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Level sets, a Gauss–Fourier conjecture, and a counter-example to a conjecture of Borcea and Shapiro

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Abstract. In this paper we use level sets to give a counter-example to a conjecture of J. Borcea and B. Shapiro. In particular we show that there are real polynomials P such that there is a chord of the level set $\operatorname{Im} P'(z)/P(z) = 0$ that does not have a non-real zero of P lying on it. In addition, we use level sets to show that there are no bounded components of the set $\{z : \operatorname{Im} z > 0 \text{ and } \operatorname{Im} Q(z) > 0\}$ where $Q = z - f/f'$ and f is a real entire function.

Keywords. real polynomials, level sets, zeros, Hawai'i conjecture.

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1. Introduction

In their paper, "Classifying Real Polynomial Pencils" [1], J. Borcea and B. Shapiro determine for which pencils $L = \{\alpha P + \beta Q : (\alpha, \beta) \in \mathbb{RP}^1\}$ the number of real solutions (counted with multiplicities) of the equation $\alpha P + \beta Q = 0$ is independent of α/β, thereby enumerating the equivalence classes of all generic pencils in $\mathbb{RP}^n$. Here P and Q are real polynomials, that is, polynomials in one complex variable with real coefficients. By a *pencil*, they mean a real polynomial pencil of degree n homogeneous polynomials in two real variables (a real line in $\mathbb{RP}^n$ identified with the space of all homogeneous degree n real polynomials considered up to a constant factor).

These results stemmed from their attempt to prove a conjecture of T. Craven, G. Csordas, and W. Smith called the "Hawai'i Conjecture" [3], which states the following. Let P be a real polynomial of degree $n \geq 2$. Then the number of real zeros of $(P'/P)'$ does not exceed the number of non-real zeros of P.

The conjecture of Borcea and Shapiro, if true, would have affirmed the Hawai'i Conjecture. Their classification involves *gardens* and *chords*. The *garden* $\mathcal{G}_P$ of

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the rational function P'/P is the set of all $z \in \mathbb{CP}^1$ for which the rational function attains real values. Further, the *chord part*, $\mathcal{C}$, of $\mathcal{G}_P$ is the connected component of the garden containing the real axis. A *chord* is a connected component of $\mathcal{C} \cap H^+$. Here $H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the upper half plane. Their conjecture is:

Conjecture 1 (Borcea and Shapiro)**.** Let P be a real polynomial. Each chord of $\mathcal{G}_P$ contains at least one non-real zero of P.

In this paper, we address two problems related to the level sets of certain meromorphic functions. Firstly, we give a counter-example to the conjecture of Borcea and Shapiro, and verify the following theorem.

Theorem 2. *There exists a real polynomial P such that there is a chord of $\mathcal{G}_P$ containing no non-real zeros of P.*

In addition we give the background behind the Hawai'i Conjecture, and state a conjecture, stronger than the Hawai'i Conjecture which we believe to be closer to the spirit behind the work of C.F. Gauss and J. Fourier.

Our counter-example to Conjecture 1 involves the study of the geometry of the sets $\Lambda = \{z \in H^+ : \operatorname{Im}(P'/P)(z) > 0\}$ and their boundaries, which are level sets of P'/P.

Another question, similar in flavor, yet unrelated to the first counterexample, involves the sets $K = \{z \in H^+ : \operatorname{Im} Q(z) > 0\}$ where $Q = z - f/f'$ and f is a real entire function. We show that there are no bounded components of K, answering Question 6.2 in [5].

2. History

The Hawai'i Conjecture has its roots in the work of Gauss and Fourier. In his correspondence, "Göttingische gelehrte Anzeigen" (G.G.A.) from February 25, 1833 [7], Gauss describes some observations and a conjecture of Fourier. Given a real polynomial, P, Gauss discusses how to determine the number of real and non-real zeros of P. Horizontally translating the polynomial by a units, where $a \in \mathbb{R}$, that is, by considering $P(x + a)$ instead of $P(x)$, he is able to apply Descartes' Rule of Signs and determine that the number of positive real zeros of P which lie to the right of the point $z = a$ is either equal to the number of sign differences (n_a) between consecutive non-zero coefficients, or less than it by an integral multiple of 2 (multiple roots of the same value are counted according to multiplicity). Then, horizontally translating the polynomial by a' units (where $a' \in \mathbb{R}$ and $a' > a$), he finds $n_{a'}$, the number of sign changes between consecutive non-zero coefficients of the translated polynomial. Subtracting n_a from $n_{a'}$ gives either the number of real zeros of P between a and a', or less that it by an integral multiple of 2. Gauss goes further to say that a and a' can be chosen so that $n_{a'} - n_a$ is equal to either 0, 1, or 2. The case where the difference is 2 was particularly interesting to him.

