n+6) \equiv 0 \pmod{11}$.

These follow from letting $l \geq 5$ be prime and setting $\delta_l = \frac{l^2-1}{24}$
Then we have $\sum_{n=0}^{\infty} p(ln - \delta_l)q^n \equiv 0 \pmod{l}$ for $l = 5, 7, 11$.

Ramanujan also speculated that there were no**Theorem** equally simple properties for primes $l \neq 5, 7, 11$.

Formally:

Conjecture: If $p(ln + b) \equiv 0 \pmod{l}$ $\forall n$, then $b \equiv -\delta_l \pmod{l}$ and $l = 5, 7, 11$.

Kiming and Olsson proved that $b \equiv -\delta_l \pmod{l}$

Theorem. *(Ahlgren, Boylan). The conjecture is true.*

Proof. It is enough to prove :

$\sum_{n=0}^{\infty} p(l_n - \delta_l)q^n \equiv 0 \pmod{l} \Rightarrow l = 5, 7, 11.$
Set $f_l := \Delta^{\delta_l} = (q \prod_{n=1}^{\infty} (1-q^n)^{24})^{\delta_l}$.
Note that $\delta_l = \frac{l^2-1}{24}$
So,

$$f_l = q^{\delta_l} \frac{\prod_{n=1}^{\infty} (1-q^n)^{l^2}}{\prod_{n=1}^{\infty} (1-q^n)}$$

$$= \prod_{n=1}^{\infty} (1-q^n)^{l^2} \sum_{n=0}^{\infty} p(n)q^{n+\delta_l}$$

$$\equiv \prod_{n=1}^{\infty} (1-q^{ln})^{l} \sum_{n=0}^{\infty} p(n-\delta_l)q^{n} \pmod{l}.$$

So $f_l|U_l \equiv \prod_{n=1}^{\infty} (1-q^n)^{l} \sum_{n=0}^{\infty} p(ln-\delta_l)q^{n} \pmod{l}$.

Note $f_l|U_l \equiv 0 \pmod{l} \Leftrightarrow \sum_{n=0}^{\infty} p(ln-\delta_n)q^{n} \equiv 0 \pmod{l}$.

So it is enough to show $f_l|U_l \equiv 0 \pmod{l} \Rightarrow l = 5, 7, 11$.

Now consider the connections to the $\theta-$ operator.
Suppose $f = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]]$.

Then $(f|U_l)^l = \sum_{n=0}^{\infty} a(ln)q^{ln} \Rightarrow (f|U_l)^l \equiv \sum_{n=0}^{\infty} a(ln)q^{ln} \pmod{l}$.

Now $(x+y)^l \equiv x^l + y^l \pmod{l}$ and Fermat's Little Theorem gives that $(f|U_l)^l \equiv \sum_{l|n} a(n)q^n \pmod{l}$.

Then,
$\theta^{l-1} f = \sum_{n=0}^{\infty} n^{l-1} a(n)q^n \equiv \sum_{l\nmid n} a(n)q^n \pmod{l}$.

This implies $f - \theta^{l-1}f \equiv (f|U_l)^l \pmod{l}$.

Now it is enough to prove:

Proposition 62. *If* $f_l \equiv \theta^{l-1} f_l \pmod{l}$, *then* $l = 5, 7, 11$.

Remark. $\theta^l f = \sum\limits_{n=0}^{\infty} n^l a(n) q^n$
$\equiv \theta f \pmod{l}$

This gives the "Tate theta cycle":

$$\begin{tikzcd}
f_l, & \theta f_l, & \ldots, & \theta^{l-1} f_l, & \theta^l f_l \arrow[lll, bend right, "\equiv"']
\end{tikzcd}$$

Now we require a lemma.

Lemma 63. *If $m \in \mathbb{N}$ then $w(\theta^m f_l) \geq w(f_l) = \frac{l^2-1}{2}$.*

Proof. <u>Fact</u>: If $f \in M_k$ then $w(f^i) = iw(f)$. (Proof: Exercise)

This fact implies
$w(f_l) = w(\Delta^{\delta_l})$

$= \delta_l w(\Delta)$

$= \frac{l^2-1}{24} \cdot 12.$

For the inequality case, let $k = w(\theta^m f_l)$ and set $d = \dim(M_k)$.
Then recall the handy basis:
$\{1 + \ldots, q + \ldots, \ldots, q^{d-1} + \ldots\} =: \{g_1, g_2, \ldots, g_d\}$ where all the g_i have integer coefficients.

Remark. $\theta^m f_l = \theta^m(q_l^\delta + \ldots) = \delta_l^m q^{\delta_l} + \ldots.$

Since $\delta_l \not\equiv 0 \pmod{l}$ and $\theta^m f_l \equiv \sum\limits_{i=0}^{\infty} a_i g_i \pmod{l}$ with $a_i \in \mathbb{Z}$, we have:
$\delta_l \leq d - 1$.

Remark. $d \leq \frac{k}{12} + 1 \Rightarrow \frac{l^2-1}{24} \leq \frac{k}{12}$.

□

Now we can continue with the proof of Proposition 62.
Without loss of generality we may assume $l \geq 13$.
Suppose $f_l \equiv \theta^{l-1} f_l \pmod{l}$. Then $w(f_l) = w(\theta^{l-1} f_l)$.
Also suppose that $w(\theta^{l-2} f_l) \not\equiv 0 \pmod{l}$.
Then $w(\theta^{l-1} f_l) = w(\theta^{l-2}) + l + 1$. This implies $w(\theta^{l-2} f_l) < w(f_l)$, which contradicts Lemma 63.

So we have that $w(\theta^{l-2} f_l) \equiv 0 \pmod{l}$.

Now looking at the filtrations, we can represent them in this table:

	f_l	θf_l	$\theta^2 f_l$	$\cdots$	$\theta^{l-2} f_l$	$\theta^{l-1} f_l$
Filtration	$\frac{l^2-1}{2}$	$\frac{l^2-1}{2}+(l-1)$	$\frac{l^2-1}{2}+2(l-1)$	$\cdots$	$0 \pmod{l}$	$\frac{l^2-1}{2}$

Now we wish to find the least j such that $w(\theta^j f_l) \equiv 0 \pmod{l}$.
Then looking at $w(\theta^j f_l)$, we see

$$w(\theta^j f_l) = \frac{l^2-1}{2} + j(l+1)$$

$$\Rightarrow j \equiv \frac{1}{2} \pmod{l}$$

$$\Rightarrow j = \frac{l+1}{2}.$$

$$\Rightarrow w(\theta^{\frac{l+3}{2}} f_l) = \frac{l^2-1}{2} + (\frac{l+3}{2})(l+1) - \alpha(l-1) \text{ for some } \alpha > 0.$$

Then Lemma 63
$\Rightarrow (\frac{l+3}{2})(l+1) - \alpha(l-1) \geq 0$

$$\Rightarrow \alpha \leq \frac{l+5}{2} + \frac{4}{l-1}$$

$$\Rightarrow \alpha \leq \frac{l+5}{2} \text{ since } l > 5.$$

So $\theta^{\frac{l+3}{2}} f_l$ has filtration:
$\frac{l^2-1}{2} + (\frac{l+3}{2})(l+1) - \alpha(l-1)$ for $0 < \alpha \leq \frac{l+5}{2}$.
Now we wish to show that this is the only such drop.
Let j be least with $1 \leq j \leq \frac{l-5}{2}$ such that $w(\theta^{\frac{l+1}{2}+j} f_l) \equiv 0 \pmod{l}$. Note that $j = \frac{l-5}{2}$ works.

Then,
$w(\theta^{\frac{l+1}{2}+j} f_l) = \frac{l^2-1}{2} + (\frac{l+1}{2}+j)(l+1) - \alpha(l-1)$.

$$\equiv \frac{-1}{2} + \frac{1}{2} + j + \alpha \pmod{l}$$

$$\equiv (j+\alpha) \pmod{l}.$$

So $j + \alpha \equiv 0 \pmod{l}$. Then the size of j and α implies $\alpha = \frac{l+5}{2}$.
Therefore $w(\theta^{\frac{l+3}{2}} f_l) = \frac{l^2-1}{2} + 4$.

20. S. Chaiya. 11/14 and 11/17

Proof. Continued Poof of Theorem.
Suppose that $\omega(\theta^{\frac{\ell^2+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + 4$.

Remark. $\frac{\ell^2-1}{2} \equiv 0 \pmod{12}$.

Handy basis for $M_{\frac{\ell^2-1}{2}+4} : \{E_4 \cdot E_4^{\frac{\ell^2-1}{8}}, E_4 \cdot \triangle E_4^{\frac{\ell^2-1}{8}-3}, ..., E_4 \triangle^{\frac{\ell^2-1}{24}} = E_4 \triangle^{\delta_\ell} = E_4 f_\ell\}$.

Remark. $\theta^{\frac{\ell^2+3}{2}} f_\ell = \theta^{\frac{\ell^2+3}{2}}(q^{\delta_\ell} + ...) = (\delta_\ell)^{\frac{\ell^2+3}{2}} q^{\delta_\ell} + ...$

So $\quad \theta^{\frac{\ell^2+3}{2}} f_\ell \equiv (\delta_\ell)^{\frac{\ell^2+3}{2}} E_4 f_\ell \mod \ell.$
Since

$$\begin{aligned} f_\ell = \triangle^{\delta_\ell} = q^{\delta_\ell}(1-q)^{\ell^2-1} \cdots &= q^{\delta_\ell}(1 - (\ell^2-1)q + ...) \\ &\equiv q^{\delta_\ell} + q^{\delta_\ell+1} ... \mod \ell, \end{aligned}$$

so $\quad \theta^{\frac{\ell^2+3}{2}} f_\ell \equiv \delta_\ell^{\frac{\ell^2+3}{2}} q^{\delta_\ell} + (\delta_\ell+1)^{\frac{\ell^2+3}{2}} q^{\delta_\ell+1} + ... \mod \ell.$
Also,

$$\begin{aligned} (\delta_\ell)^{\frac{\ell^2+3}{2}} E_4 f_\ell &\equiv \delta_\ell^{\frac{\ell^2+3}{2}} (1 + 240q + ...)(q^{\delta_\ell} + q^{\delta_\ell+1} + ...) \\ &\equiv \delta_\ell^{\frac{\ell^2+3}{2}} q^{\delta_\ell} + \delta_\ell^{\frac{\ell^2+3}{2}} 241 q^{\delta_\ell+1} + ... \qquad \mod \ell \end{aligned}$$

By comparing coefficient of $q^{\delta_\ell+1}$, we have

$$\begin{aligned} (\delta_\ell+1)^{\frac{\ell^2+3}{2}} &\equiv 241 \delta_\ell^{\frac{\ell^2+3}{2}} && \mod \ell \\ (1+\frac{1}{\delta_\ell})^{\frac{\ell+3}{2}} &\equiv 241 && \mod \ell \\ (-23)^{\frac{\ell+3}{2}} &\equiv 241 && \mod \ell \\ (-23)^{\frac{\ell-1}{2}}(-23)^2 &\equiv 241 && \mod \ell \\ \pm 529 &\equiv 241 && \mod \ell \end{aligned}$$

Since $529 + 241 = 770 = 2 \cdot 5 \cdot 7 \cdot 11$ and $259 - 241 = 288 = 2^5 \cdot 3^2$, and $\ell \geq 5$, so ℓ can be only 5,7, or 11. We complete the proof. □

Modular forms of half-integral weight (Shimura 1973, Ann. paper).

Prototype: $\Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$. We know $\Theta^2 \in M_1(\Gamma_0(4), \chi_{-4})$, where $\chi_{-4}(d) = (\frac{-4}{d})$. One wants Θ to be in $M_{\frac{1}{2}}(\Gamma_0(4))$. Some reasons to look at half-integral weight case: Fourier coefficients interesting,

Example. $\Theta^k(z) = \sum r_k(n) q^n$ *, where* $r_k(n)$ *= the number of representations of* n *as sum of* k *squares of positive integers.*

Recall. $\quad j(\gamma, z) := \frac{\Theta(\gamma z)}{\Theta(z)}$, $\quad \gamma \in \Gamma_0(4)$, $\quad z \in \mathbb{H}$. Then

$$j(\gamma, z) = (\frac{c}{d}) \varepsilon_d^{-1} \sqrt{cz+d} \qquad ,$$

where $\gamma = \begin{pmatrix} a & c \\ c & d \end{pmatrix}$, $\sqrt{cz+d}$ has positive real part,

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4 \\ i & \text{if } d \equiv 3 \bmod 4 \end{cases}, \text{ and } (\tfrac{c}{d}) = \begin{cases} \text{Jacobi symbol} & \text{if } d > 0, \text{ odd} \\ (\frac{c}{|d|}) & \text{if } d < 0, c > 0 \\ -(\frac{c}{|d|}) & \text{if } d < 0, c < 0 \\ 1 & \text{if } c = 0 \text{ and } d = \pm 1. \end{cases}$$

Remark. $(\frac{-1}{d}) = (-1)^{\frac{d-1}{2}}$ *for all odd* d.

In general, call $J(\gamma, z)$ an *automorphy factor* for f if

$$f(\gamma z) = J(\gamma, z) f(z) \qquad \forall \gamma \in \Gamma' \qquad \text{(some group)}.$$

Remark.

$$\frac{f(\alpha\beta z)}{f(z)} = \frac{f(\alpha(\beta z))}{f(\beta z)} \cdot \frac{f(\beta z)}{f(z)} \quad ,$$

so

$$J(\alpha\beta, z) = J(\alpha, \beta z) \cdot J(\beta, z). \tag{A}$$

One can check that if $J(\alpha, z) = \sqrt{cz+d}$, then (A) fails for any congruence subgroup Γ'. We showed that if $\Gamma' \subset \Gamma_0(4)$, then (A) holds for $j(\alpha, z)$.

Transformation law.

f transforms in weight $k/2$ for $\Gamma' < \Gamma_0(4)$ if

$$f(\gamma z) = j(\gamma, z)^k \cdot f(z)\,, \qquad \forall \gamma \in \Gamma'\,.$$

Definition. $T := \{\pm 1, \pm i\}$ *and*

$G := \{(\alpha, \phi(z)) : \alpha \in GL_2^+(\mathbb{Q}), \phi$ *is a holomorphic function on* $\mathbb{H}$ *such that* $\phi^2(z) = \frac{t(cz+d)}{\sqrt{\det \alpha}}$ *for some* $t \in \{\pm 1\}\}$.

So, $\forall\, \alpha$ and $\forall\, t \in \{\pm 1\}$, one gets two elements $(\alpha, \pm\phi(z)) \in G$.