Gauss explains that Fourier believed that for each real zero of $(P'/P)'$ (or, as Fourier calls them, "critical points"), there existed an associated *pair* of non-real zeros of the polynomial P. We will call the zeros of $(P'/P)'$ Fourier critical points of P. The real Fourier critical points can be loosely associated with real inflection points of P. Suppose that the interval (a, a') does not contain any real zeros of P. Then, Rolle's Theorem gives that between any two real zeros of P' on (a, a'), there is a real zero of P'', an inflection point. Further, on (a, a'), the real zeros of P' are precisely the real zeros of P'/P, and again, Rolle's Theorem gives that there is real zero of $(P'/P)'$ between the real zeros of P'. Thus, there is a correspondence between *some* of the inflection points of P and the real Fourier critical points.

In this correspondence, Gauss notes that this association, occurring in a natural way and not involving arbitrary choices, of specific pairs of non-real zeros of P to real Fourier critical points of P is not proven and more investigation is needed. Later, in a correspondence with Schumacher on April 2, 1833 [8], Gauss explains that he did not comment on his belief or disbelief of Fourier's ideas because he feels that the G.G.A. is not the appropriate place to do so [8]. A few years later, he wrote to Schumacher (June 20, 1836), stating that he believed that a connection could be made between the real Fourier critical points and certain pairs of non-real zeros of the polynomial. A few days later (June 24, 1836), however, Gauss changed his mind and wrote to Schumacher that upon further reflection, he has found, to the contrary, that a relationship between the individual real Fourier critical points and the individual pairs of non-real zeros of P does not exist.

The polynomial $P(z) = z^4 - 1$ has two non-real zeros. Since $(P'/P)'(z) = 4z^2(z^4+3)/(z^4-1)^2$, there is a double Fourier critical point at the origin. An example with only simple critical points is obtained by taking $P_1(z) = P(z) - 10^{-3}z^2$. Now P_1 has two non-real zeros, both simple as is the case also for P, and $(P_1'/P_1)'$ has two real zeros, both simple (at about $z = \pm 0.0129$). Hence one cannot start with the real Fourier critical points of P_1 and associate distinct complex conjugate pairs of non-real zeros of P_1 to each.

A restatement of the conjecture could be:

Conjecture 3. Let P be a real polynomial. Real zeros of $(P'/P)'$ (Fourier critical points) can be grouped in pairs so that to each such pair there is an associated pair of complex conjugate zeros of P.

This conjecture changes the association from *one* real Fourier critical point and a pair of non-real zeros to the association between a *pair* of real Fourier critical points and a pair of non-real zeros. Note that with this conjecture, we can have more non-real zeros of P than real Fourier critical points, but we may not have more real Fourier critical points than non-real zeros. Of course, it is intended that distinct pairs of non-real zeros be used for distinct pairs of real Fourier critical points, with obvious exceptions allowed in case there are multiple real Fourier critical points or multiple non-real zeros. So if all these points are simple, then to distinct pairs of real Fourier critical points we must associate distinct pairs of non-real zeros.

The Hawai'i Conjecture is an attempt by Craven, Csordas, and Smith to quantify the ideas that Gauss and Fourier were investigating. The statement of their conjecture is that the number of real zeros of $(P'/P)'$ does not exceed the number of non-real zeros of P. They make no mention of an association between zeros.