Proposition 64. *G is a group under the operation:*

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)) \qquad ,$$

with identity $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$.

Proof. Check closure, associativity, inverses:

Closure: $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\beta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $\alpha\beta = \begin{pmatrix} * & * \\ ce+dg & cf+dh \end{pmatrix}$.

$$\begin{aligned} (\phi(\beta z)\psi(z))^2 &= \frac{t_1(c\beta z + d)}{\sqrt{\det \alpha}} \cdot \frac{t_2(gz+h)}{\sqrt{\det \beta}} \\ &= \frac{t_1 t_2}{\sqrt{\det(\alpha\beta)}} (c(\frac{ez+f}{gz+h}) + d)(gz+h) \\ &= \frac{t_1 t_2}{\sqrt{\det(\alpha\beta)}} ((ce+dg)z + cf + dh). \end{aligned}$$

□

If $\gamma \in \Gamma_0(4)$ then $(\gamma,\, j(\gamma, z)) \in G$ where $j(\gamma, z) = (\frac{c}{d})\varepsilon_d^{-1}\sqrt{cz+d}$ (a canonical choice of $\phi(z)$).

Definition. *If* $\gamma \in \Gamma_0(4)$*, define* $\tilde{\gamma} \in G$ *by* $\tilde{\gamma} := (\gamma, j(\gamma, z))$.

If $\Gamma' < \Gamma_0(4)$, let $\tilde{\Gamma}' := \{\tilde{\gamma} : \gamma \in \Gamma'\} < G$.

Definition. *Projection map* $P : G \to GL_2^+(\mathbb{Q})$ *is defined by* $(\alpha, \phi(z)) \mapsto \alpha$.

Then $P : \tilde{\Gamma}' \to \Gamma'$ is an isomorphism, i.e., $\tilde{\Gamma}' \approx \Gamma'$.

Slash operator.
Suppose $k \in \mathbb{Z}$ and $\xi = (\alpha, \phi(z)) \in G$. Define

$$f\,|_{k/2}\,[\xi] := \phi(z)^{-k} f(\alpha z).$$

Transformation law. Say f transforms in weight k/2 w.r.t. $\Gamma' < \Gamma_0(4)$ if

$$f\,|_{k/2}\,[\tilde{\gamma}] = f \qquad\qquad \forall \tilde{\gamma} \in \tilde{\Gamma}'$$

i.e., $$f(\gamma z) = j(\gamma, z)^k f(z).$$

Remark. $(f\,|_{k/2}\,[\xi_1])\,|_{k/2}\,[\xi_2] = f\,|_{k/2}\,[\xi_1\xi_2]$.

Fourier expansion at cusps.
$G' := \{(\gamma, \phi(z)) \in G : \gamma \in SL_2(\mathbb{Z})\}$.
Abuse of notation : If $\xi = (\alpha, \phi(z))$, write $\xi z = \alpha z$.

Goal : for $\xi = (\gamma, \phi(z)) \in G'$, find Fourier expansion for $f\,|_{k/2}\,[\xi]$.
At ∞ : Suppose $f\,|_{k/2}\,[\tilde{\gamma}] = f \quad \forall \tilde{\gamma} \in \tilde{\Gamma}', \quad [\Gamma_0(4) : \Gamma'] < \infty$.
Set $\Gamma_\infty := \{\pm\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle\}$ the stabilizer of ∞ in Γ. $\Gamma_\infty \cap \Gamma'$ is non-trivial (since Γ_∞ is an infinite cyclic, $[\Gamma : \Gamma'] < \infty$). So $\Gamma_\infty \cap \Gamma'$ has one of the forms :
$\{\pm\langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rangle\}, \langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rangle, \langle -\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rangle \quad (h > 0)$. $z \mapsto z + h$ is the smallest translation in group.

Remark. $-\tilde{I} = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, (\frac{0}{-1})i^{-1}\sqrt{-1}\right) = (-I, 1)$. *So* $f\,|_{k/2}\,[-\tilde{I}] = f$.

Also $j\left(\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, z\right) = 1 \quad \forall h$. *So* $f\,|_{k/2}\,[\widetilde{\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}}] = f(z+h)$.

Since $\pm\begin{pmatrix} 1 & h \\ 1 & 1 \end{pmatrix} \in \Gamma'$, one gets $f(z+h) = f(z)$. So $f(z) = \sum a(n) q_h^n \quad , q_h = e^{2\pi i z/h}$, is a *Fourier expansion* at ∞.

Now suppose $s \in \mathbb{Q} \cup \{\infty\}$, $s = \alpha\infty$, $\alpha \in \Gamma$. Choose $\xi = (\alpha, \phi(z)) \in G$.
Set $G'_\infty := \{(\gamma, \phi(z)) : \gamma \in \Gamma_\infty\} = \left\{\left(\pm\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, t\right) : j \in \mathbb{Z}, t \in T = \{\pm 1, \pm i\}\right\}$.
Set $\tilde{\Gamma}'_s := \{\tilde{\gamma} \in \tilde{\Gamma}' : \gamma s = s\}$.

Remark.
- $\xi^{-1}\tilde{\Gamma}'_s\xi$ *fixes* ∞ *, i.e.,* $\xi^{-1}\tilde{\Gamma}'_s\xi \subseteq G'_s$.
- $\xi^{-1}\tilde{\Gamma}'_s\xi$ *is isomorphic to* $\alpha^{-1}\Gamma'_s\alpha \subseteq \Gamma_\infty$ *under the projection map.*

- *same argument as before, show that* $\alpha^{-1}\Gamma'_s\alpha\cap\Gamma_\infty$ *is non-trivial* ($[\Gamma:\Gamma']<\infty$). *So,* $\alpha^{-1}\Gamma'_s\alpha$ *has one of the forms:*$\{\pm\langle\begin{pmatrix}1&h\\0&1\end{pmatrix}\rangle\}$, $\langle\begin{pmatrix}1&h\\0&1\end{pmatrix}\rangle$, $\langle-\begin{pmatrix}1&h\\0&1\end{pmatrix}\rangle$, *h>0. Hence,* $\pm\xi^{-1}\tilde{\Gamma}'_s\xi$ *has the form* $\langle(\pm\begin{pmatrix}1&h\\0&1\end{pmatrix},t)\rangle$ *for some* $t\in T$*. Then cusp* s *associates a number* $h>0$ *and an element* $t\in\{\pm1.\pm i\}$*.*

Proposition 65. *h,t depend only on the* Γ' *equivalence class of s (and not on choice of* ξ*).*

Proof. See Koblitz. □

Expansion at cusps.

Suppose $f\ |_{k/2}\ [\tilde{\gamma}]=f$, $\forall\tilde{\gamma}\in\tilde{\Gamma}'$, and $s:=\xi\infty$. Set $g:=f\ |_{k/2}\ [\xi]$. Then g is invariant under $\pm\xi^{-1}\tilde{\Gamma}'_s\xi$.

$$\begin{aligned} g\ |_{k/2}\ [\xi^{-1}\tilde{\gamma}_s\xi] &= f\ |_{k/2}\ [\pm\xi\xi^{-1}\tilde{\gamma}_s\xi] \\ &= f\ |_{k/2}\ [\xi]=g \quad . \end{aligned}$$

21. J. ATKINSON. 11/19 AND 11/21

Review: Let $\Gamma'\prec\Gamma_0(4)$ of finite index, $s\in\mathbb{Q}\cup\{\infty\}$, $\Gamma'_s=\{\gamma\in\Gamma':\gamma s=s\}$, and $\widetilde{\Gamma}'_s=\{(\gamma,j(\gamma,z))\in G:\gamma\in\Gamma'_s\}$. Then $\Gamma'_s\approx\widetilde{\Gamma}'_s$ and we have shown that if $\xi=(\alpha,\phi(z))$ with $\alpha\in\Gamma$ and $\alpha\infty=s$, then $\pm\xi^{-1}\widetilde{\Gamma}'_s\xi$ has the form $\langle(\pm\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right),t)\rangle$ with $h>0$ and $t\in T$ depending only on the equivalence class of s.

Expansions at cusps: Suppose $f|_{k/2}[\widetilde{\gamma}]=f$ for all $\widetilde{\gamma}\in\widetilde{\Gamma}'$ and $s=\xi\infty$. Set $g=f|_{k/2}[\xi]$. Then g invariant under $\pm\xi^{-1}\widetilde{\Gamma}'_s\xi$. But $(\pm\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right),t)\in\pm\xi^{-1}\widetilde{\Gamma}'_s\xi$, so

$$g(z)=g(z)\big|_{k/2}[(\pm\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right),t)]=t^{-k}g(z+h)$$

Set $t^k=e^{2\pi ir}$ where r is 0, 1/4, 1/2, or 3/4. Then $e^{-\frac{2\pi irz}{h}}g(z)$ is invariant under $z\to z+h$ and has expansion $\sum a(n)q_h^n$. So $g(z)$ has expansion $g(z)=\sum a(n)q_h^{n+r}$. We say f is holomorphic at s if $a(n)=0$ for all negative n.

Note: If $r\neq0$, then a form that is holomorphic at s must vanish at s.

It may be verified that the order of vanishing depends only on the equivalence class of s, and not on ξ. However, the expansion does depend on ξ.

Definition. *The cusp s is k-regular if* $r=0$*, and k-irregular if* $r\neq0$

Example. *For* $\Gamma_0(4)$*,*

∞	$h=1,\ t=1$
0	$h=4,\ t=1$
$1/2$	$h=1,\ t=i$

So ∞ *and* 0 *are k-regular for all k, and* $1/2$ *is k-regular iff* $\frac{k}{2}\in2\mathbb{Z}$*.*

Definition. *Suppose* $\Gamma' \prec \Gamma_0(4)$ *of finite index. We write* $f \in M_{k/2}\left(\widetilde{\Gamma'}\right)$ *if* f *holomorphic on* $\mathbb{H}$ *and at the cusps and* $f|_{k/2}[\widetilde{\gamma}] = f$ *for all* $\widetilde{\gamma} \in \widetilde{\Gamma'}$*. We write* $f \in S_{k/2}\left(\widetilde{\Gamma'}\right)$ *if, in addition, it vanishes at the cusps.*

Suppose $4|N$ and χ a character modulo N. Define $M_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$ to be those $f \in M_{k/2}\left(\widetilde{\Gamma_1(N)}\right)$ such that

$$f\big|_{k/2}\left[\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\right] = \chi(d) f \qquad\qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$$

As before,

$$M_{k/2}\left(\widetilde{\Gamma_1(N)}\right) = \bigoplus_{\chi} M_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right),$$

and the same holds for $S_{k/2}$.

Example. *Note that* $\theta(z) \in M_{1/2}\left(\widetilde{\Gamma_0(4)}\right)$*, so* $\theta^2(z) \in M_1\left(\widetilde{\Gamma_0(4)}\right)$*. Also, we have seen that* $\theta^2(z) \in M_1\left(\Gamma_0(4), \chi_{-4}\right)$*.*

Suppose $\frac{k}{2} \in \mathbb{Z}$ and $\gamma \in \Gamma_0(4)$. We compute

$$\begin{aligned} f(z)\big|_{k/2}[\widetilde{\gamma}] &= j(\gamma, z)^{-k} f(z) \\ &= \left[\left(\frac{c}{d}\right) \epsilon_d^{-1} \sqrt{cz+d}\right]^{-k} f(z) \\ &= \epsilon_d^k (cz+d)^{-k/2} f(z) \\ &= \left(\frac{-1}{d}\right)^{k/2} (cz+d)^{-k/2} f(z) \\ &= \chi_{-4}(d)^{k/2} f\big|_{k/2}\gamma. \end{aligned}$$

This gives the following.

Proposition 66. *If* $4|N$ *and* $\frac{k}{2} \in \mathbb{Z}$ *then* $M_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right) = M_{k/2}\left(\Gamma_0(N), \chi\chi_{-4}^{k/2}\right)$*.*

Structure of $M_{k/2}\left(\widetilde{\Gamma_0(4)}\right)$**:** Let $\sigma_1(n) = \sum_{d|n} d$ and set

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1} \\ &= -\frac{1}{24}\left(E_2(z) - 3E_2(2z) + 2E_2(4z)\right) \\ &= \frac{\eta^8(4z)}{\eta^8(2z)} \\ &\in M_2\left(\Gamma_0(4)\right) = M_2\left(\widetilde{\Gamma_0(4)}\right). \end{aligned}$$

Theorem 67. *The dimension of* $M_{k/2}\left(\widetilde{\Gamma_0(4)}\right)$ *is given by* $d = 1 + \lfloor \frac{k}{4} \rfloor$ *and*

$$\{\theta^k, F\theta^{k-4}, F^2\theta^{k-8}, \ldots, F^{d-1}\theta^{k-4d+4}\}$$

is a basis.

Note that this basis has the handy form $1+\dots, q+\dots, q^2+\dots, \dots$.

To prove Theorem 67, we need the following fact.

Remark. *If $f \in M_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$ vanishes to order $> \frac{k/2}{12}[\Gamma:\Gamma_0(N)]$ at ∞, then f is identically zero.*

Proof. Squaring f gives an element of $M_k(\Gamma_0(N), \chi\chi_{-4}^k)$ that vanishes to order $> \frac{k}{12}[\Gamma:\Gamma_0(N)]$. So $f^2 \equiv 0$ and the result follows. $\square$

Proof of Theorem 67. Since $\theta^k, F\theta^{k-4}, F^2\theta^{k-8}, \dots, F^{d-1}\theta^{k-4d+4}$ are linearly independent, given $f \in M_{k/2}\left(\widetilde{\Gamma_0(4)}\right)$, there is a linear combination so that $f-$ (linear comb.) vanishes to order at least d. But $\frac{k/2}{12}[\Gamma:\Gamma_0(4)] = \frac{k}{4} < 1 + \lfloor\frac{k}{4}\rfloor = d$, so by the remark it follows that f actually equals this linear combination. $\square$

Theta Series (see Shimura, Annals 73; Serre-Stark, SLN 627)

Suppose ψ is an even primitive character of conductor $r = r(\psi)$ (recall that ψ is even if $\psi(-1)=1$). Set $\theta_\psi(z) := \sum_{n=-\infty}^{\infty} \psi(n) q^{n^2}$.

Notes:

- $\theta_1(z) = \theta(z)$
- If $\psi \neq 1$, then $\theta_\psi(z) = 2\sum_{n=1}^{\infty}\psi(n)q^{n^2}$.
- $\theta_\psi(z) \neq \theta(z) \otimes \psi = \sum \psi(n^2) q^{n^2}$

Fact (Shimura): $\theta_\psi(z) \in M_{1/2}\left(\widetilde{\Gamma_0(4r^2)}, \psi\right)$.