Gauss's and Fourier's attempts to find a specific association made their ideas and conjectures difficult to articulate precisely; however, it is evident from the correspondence that they were trying to understand exactly how these zeros and critical points were related. Both Gauss and Fourier seemed to express frustrations about not being able to precisely state and prove these conjectures. The Hawai'i Conjecture, a conjecture on merely counting numbers of zeros, is much weaker than what Gauss and Fourier were considering.

It should be noted that M. Tyaglov [17] has posted a preprint which claims a complete proof of the Hawai'i conjecture.

We would like to thank the referee for helpful comments, and in particular, for suggesting the formulation of Conjecture 3 in terms of pairs of zeros, our original association of one real Fourier critical point to one non-real zero being more ambiguous.

3. The Conjecture of Borcea and Shapiro

As mentioned in the introduction, the attempt by Borcea and Shapiro to prove the Hawai'i Conjecture led to their classification of real polynomial pencils. If one looks at a special case where the basis of the pencil is a real polynomial, P, and its derivative, P', the structure of the set of points in the complex plane $\mathbb{C}$ where the rational function P'/P attains real values provides a natural setting to consider the Hawai'i conjecture. Affirmation of their conjecture 5.1 [1] would verify the Hawai'i conjecture.

Recall the conjecture of Borcea and Shapiro:

Conjecture 1 Let P be a real polynomial. Each chord of $\mathcal{G}_P$ contains at least one non-real zero of P.

In order to understand this conjecture we will give some background in the context of level sets. We will study the components of the sets

$$\Lambda = \{z \in H^+ : \operatorname{Im}(P'/P)(z) > 0\} \text{ and } \Lambda^- = \{z \in H^+ : \operatorname{Im}(P'/P)(z) < 0\}.$$

The geometry of these and other level sets was first studied by T.B. Sheil-Small [16] when he proved A. Wiman's 1915 conjecture. After that the second author used them to study the analogue of G. Pólya's Conjecture for meromorphic functions in [9], [10], [11] and [12]. S. Hellerstein and the first author extended the techniques of Sheil-Small in [5] to prove an analogue of Wiman's conjecture for higher derivatives and a variety of other results. And, more recently, a survey by the first author appeared in [4].

The boundary components of Λ are the level curves where $\operatorname{Im}(P'/P)(z) = 0$ in $H^+ \cup \mathbb{R}$. These curves are the same as Borcea and Shapiro's *garden*, $\mathcal{G}_P$, when restricted to $H^+ \cup \mathbb{R}$. In terms of the geometry of level curves, the chord part is equivalent to the boundary of those bounded components of Λ and Λ^- whose boundaries intersect the real axis, together with the real axis. We call the bounded components of Λ and Λ^- whose boundaries intersect the real axis *bumps*. Those bounded components of Λ and Λ^- whose boundaries do not intersect the real axis are called *holes* and their boundaries are called the *oval* part of the garden.

3.1. The polynomial $P(z) = z(z^2+1)^5$. The particular function that we are considering is $P(z) = z(z^2+1)^5$. We will show that this function has a bump without a non-real zero of P on its boundary. Note that this is equivalent to saying that there is a chord which does not have a non-real zero of P on it.

The function P has a real zero at 0 and non-real zeros of multiplicity 5 at $\pm i$. The function $P'(z) = (11\,z^2+1)\,(z^2+1)^4$ has no real zeros, simple non-real zeros at $\pm i\sqrt{1/11}$, and non-real zeros of multiplicity 4 at $\pm i$. We note that $(P'/P)(z) = (11z^2+1)/(z(z^2+1))$ has two finite zeros (at $\pm i\sqrt{1/11}$) and three poles (at $0, \pm i$). Further, we see that the finite zeros of $(P'/P)'(z) = -(11\,z^4 - 8\,z^2+1)/((z^2+1)^2\,z^2)$ are all real (at $\pm\dfrac{\sqrt{44+11\sqrt{5}}}{11}$, $\pm\dfrac{\sqrt{44-11\sqrt{5}}}{11}$).