Define $\chi_t :=$ Kronecker character for $\mathbb{Q}\left(\sqrt{t}\right)$, i.e., if $D = \operatorname{disc}\left(\mathbb{Q}\left(\sqrt{t}\right)\right)$, then $\chi_t(n) = \left(\frac{D}{n}\right)$ with conductor $|D|$.

Example. $t=-1$,$D=-4$, $\chi_{-1}(n) = \left(\frac{-4}{n}\right)$ *with conductor* 4. $\chi_{-4} = \chi_{-1}$.

Fact: The operator V_t (given by $f(z) \mapsto f(tz)$) takes $M_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$ to $M_{k/2}\left(\widetilde{\Gamma_0(Nt)}, \chi\chi_t^k\right)$. Note that $\chi\chi_t^k = \chi$ for k an even integer.

Definition.

$$\theta_{\psi,t}(z) := \theta_\psi(z)\big|V_t = \sum_{n=-\infty}^{\infty} \psi(n) q^{tn^2} \in M_{1/2}\left(\widetilde{\Gamma_0(4r^2t)}, \psi\chi_t\right)$$

Recall: $f\big|_{k/2}\left[-\widetilde{I}\right] = f$ for all f, so $M_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right) = \{0\}$ unless χ is even.

Notation: Suppose $4|N$ and χ an even character modulo N. Define

$$\Omega(N,\chi) := \{(\psi,t) : t \geqslant 1, \psi \text{ even and primitive, } 4r(\psi)^2 t | N,$$
$$\chi(n) = \psi(n)\chi_t(n) \text{ if } (n,N)=1\}.$$

In other words, $\Omega(N,\chi)$ is the set of all (ψ,t) such that $\theta_{\psi,t} \in M_{1/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$.

Theorem 68. *(Serre-Stark) The theta series $\theta_{\psi,t}$ with $(\psi,t) \in \Omega(N,\chi)$ form a basis for $M_{1/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$.*

Set $\Omega(N) = \cup_{\chi \pmod N}\Omega(N,\chi)$.

Corollary 69. *The theta series* $\theta_{\psi,t}$ *with* $(\psi,t) \in \Omega(N)$ *form a basis for* $M_{1/2}\left(\widetilde{\Gamma_1(N)}\right)$.

Corollary 70. *If* $f(z) = \sum a(n)q^n \in M_{1/2}\left(\widetilde{\Gamma_1(4)}\right)$ *then* $a(n) = 0$ *unless* $n = tm^2$ *for some* $t|\frac{N}{4}$ *and some* m.

Example. *Let* $N = 4P_1P_2\dots P_s$ *where* $P_1, P_2, \dots, P_s$ *distinct primes. Then* $(\psi,t) \in \Omega(N)$ *implies that* $r(\psi) = 1$*, and so* $\psi = 1$*. Also, we must have* $t|P_1P_2\dots P_s$*. So a basis for* $M_{1/2}\left(\widetilde{\Gamma_1(N)}\right)$ *is given by* $\{\theta(tz) : t|P_1P_2\dots P_s\}$*. Note that* $\theta(tz) \in M_{1/2}\left(\widetilde{\Gamma_0(N)}, \chi_t\right)$ *and the* χ_t *are distinct, so each* $M_{1/2}\left(\widetilde{\Gamma_0(N)}, \chi_t\right)$ *is one-dimensional.*

Cusp forms

Definition. *If* ψ *has conductor* r *and* $r = \prod_{p|r} p^{\alpha_p}$*, then we can write* $\psi = \prod_{p|r}\psi_{p^{\alpha_p}}$ *where* $\psi_{p^{\alpha_p}}$ *is primitive modulo* p^{α_p}*. We call* ψ *totally even if all the* $\psi_{p^{\alpha_p}}$ *are even.*

Set $\Omega_C(N,\chi) := \{(\psi,t) \in \Omega(N,\chi) : \psi \text{ not totally even}\}$.

Theorem 71. *The theta series* $\theta_{\psi,t}$ *with* $(\psi,t) \in \Omega_C(N,\chi)$ *form a basis for* $S_{1/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$.

The first cusp form: The first even, non totally even ψ is $\psi = \chi_3$ with conductor 12 ($\chi_3(n) = \left(\frac{12}{n}\right) = \left(\frac{-3}{n}\right)\left(\frac{-4}{n}\right)$). So the first N with a nonzero cusp form is $N = 4(12)^2 = 576$. This is the only such N that is smaller than 900.

$$\theta_{\chi_3}(z) = 2\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)q^{n^2} = 2(q - q^{25} - q^{49} + q^{121} + q^{169} - \dots) \in S_{1/2}\left(\widetilde{\Gamma_0(576)}, \chi_3\right)$$

Euler showed that $\frac{1}{2}\theta_{\chi_3}(z) = \eta(24z)$.

Weight $3/2$**:** Suppose ψ is a primitive odd character with conductor r. Set $h_\psi(z) := \sum_{n=-\infty}^{\infty}\psi(n)nq^{n^2}$. Then $h_\psi(z) \in S_{3/2}\left(\widetilde{\Gamma_0(4r^2)}, \psi\chi_{-4}\right)$ (due to Shimura).

Example.

$$h_{\chi_{-4}}(z) = \sum_{n=-\infty}^{\infty}\left(\frac{-4}{n}\right)nq^{n^2} = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(2k+1)^2}$$

It turns out that $h_{\chi_{-4}}(z) = \eta^3(8z) \in S_{3/2}\left(\widetilde{\Gamma_0(64)}\right)$.

22. B. WALKER. 12/1 AND 12/3

Eisenstein Series of Half-Integral Weight

Recall: In the integral weight case with $k > 2$ even we have

$$E_k(z) = \frac{1}{2\zeta(k)}\sum_{m,n}\frac{1}{(mz+n)^k} = \sum_{(m,n)=1}(mz+n)^{-k}.$$

In the last sum, only one of (m,n) and $(-m,-n)$ is taken into consideration. The last equality is an exercise in Koblitz.

Set $J(\gamma,z) = mz+n$ where $\gamma = \begin{pmatrix} a & b \\ m & n \end{pmatrix}$. Its easy to check that γ_1 and γ_2 have the same bottom row up to sign if and only if $\gamma_1 = \pm \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \gamma_2$ i.e. γ_1, γ_2 are in the same coset of $\Gamma_\infty \backslash \Gamma$. Thus we obtain

$$E_k(z) = \sum_{\Gamma_\infty \backslash \Gamma} J(\gamma,z)^{-k}$$

We copy this in the half-integral weight case for $\Gamma_0(4)$. A quick check shows that $\Gamma_0(4)_\infty = \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \right\}$. Now we need to modify the automorphy factor, we set $j(\gamma,z) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (cz+d)^{1/2}$ where c, d, and ϵ_d^{-1} are as usual.

Definition. *For $k > 5$ and odd we define*

$$E_{k/2} = \sum_{\Gamma_\infty \backslash \Gamma_0(4)} j(\gamma,z)^{-k}.$$

Proposition 72. $E_{k/2}(z) \in M_{k/2}(\widetilde{\Gamma_0(4)})$ *Note that the series converges absolutely because $k/2 > 2$.*

Proof. If $\gamma_1 \in \Gamma_0(4)$ then we get

$$E_{k/2} \mid_{k/2} [\widetilde{\gamma_1}] = E_{k/2}(\gamma_1 z) j(\gamma_1,z)^{-k} = \sum_{\Gamma_\infty \backslash \Gamma_0(4)} j(\gamma, \gamma_1 z)^{-k} j(\gamma_1,z)^{-k}.$$

Since $j(\gamma,z)$ is an automophy factor we know this last sum is equal to

$$\sum_{\Gamma_\infty \backslash \Gamma_0(4)} j(\gamma\gamma_1,z)^{-k}.$$

This last sum is just $E_{k/2}$ since the sum ranges over the same set. The remaining cusp condition is an exercise in Koblitz. ☐

Coset representatives of $\Gamma_\infty \backslash \Gamma_0(4)$

A set of coset representatives for $\Gamma_\infty \backslash \Gamma_0(4)$ is $\begin{pmatrix} a & b \\ m & n \end{pmatrix}$ where $4 \mid m$, $(m,n) = 1$ and n is odd and strictly greater than zero. Loosely, cusps come in two types, k-regular and k-irregular. If f is holomorphic at an irregular cusp, then it vanishes. Given a regular cusp s, one can build an Eisenstein series that vanishes at every other cusp except s. Thus we have

$$\text{dim(modular forms)} = \text{dim(cusp forms)} + \text{number of regular cusps.}$$

The last part is spanned by the Eisenstein series. On $\Gamma_0(4)$, there are two k-regular cusps, 0 and ∞ when k is odd. The above E corresponds to the cusp at ∞.

Fricke Involution

Recall $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Now define $\rho_N = \left(\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{1/4} z^{1/2} \right) \in G$. Then $|_{k/2}\, \rho_N$

maps $M_{k/2}(\widetilde{\Gamma_0(N)}, \chi)$ to $M_{k/2}(\widetilde{\Gamma_0(N)}, \bar{\chi}\chi_N^k)$ where χ_N is the quadratic character for $\mathbb{Q}(\sqrt{N})$. If $N = 4$, then ρ_4 is an automorphism.

Definition. *Define* $F_{k/2} = E_{k/2} \mid_{k/2} [\rho_4]$.

With some work, one can show that

$$F_{k/2} = 2^{-k/2} \sum_{(m,n)=1} \left(\frac{-m}{n}\right) \epsilon_n^k (nz+m)^{-k/2}$$

with n odd and positive.

What do the q-expansions of these things look like and is there a connection with L-series? For a Dirichlet character, recall that $L(s,\chi) = \sum_n \frac{\chi(s)}{n^s}$ has an analytic continuation to the complex plane if χ is non-trivial. If $D = \mathrm{disc}(\mathbb{Q}(\sqrt{t}))$ then D is a fundamental discriminant. Let $\chi_t = \chi_D = (\frac{D}{\bullet})$, the Kronecker character.

Proposition 73. *Suppose* $k/2 = \lambda + 1/2$*,* $\lambda > 1$*, and* $\lambda \in \mathbb{Z}$*. Then* $E_{k/2} = \sum a(n)q^n$ *and* $F_{k/2} = \sum b(n)q^n$ *where if* D *is a fundamental discriminant such that* $D = (-1)^\lambda l$ *or* $D = 4(-1)^\lambda l$ *then*

$$a(|D|) = L(1-\lambda, \chi_D) \times \textit{Factor depending on } \lambda;\ l \textit{ mod } 8,$$

$$b(|D|) = L(1-\lambda, \chi_D) \times \textit{Different such factor.}$$

Cohen found linear combinations whose coefficient are the L-values.

Definition (Cohen-Eisenstein Series)**.**

$$H_{k/2} = \zeta(1-2\lambda)[E_{k/2} + (1+i^k)2^{-k/2}F_{k/2}]$$

Proposition 74. *If* $H_{k/2} = \sum c(n)q^n$ *and* D *is as in Proposition 73, then*

$$c(|D|) = L(1-\lambda, \chi_D).$$

Example. *Let*

$$F = \sum_{n>0,\ odd} \sigma_1(n)q^n.$$

Note that F *has weight 2. It turns out that* $H_{5/2} = \frac{1}{120}\Theta^5 - \frac{1}{6}\Theta F$ *and in this case* $\lambda = 2$*.*

Corollary. *If* $D > 0$ *is any fundamental discriminant, then*

$$L(-1, \chi_D) = \frac{1}{120} r_5(D) - \frac{1}{6} \sum_{|j|<\sqrt{|D|},\ D-j^2 \ odd} \sigma_1(D-j^2).$$

Another fact is that $E_{k/2}$, $F_{k/2}$ and $H_{k/2}$ all have Euler product representations.

Proposition 75. *Suppose* $(-1)^\lambda l_0$ *is a fundamental discriminant. Then*

$$\sum_{n=1}^{\infty} \frac{c(l_0 n^2)}{n^s} = L(1-\lambda, \chi_{(-1)^\lambda l_0}) \prod_p \frac{1 - \chi_{(-1)^\lambda l_0}(p) p^{\lambda-1-s}}{(1-p^{-s})(1-p^{2\lambda - s})}.$$

An important observation here is that the denominator of the Euler factor does not depend on l_0. Also $c(l) = 0$ if l is not of the form $l_0 n^2$ for some fundamental discriminant $(-1)^\lambda l_0$. In particular, $c(l) = 0$ if $(-1)^\lambda l \equiv 2, 3 \pmod 4$. A final note:

$$\prod_p \frac{1}{(1-p^{-s})(1-p^{2\lambda-1-s})} = \sum_{n=1}^{\infty} \sigma_{2\lambda-1}(n) n^{-s}.$$

Notice here that the right hand side looks like $\frac{E_{2\lambda}}{2\zeta(2\lambda)}$. The Euler product is equivalent to $E_{2\lambda}$ being an eigenform of all the T_p with eigenvalue $\sigma_{2\lambda-1}(p)$. Turns out $H_{k/2}$ is an eigenform of the T_{p^2} with the same eigenvalues.

Half-Integral Weight Hecke Operators

Reference: Shimura 1973. Goal: Define $T_n : M_{k/2}(\widetilde{\Gamma_1(N)}) \to M_{k/2}(\widetilde{\Gamma_1(N)})$. We start by setting

$$\zeta_n = \left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, n^{1/4} \right) \in G$$

Next define T_n on $M_{k/2}(\widetilde{\Gamma_1(N)})$ by

$$T_n f = n^{k/4-1} f \mid_{k/2} [\widetilde{\Gamma_1(N)} \zeta_n \widetilde{\Gamma_1(N)}]$$

But what in the world does this mean? If $\widetilde{\Gamma_1(N)} \zeta_n \widetilde{\Gamma_1(N)} = \bigcup_j \widetilde{\Gamma_1(N)} \zeta_n \tilde{\gamma}_j$ then

$$T_n f = n^{k/4-1} \sum_j f \mid_{k/2} [\zeta_n \tilde{\gamma}_j]$$

Alternatively set $\widetilde{\Gamma''} = \zeta_n^{-1} \widetilde{\Gamma_1(N)} \zeta_n \cap \widetilde{\Gamma_1(N)}$. We can then choose $\tilde{\gamma}_j$ as coset reps of $\widetilde{\Gamma_1(N)} / \widetilde{\Gamma''}$. One fact about these operators is that when $(n, N) = 1$ and n is not a perfect square, then T_n is the 0 operator. Why? Set $\alpha_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ and then set $\Gamma' = \alpha_n^{-1} \Gamma_1(N) \alpha_n \cap \Gamma_1(N)$. Next we would like to compare $\widetilde{\Gamma'}$ to $\widetilde{\Gamma''}$.