3.2. Geometry and Zeros. In order to prove that there is a bump that does not have a non-real zero of P on its boundary, we need to know about the geometry of the level curves. The following results are noted in Sheil-Small [15], Chapter 9, pp. 306-307:

Lemma 4. *Let $z_0 \in \mathbb{R}$ be a zero of $(P'/P)'$. Then $z_0 \in \partial\Lambda \cap \partial\Lambda^-$. In particular, this statement applies if z_0 is a real multiple zero of P'/P.*

Proof. Let $z_0 \in \mathbb{R}$ be a critical point of P'/P. Then z_0 is a branch point of the level curves $\operatorname{Im} P'(z)/P(z) = 0$. ∎

Lemma 5. *Each component of the level set $\{z \in H^+ : \operatorname{Im}(P'/P)(z) = 0\}$ consists of either a level curve with two (possibly coincident) end points on the real axis that are critical points of P'/P (a bump), or a loop in H^+ (a hole).*

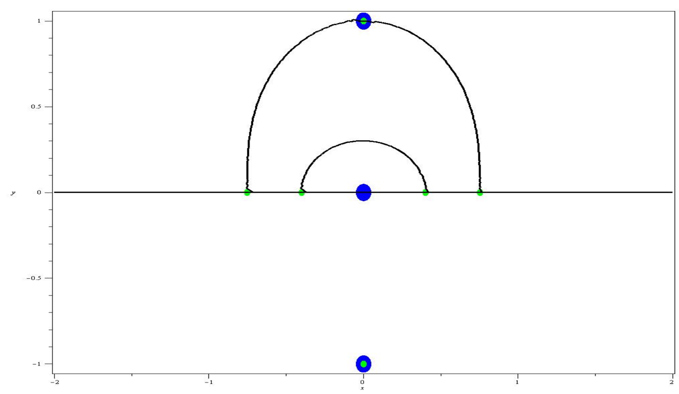

FIGURE 1. The level curves of $\operatorname{Im} P'(z)/P(z) = 0$ for $P(z) = z(z^2+1)^5$

We are now ready to prove Theorem 2, using $P(z) = z(z^2+1)^5$.

Proof of Theorem 2. Since there are four real critical points of P'/P, Lemma 4 shows that these zeros of $(P'/P)'$ are branch points of the level curves $\operatorname{Im} P'(z)/P(z) = 0$. The critical points are simple and so a chord connects these points in pairs by Lemma 5. Thus there are two chords. We know that there is one non-real zero of P in H^+, at $z = i$. Suppose that it lies on the boundary of both chords. Then there would be two level curves intersecting at this point, indicating that the point i is a branch point of the level curves and hence it must be a pole of multiplicity greater than one of P'/P. However, $z = i$ is a simple pole of P'/P. Thus, $z = i$ can not lie on both chords. Hence there must be one chord that does not have non-real zero of P on its boundary. This completes the proof of Theorem 2.

One can see from Figure 1 (generated by *Maple*) that the smaller chord does not have a non-zero of P on its boundary and the larger chord does so.

It should also be noted that Lemma 5 suggests a natural pairing of real critical points of P'/P. This grouping was also used by Borcea and Shapiro [1]. They followed the work of A. Eremenko and A. Gabrielov in [6].

It is not the multiple zeros of the polynomial, $P(z) = z(z^2+1)^5$, that enable us to find a chord without a non-real zero of P on it. One can see this by considering a similar polynomial which spreads out the non-real zeros of P. We consider $P_1(z) = z((z-1)^2+1)((z-0.5)^2+1)((z+1)^2+1)((z+0.5)^2+1)(z^2+1)$. This polynomial has only simple zeros and the geometry to the level curves is very similar to that of P yet still the inside chord has no non-real zeros of P_1 on it (see Figure 2). Further, we can see that if the non-real zeros of the polynomial are spread out "too much" (e.g., $P_2(z) = z((z-1)^2+1)((z-2)^2+1)((z+1)^2+1)((z+2)^2+1)(z^2+1))$, that there is no longer a chord without a non-real zero of the polynomial on it (see Figure 3).