Fact. $\widetilde{\Gamma''} < \widetilde{\Gamma'}$ is the kernel of the map

$$t : \widetilde{\Gamma'} \to \pm 1$$

given by

$$t(\tilde{\gamma}) = \left(\frac{d}{n} \right) \text{ when } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If n is not a square then $\widetilde{\Gamma''}$ has index 2 in $\widetilde{\Gamma'}$. Thus each coset appears twice, one with plus and the other with minus. This then implies the whole sum is zero.

Facts.

(1) T_{n^2} commute
(2) $T_{n^2 m^2} = T_{n^2} T_{m^2}$ if $(n, m) = 1$
(3) They preserve $M_{k/2}(\widetilde{\Gamma_0(N)}, \chi)$
(4) Can always find a basis of eigenforms of all T_{p^2} with $p \nmid n$
(5) $T_{p^2} f = U_{p^2} f$ if p divides n
(6) T_{p^2} generate all T_{n^2}

Theorem 76. *Suppose* $4|N$, χ *a character mod* N, *and* $f = \sum a(n)q^n \in M_{k/2}(\widetilde{\Gamma_1(N)}, \chi)$. *Then if* $k/2 = \lambda + 1/2$ *we have*

$$T_{p^2}f = \sum \left[a(p^2n) + \chi(p)\left(\frac{(-1)^\lambda n}{p}\right)p^{\lambda-1}a(n) + \chi^2(p)p^{2\lambda-1}a\left(\frac{n}{p^2}\right)\right]q^n.$$

23. R. Astudillo. 12/5 and 12/8

NOTES FROM 12/5/03 AND 12/8/03

Recall Theorem 76: Suppose that $f = \sum a(n)q^n \in M_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)}, \chi)$. Then

$$T_{p^2}f = \sum (a(p^2n) + \chi(p)\chi_{(-1)^\lambda n}(p)p^{\lambda-1}a(n) + \chi^2(p)p^{2\lambda-1}a(n/p^2))q^n.$$

Example. *The function* $\eta(24z) = \sum \chi_{12}(n)q^{n^2} \in S_{\frac{1}{2}}(\widetilde{\Gamma_0(576)}, \chi_{12})$, *which is a 1-dimensional space. Hence for all primes* p *there exists* λ_p *such that* $T_{p^2}\eta(24z) = \lambda_p\eta(24z)$. *In this case we can compute* λ_p, *but in general it is not possible.*

It is clear that $\lambda_2 = \lambda_3 = 0$ *(since* $T_4 = U_4$ *and* $T_9 = U_9$*). For* $p \geq 5$, *write* $\sum \chi_{12}(n)q^{n^2} = \sum a(n)q^n$. *Then*

$$\begin{aligned} T_{p^2}\sum a(n)q^n &= \sum (a(p^2n) + \chi_{12}(p)\left(\frac{n}{p}\right)p^{-1}a(n) + \chi_{12}^2(p)p^{-1}a(n/p^2))q^n \\ &= \sum (a(p^2n^2) + \chi_{12}(p)\left(\frac{n^2}{p}\right)p^{-1}a(n^2) + \chi_{12}^2(p)p^{-1}a(n^2/p^2))q^n, \end{aligned}$$

since $a(n) = 0$ *when* n *is not a square. The coefficient of the* q^{n^2} *when* $p \mid n$ *is thus*

$$\chi_{12}(pn) + p^{-1}\chi_{12}\left(\frac{n}{p}\right) = (\chi_{12}(p) + p^{-1}\chi_{12}(p))\chi_{12}(n),$$

and so $\lambda_p = \chi_{12}(p) + p^{-1}\chi_{12}(p)$.

We now establish the following Euler Product:

Theorem 77. *Suppose that* $t \in \mathbb{N}$, p *is a prime, and that* $f = \sum a(n)q^n \in M_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ *has* $T_{p^2}f = \lambda_p f$. *Suppose also that* $p^2 \mid t$ *implies that* $p \mid N$. *Then*

$$\sum a(tn^2)n^{-s} = \sum_{(n,p)=1} a(tn^2)n^{-s}\left(\frac{1 - \chi(p)\chi_{(-1)^\lambda t}(p)p^{\lambda-1-s}}{1 - \lambda_p p^{-s} + \chi^2(p)p^{2\lambda-1-2s}}\right).$$

Corollary 78. *If* f *is an eigenform of all* T_{p^2} *and* t *is not divisible by any square co-prime to* N, *then*

$$\sum a(tn^2)n^{-s} = a(t)\prod_p \frac{1 - \chi(p)\chi_{(-1)^\lambda t}(p)p^{\lambda-1-s}}{1 - \lambda_p p^{-s} + \chi^2(p)p^{2\lambda-1-2s}}.$$

Proof of Theorem 77. Let x be an indeterminant. Define $H_n(x) := \sum\limits_{m=0}^{\infty} a(tp^{2m}n^2)x^m$. Then

$$\begin{aligned}\sum_{(n,p)=1} H_n(p^{-s})n^{-s} &= \sum_{(n,p)=1}\sum_{m=0}^{\infty} a(tp^{2m}n^2)p^{-ms}n^{-s}\\ &= \sum_{n=1}^{\infty} a(tn^2)n^{-s}.\end{aligned}$$

Claim: $H_n(x) = a(tn^2)\Lambda$, where Λ is the given Euler factor at p with p^{-s} replaced by x. By the last displayed equation, it suffices to prove the claim.

Suppose that $T_{p^2} = \lambda_p f$. Theorem 76 says that if $p \nmid n$ then:

$$a(tp^2n^2) + \chi(p)\chi_{(-1)^\lambda t}(p)p^{\lambda-1}a(tn^2) = \lambda_p a(tn^2) \tag{11}$$

and

$$a(tp^{2m+2}n^2) + \chi^2(p)p^{2\lambda-1}a(tp^{2m-2}n^2) = \lambda_p a(tp^{2m}n^2). \tag{12}$$

Let E_L and E_R denote the left and right-hand sides of (11), respectively. Similarly, let $E_{m,L}$ and $E_{m,R}$ denote the left and right-hand sides of (12). The claim follows after considering the equation

$$xE_L + \sum_{m=1}^{\infty} x^{m+1}E_{m,L} = xE_R + \sum_{m=1}^{\infty} x^{m+1}E_{m,R}$$

and collecting like terms. □

SHIMURA CORRESPONDENCE

Recall the following three facts:

(i) $F(z) = \sum\limits_{n=1}^{\infty} a(n)q^n \in S_{2\lambda}(\Gamma_0(N), \chi)$ being an eigenform of all T_p with eigenvalue λ_p is equivalent to

$$\sum a(n)n^{-s} = \prod_p \frac{1}{1-\lambda_p p^{-s} + \chi(p)p^{2\lambda-1-2s}}.$$

(ii) $L(s,\chi) = \prod\limits_p \frac{1}{1-\chi(p)p^{-s}}$.

(iii) $f(z) = \sum\limits_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)}, \chi)$ is an eigenform for all T_{p^2} with eigenvalue λ_p if and only if for all squarefree t we have

$$\begin{aligned}\sum a(tn^2)n^{-s} &= a(t)\prod_p \frac{1-\chi(p)\chi_{(-1)^\lambda t}(p)p^{\lambda-1-s}}{1-\lambda_p p^{-s}+\chi^2(p)p^{2\lambda-1-2s}}\\ &= a(t)L(s-\lambda+1,\chi\chi_{(-1)^\lambda t})^{-1}\prod_p \frac{1}{1-\lambda_p p^{-s}+\chi^2(p)p^{2\lambda-1-2s}}.\end{aligned}$$

Theorem 79 (Shimura, refined by Cipra, Niwa)**.** *Let* $f(z) = \sum\limits_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\widetilde{\Gamma_0(N)},\chi)$ *with* $\lambda \geq 1$*, and* $t > 0$ *squarefree. Define the* Shimura lift $S_t f(z) := \sum\limits_{n=1}^{\infty} A_t(n)q^n$*, where*

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = L(s-\lambda+1,\chi\chi_{(-1)^\lambda t})\sum_{n=1}^{\infty} a(tn^2)n^{-s}.$$

Then:

1. $S_t f(z) \in \begin{cases} M_{2\lambda}(\Gamma_0(N/2),\chi^2) & \text{if } \lambda = 1,\\ S_{2\lambda}(\Gamma_0(N/2),\chi^2) & \text{if } \lambda > 1.\end{cases}$
2. S_t *commutes with the Hecke operators, i.e.,* $S_t(T_{p^2}f) = T_p(S_t f)$*. In particular, if* $T_{p^2}f = \lambda_p f$*, then* $T_p(S_t f) = \lambda_p(S_t f)$*.*
3. *If* f *is an eigenform of all* T_{p^2}*, then for* t_1, t_2 *squarefree we have* $a(t_2)S_{t_1}f = a(t_1)S_{t_2}f$*.*

Example. *It can be shown that* $S_{\frac{9}{2}}(\widetilde{\Gamma_0(4)})$ *is the first non-empty space of cuspforms. We have*

$$f := \frac{\eta^{12}(2z)}{\theta^3(z)} \in S_{\frac{9}{2}}(\widetilde{\Gamma_0(4)}).$$

Since the space is of dimension 1*, it follows that* $T_{p^2}f = \lambda_p f$ *for all* p*. Note that in our case* $\lambda = 4$*. Then* $S_1 f \in S_8(\Gamma_0(2))$*, which is 1-dimensional and spanned by* $F(z) = \eta^8(z)\eta^8(2z)$*.*

Write $f = \sum a(n)q^n$*, and* $F = \sum A(n)q^n$*. By checking the first coefficient we can see that* $S_1 f = F$*. Since* F *is a normalized newform the eigenvalues are the coefficients, i.e.,* $T_p F = A(p)F$ *for all* p*. Theorem 79 implies that* $T_{p^2}f = A(p)f$ *for all* p*. Let* χ_0 *be the trivial character modulo 2, then*

$$\sum \left(a(p^2n) + \chi_0(p)\left(\frac{n}{p}\right)p^3 a(n) + \chi_0^2(p)p^7 a(n/p^2)\right)q^n = A(p)\sum a(n)q^n. \tag{13}$$

Setting $n = 1$ *in* (13) *yields the following identity:*

$$A(p) = \begin{cases} a(4) & \text{if } p = 2,\\ a(p^2) + p^3 & \text{else.}\end{cases}$$

24. H. Hahn. 12/10 and 12/12

Forms of half integral weight and special values of L-functions

Suppose that

$$f(z) = \sum a(n)q^n \in S_{2\lambda}^{\text{new}}(\Gamma_0(N))$$

is a newform and

$$L(f,s) = \sum a(n)n^{-s}$$

converges for Re $(s) \geq 0$. Let D be a fundamental discriminant and let $f_D := f \otimes \chi_D$ be a quadratic twist. If we set

$$\Lambda(f,s) := (2\pi)^{-s}\Gamma(s)N^{s/2}L(f,s),$$

then there exists $\varepsilon \in \{\pm 1\}$ such that for any D we have the following functional equation

$$\Lambda(f_D,s) = \varepsilon\chi_D(-N)\Lambda(f_D, 2\lambda - s). \tag{14}$$

The central critical value $L(f_D,\lambda)$ is the value taken at the fixed point under $s \leftrightarrow 2\lambda - s$. Note that $L(f_D,\lambda) = 0$ if $\varepsilon\chi_D(-N) = -1$. Let E be an elliptic curve over $\mathbb{Q}$ and $L(E,s)$ be the L-function of the curve E. Wiles *et al.* proved that there exists a weight 2 newform f_E such that $L(f_E,s) = L(E,s)$.

Weak Birch-Swinnerton-Dyer Conjecture: Let E be an elliptic curve over $\mathbb{Q}$. Then $L(E,1) = 0$ if and only if E has infinitely many rational points (i.e. rank$(E) \neq 0$). The order of $L(E,s)$ at $s = 1$ is the same as the rank of E.

Congruent number problem

A congruent number d is an integer d arising as area of a right triangle with rational side lengths. Assume that d is square free, and define

$$E : y^2 = x^3 - x \quad \text{and} \quad E(d) : y^2 = x^3 - d^2x.$$

Then it is not hard to see that d is a congruent number if and only if $E(d)$ has infinitely many rational points.

If we assume the Birch-Swinnerton-Dyer conjecture, then we have that d is congruent $\Leftrightarrow L(E(d),1) = 0$. Unconditionally the direction $\Rightarrow$ is known.

If we set

$$f(z) := \eta^2(4z)\eta^2(8z) \in S_2^{\text{new}}(\Gamma_0(32))$$

then it turns out that $L(f,s) = L(E,s)$.

Interested in $L(f_D,1)$ as D varies.

Set

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd} \\ \frac{|D|}{4} & \text{if } D \text{ is even.} \end{cases}$$

Then we have the following Waldspurger's theorem

Theorem 80. *Suppose* $f(z) \in S_{2\lambda}^{new}(\Gamma_0(N))$ *is a new form. Then there exist a positive integer* M *with* $N|M$*, a character* χ *modulo* $4M$*,* $\Omega_f \in \mathbb{C}^{\times}$*, and an eigenform* $g(z) = \sum b(n)q^n \in S_{\lambda+\frac{1}{2}}\left(\tilde{\Gamma_0}(4M), \chi\right)$ *such that there exists arithmetic progressions of* D *with*

$$b(D_0)^2 = \frac{\varepsilon_D L(f_D, \lambda) D_0^{\lambda-\frac{1}{2}}}{\Omega_f},$$

where ε_D *is a root of unity. In particular,* $b(D_0) = 0 \Leftrightarrow L(f_D, \lambda) = 0$.

Tunnell found two forms g_1, g_2 corresponding to $f(z) = \eta^2(4z)\eta^2(8z)$, where

$$g_1(z) = \eta(8z)\eta(16z)\Theta(2z) = \sum b_1(n)q^n,$$

$$g_2(z) = \eta(8z)\eta(16z)\Theta(4z) = \sum b_2(n)q^n.$$

Tunnell computes that if d is odd and square free, then

$$L(E(d), 1) = \frac{b_1(d^2)\Omega}{4\sqrt{d}} \quad \text{and} \quad L(E(2d), 1) = \frac{b_2(d^2)\Omega}{2\sqrt{2d}},$$

where

$$\Omega = \int_1^{\infty} \frac{dx}{\sqrt{x^3 - x}} \approx 2.62\cdots.$$

Note that if we assume the Birch-Swinnerton-Dyer Conjecture is true, then $b_1(d) = 0 \Leftrightarrow d$ is congruent.