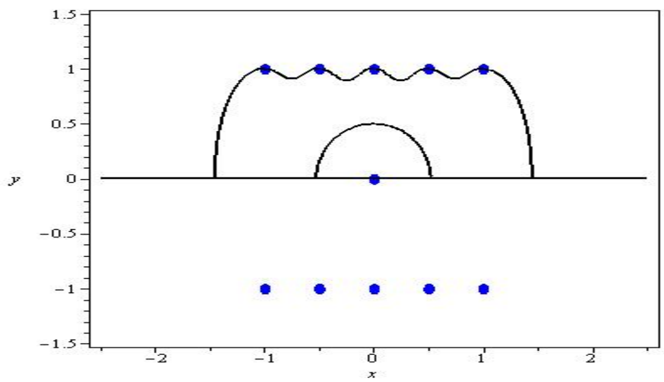

FIGURE 2. The level curves of $\operatorname{Im} P_1'(z)/P_1(z) = 0$ for $P_1(z) = z((z-1)^2+1)((z-0.5)^2+1)((z+1)^2+1)((z+0.5)^2+1)(z^2+1)$

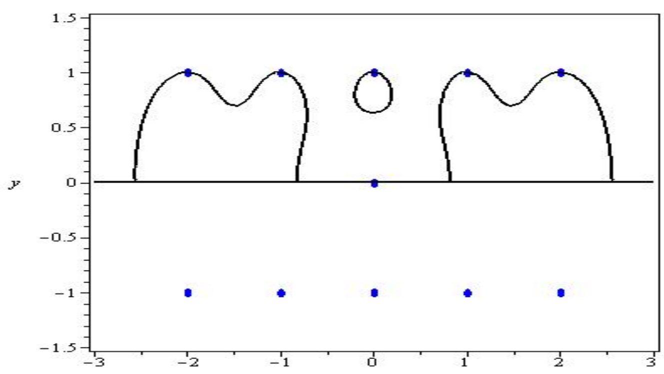

FIGURE 3. The level curves of $\operatorname{Im} P_2'(z)/P_2(z) = 0$ for $P_2(z) = z((z-1)^2+1)((z-2)^2+1)((z+1)^2+1)((z+2)^2+1)(z^2+1)$

4. Fixed point problems

We now present a solution to a question that the first author and S. Hellerstein posed in [5]. The question asked is if f is a real entire function of finite order, can the set $K = \{z \in H^+ : \operatorname{Im} Q(z) > 0\}$, where $Q = z - f/f'$, have a bounded component? We show that there are no bounded components of the set K.

Recall that an entire function f is called real if $f(z)$ is real whenever z is real.

Answering a question raised by the first author and Hellerstein ([5], Question 6.1), Langley ([13], Theorem 4, p. 244) has shown that the set $K^- = \{z \in H^+ : \operatorname{Im} Q(z) < 0\}$ has no unbounded components and the set K has exactly one unbounded component, for a real entire function f of finite order with only finitely many non-real zeros (and indeed for a real meromorphic function f of finite order with only finitely many poles and with only finitely many non-real zeros).

Theorem 6. *If f is a non-constant real entire function, then there are no bounded components of $K = \{z \in H^+ : \operatorname{Im} Q(z) > 0\}$ where $Q(z) = z-(f/f')(z)$. If f is a non-constant real entire function of finite order with only finitely many non-real zeros, then the set K is connected and unbounded, while all components of K^- are bounded.*

Proof. Let the assumptions of Theorem 6 be satisfied. To get a contradiction, suppose that U is a bounded component of K. Since f is real, the set K coincides with a component of the set $\{z \in \mathbb{C} : \operatorname{Im} Q(z) > 0\}$. Note that we are not assuming that U is simply connected. However, since U is bounded, it is necessarily of finite connectivity. Then Q maps U, firstly, into H^+. Since U is bounded, the map $Q|U$ is proper, so that $Q|U$ is a covering map: each point $w \in H^+$ has the same number, say $m \geq 1$, of inverse image points in U, with due count of multiplicity.

If ∂U contains a fixed point z_0 of Q, then $f(z_0) = 0$. Further, since Q maps ∂U into the real axis, each such z_0 must be real. There can be only finitely many such points on ∂U, say $c_1, \dots, c_k$. If there are no such points c_j, then the argument given below is simplified since certain modifications do not have to be performed at all. We proceed with the proof with the understanding that this is the case.

We write $B(z,r) = \{w \in \mathbb{C} : |w-z| < r\}$, $S(z,r) = \{w \in \mathbb{C} : |w-z| = r\}$, and $\overline{B}(z,r) = \{w \in \mathbb{C} : |w-z| \leq r\}$.

Since ∂U is the union of finitely many analytic curves, we may find $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, then for each j with $1 \leq j \leq k$, the sets $B(c_j,\varepsilon) \cap \partial U$, $B(c_j,\varepsilon) \cap U$, and $B(c_j,\varepsilon) \setminus \overline{U}$ are connected, and the closed disks $\overline{B}(c_j,\varepsilon)$ are pairwise disjoint. We may further choose ε_0 so small that Q has no fixed points other than the points $c_1, \dots, c_k$ in $\cup_{j=1}^k \overline{B}(c_j,\varepsilon_0)$.

Each c_j is an attracting or superattracting fixed point of Q (that is, $0 < |Q'(c_j)| < 1$ or $Q'(c_j) = 0$) since $Q'(c_j) = 1 - 1/s$ if c_j is a zero of f of order $s \geq 1$. Thus, if ε_0 is small enough, as we may assume it to be, for $0 < \varepsilon \leq \varepsilon_0$ the part of $Q^{-1}(S(c_j,\varepsilon) \cap H^+)$ lying close to c_j, lies strictly inside $U \setminus \overline{B}(c_j,\varepsilon)$.

Every finite fixed point of Q is a zero of f and, as above, is an attracting or superattracting fixed point of Q. Thus, a contradiction can now be reached by showing that the existence of a bounded component U of K implies, perhaps indirectly, the existence of a finite repelling or indifferent fixed point of Q. Recall that a fixed point z_0 of Q is called repelling if $|Q'(z_0)| > 1$, and indifferent if $|Q'(z_0)| = 1$. As we shall see, a contradiction of that nature can be obtained by considering certain integrals.

Define $V = V(\varepsilon) = H^+ \setminus \cup_{j=1}^k \overline{B}(c_j,\varepsilon)$, so that $V(\varepsilon)$ is simply connected and indeed a Jordan domain.

There is a component $W = W(\varepsilon)$ of $Q^{-1}(V(\varepsilon))$ contained in U. Since Q fixes each c_j, it follows that each c_j is outside the closure of W. Also, Q has no fixed points on ∂W. If ε is small enough, then W and U are almost equal and have the same connectivity: to get W, we have merely deleted from U small neighborhoods of the points $c_1, \ldots, c_k$, as well as small neighborhoods of the finitely many other inverse image points of the c_j that may lie on ∂U. Since Q maps U into H^+, there is no point in U that Q maps onto any c_j.

Furthermore, since the part of $Q^{-1}(S(c_j, \varepsilon) \cap H^+)$ lying close to c_j, lies strictly inside $U \setminus \overline{B}(c_j, \varepsilon)$, we have $W \subset V$ as well.

It is still the case that $Q|W$ is an $m-$to-1 covering map of W onto V, for the same $m \geq 1$ as before.

Now we follow an argument due to Buff ([2], proof of Theorem 2). There are so many details to be considered that it seems best to give a complete proof even though the idea can be read off from [2].

We fix a sufficiently small ε, write $V = V(\varepsilon)$ and $W = W(\varepsilon)$, and note that the function $g(z) = z - Q(z)$ has no zeros on ∂W, which consist of finitely many pairwise disjoint rectifiable Jordan curves.

Let φ be a conformal mapping of V onto the unit disk $\mathbb{D}$. Note that φ extends to a homeomorphism of $\overline{V}$ onto $\overline{\mathbb{D}}$ since V is a Jordan domain. Write $W' = \varphi(W) \subset \mathbb{D}$. Set $h = \varphi \circ Q \circ \varphi^{-1}$. Then h maps W' onto $\mathbb{D}$ as an $m-$to-1 covering map. Also, h has no fixed points on $\partial W'$.

Next, set $F = z - h$ and $G = h$, so that F and G are analytic in W' and continuous in the closure of W'. If $z \in \partial W'$, then

$$|F(z) - G(z)| = |z| < |h(z) - z| + |h(z)| = |h(z) - z| + 1$$

since $h(z) - z \neq 0$ and $|z| \leq 1 = |h(z)|$. By a form of Rouche's Theorem ([14], Theorem 3.6, p. 341), the functions F and G have the same number of zeros, counting multiplicities, in W'. Thus the number of fixed points of h in W' is equal to the number of zeros of h in W', which is the number m mentioned before. Even though Rouche's Theorem is normally formulated for a Jordan domain, the conclusion remains valid here where $\partial W'$ consists of finitely many disjoint rectifiable Jordan curves (since ∂V consists of finitely many circular arcs, it is clear that the mapping φ is sufficiently regular to preserve sets of finite length).

If $z \in \partial W'$, then $|z| \leq 1 = |h(z)|$ and $z \neq h(z)$. Thus

$$\operatorname{Re} \frac{z}{z - h(z)} \leq \frac{1}{2}. \tag{1}$$

The same remains valid for all $z \in \partial W_r$, if we choose $r \in (0, 1)$ sufficiently close to 1, and replace W' by W_r, which is the component of $h^{-1}(B(0, r))$ contained in W'. The number of fixed points of h in W_r is still equal to m when r is close enough to 1.

At each zero z_0 of the function $z - h$, the residue of $1/(z-h)$ is equal to $1/(1-h'(z_0))$, assuming that $h'(z_0) \neq 1$. Since Q has only attracting and super-attracting fixed points, the same is true of h. Thus $|h'(z_0)| < 1$, and hence

$$\operatorname{Re} \frac{1}{1-h'(z_0)} > \frac{1}{2},$$

for all fixed points z_0 of h in W_r.

Hence by the Residue Theorem, we have

$$\frac{1}{2\pi i} \int_{\partial W_r} \frac{1}{z-h(z)}\, dz > \frac{m}{2} \tag{2}$$

since there are m zeros z_0 of $z-h$ in W_r and each is a simple zero since $|h'(z_0)| < 1$.

On the other hand, still following Buff, we write

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial W_r} \frac{1}{z-h(z)}\, dz &= \frac{1}{2\pi i} \int_{\partial W_r} \frac{1-h'(z)}{z-h(z)}\, dz + \frac{1}{2\pi i} \int_{\partial W_r} \frac{h'(z)}{z-h(z)}\, dz \\ &= \frac{1}{2\pi i} \int_{\partial W_r} \frac{h'(z)}{h(z)}\, dz + \frac{1}{2\pi i} \int_{\partial W_r} \frac{h(z)}{z-h(z)} \frac{h'(z)}{h(z)}\, dz \\ &= \frac{1}{2\pi i} \int_{\partial W_r} \frac{z}{z-h(z)} \frac{h'(z)}{h(z)}\, dz. \end{aligned} \tag{3}$$

Here we have used the fact that

$$\frac{1}{2\pi i} \int_{\partial W_r} \frac{1-h'(z)}{z-h(z)}\, dz = \frac{1}{2\pi i} \int_{\partial W_r} \frac{h'(z)}{h(z)}\, dz,$$

which follows from Rouche's Theorem as explained above.

Since h is an orientation preserving mapping of ∂W_r onto $S(0,r)$, it follows that the quantity

$$\frac{1}{2\pi i} \frac{h'(z)}{h(z)}\, dz$$

is real and positive on ∂W_r. Hence, by (1),

$$\begin{aligned} \operatorname{Re} \left(\frac{1}{2\pi i} \int_{\partial W_r} \frac{z}{z-h(z)} \frac{h'(z)}{h(z)}\, dz \right) &= \int_{\partial W_r} \operatorname{Re} \left(\frac{z}{z-h(z)} \right) \frac{1}{2\pi i} \frac{h'(z)}{h(z)}\, dz \\ &\leq \frac{1}{2} \left(\frac{1}{2\pi i} \int_{\partial W_r} \frac{h'(z)}{h(z)}\, dz \right) = \frac{m}{2}. \end{aligned}$$

By (2) and (3), this gives the desired contradiction.

The last statement of Theorem 6 follows from what we have proved so far and from the result of Langley ([13], Theorem 4, p. 244) that we have already cited. The proof of Theorem 6 is complete.

■

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