DYNAMICS OF FUNCTIONS ARISING FROM PISOT AND SALEM POLYNOMIALS

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ABSTRACT. Let $Q(z) = z^n P(z) - z^{\deg P} P(z^{-1})$ where P is the minimal polynomial of a Pisot number. Boyd [4] showed that, for $n > \deg P - 2\frac{P'(1)}{P(1)}$, Q is the product of cyclotomic polynomials and the minimal polynomial of a Salem number, say α. In this paper, we study the dynamics of the Newton map $N = z - Q/Q'$ induced by Q in the immediate basin U_α of α. We establish that N is a $2-$fold covering map of U_α onto itself. Furthermore, there exists a conformal mapping φ of U_α onto the open unit disk $\mathbb{D}$ such that

$$(\varphi \circ N \circ \varphi^{-1})(z) = cz^2$$

for all $z \in \mathbb{D}$, where c is a constant with $|c| = 1$.

1. INTRODUCTION

A Pisot number (or P.V. number) is an algebraic integer greater than 1 such that all its conjugates are strictly inside the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\mathbb{C}$ denotes the complex plane. In this paper, we call the minimal polynomial of a Pisot number a *Pisot polynomial*. Salem [9] proved that the class of all Pisot numbers is a closed set, hence it contains a smallest element. In 1944, Siegel [13] showed that the number $\theta_1 = 1.3247...$, which is a zero of $z^3 - z - 1$, is the smallest Pisot number. In [10], Salem introduced a new class of algebraic integers that are now known as *Salem numbers*. A Salem number is a real algebraic integer τ greater than 1 whose conjugates have modulus at most 1 and at least one of which has modulus 1. Here, P is called a *Salem polynomial* if it is the minimal polynomial of a Salem number. In contrast to Pisot numbers, we know less about the set $\mathcal{T}$ of Salem numbers. One of the interesting unsolved problems concerning the set $\mathcal{T}$ is whether it has a lower bound greater than 1. It has been conjectured that there is a smallest Salem number and that it is $\tau_1 = 1.17628\ldots$, which is a zero of $z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.

Let $P^*(z)$ denote the polynomial $z^{\deg P} P(z^{-1})$, where $\deg P$ denotes the degree of P. In [10], Salem considered the function

$$R_m(z) = z^m P(z) \pm P^*(z),$$

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where P is a Pisot polynomial with $P(z) \neq P^*(z)$ and $m \in \mathbb{N}$, the set of positive integers. Salem showed that R_m has at most one zero outside $\overline{\mathbb{D}}$. Moreover, if R_m has a zero τ_m with modulus greater than 1 then $\tau_m \in \mathcal{T}$. Later in 1977, Boyd [4] showed that indeed Salem's method can produce all Salem numbers. In particular, Boyd established that each Salem number is a zero of the polynomial $Q(z) := z^n P(z) - P^*(z)$ for some Pisot polynomial P and some $n \in \mathbb{N}$. In fact, Q is the product of cyclotomic polynomials and a Salem polynomial if, and only if, $n > \deg P - 2\frac{P'(1)}{P(1)}$. Therefore, to study the set $\mathcal{T}$ further, we may investigate the properties of such polynomials Q. Note that Q is anti-reciprocal (that is, $Q^*(z) \equiv -Q(z)$), hence 1 is a zero of Q. In [5], we showed that each of Q' and Q'' has exactly one zero outside $\overline{\mathbb{D}}$ and those two zeros must be real numbers greater than 1. So, if we could find a positive uniform lower bound for $\beta - 1$, where β runs over all zeros of Q' greater than 1, then this would assure that $\mathcal{T}$ has a lower bound greater than 1. This question remains open, but these observations perhaps suggest that such questions are of some interest.

The Newton map is a function named after the Newton's method which is a well known tool for finding zeros of polynomials and other functions. The study of the dynamics of the Newton map $N = z - Q/Q'$ induced by Q should give us some insight concerning Salem numbers. We will discuss the dynamics of the Newton map in the second section. In this paper, we focus on the dynamics of the Newton map N on the component of the Fatou set $\mathcal{F}(N)$ containing α, where α is a zero of Q and is a Salem number. We denote this component of $\mathcal{F}(N)$ by U_α and, as usual, call it *the immediate basin of* α. Here, we will show that N is a 2-fold covering map of U_α onto itself. More precisely, the following result is our Main Theorem.

Theorem 1. *Let f be an anti-reciprocal polynomial of degree $n \geq 3$ with real coefficients whose zeros are simple. Suppose that $\alpha > 1$ is the only zero of f lying outside $\overline{\mathbb{D}}$. Let N denote the Newton map induced by f and let U_α denote the immediate basin of α under N. Then N is a 2-fold covering map of U_α onto itself. Furthermore, there exists a conformal mapping φ of U_α onto $\mathbb{D}$ and a constant $c \in \mathbb{C}$ with $|c| = 1$ such that*

$$(\varphi \circ N \circ \varphi^{-1})(z) = cz^2$$

for all $z \in \mathbb{D}$.

2. BACKGROUND ON THE NEWTON MAP

Let f be a non-constant meromorphic function in $\mathbb{C}$, not a rational function of degree 1. The Newton map induced by f is the function $N(z) = z - \frac{f(z)}{f'(z)}$. Let α denote a zero of f. Then α is a fixed point of the function N (that is, $N(\alpha) = \alpha$) and $|N'(\alpha)| = \frac{m-1}{m} < 1$, where m is the multiplicity of α as a zero of f. Thus α is an attracting fixed point of N, and so α is in the Fatou set $\mathcal{F}(N)$ of N. Recall that $\mathcal{F}(N)$ is the maximal open subset of $\mathbb{C}$ (or of the Riemann sphere $\overline{\mathbb{C}}$ if N is a rational function) in which the iterates of N are all defined and form a normal family. We call the component of $\mathcal{F}(N)$ that contains the point α, *the immediate basin* of α, and denote it by U_α. From the corollary to a theorem of Mayer and Schleicher [6, Theorem 2.8, p. 330], if f is non-linear entire, then U_α is simply connected and unbounded. Hence, by the Riemann Mapping Theorem, there is a conformal mapping φ of U_α onto the open unit disk $\mathbb{D}$ with $\varphi(\alpha) = 0$. The function $g(z) = (\varphi \circ N \circ \varphi^{-1})(z)$ is analytic and maps $\mathbb{D}$ into itself, and onto itself in a finite-to-one fashion if N is a rational function. In this latter case, g is a finite Blaschke product, that is, g is of the form

$$g(z) = c\prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z} \tag{1}$$

for some $|c| = 1$ and $|a_k| < 1$, where n is the number of times that N maps U_α onto itself. The value of n can be determined by the number of critical points of g in $\mathbb{D}$, which is equal to the number of zeros of N' in U_α, with due count of multiplicity. Indeed, the number of critical points of g in $\mathbb{D}$ is equal to $n - 1$ [11, p. 373-374], [14, Theorem 1, p. 463].

Now suppose that f is a polynomial of degree greater than 2 and α is a simple zero of f. In this case, the point at infinity is a repelling fixed point of N, hence in $\mathcal{J}(N)$, and the zeros of f' that are not zeros of f lie in $\mathcal{J}(N)$, hence not in U_α. Thus the set of the zeros of $N' = (ff'')/((f')^2)$ in U_α consists of the zeros of f'' lying in U_α and α itself. Hence the number of zeros of f'' in U_α indicates the value of n. That is, N is an n-fold mapping of U_α onto itself if, and only if, f'' has $n - 2$ zeros in U_α. Therefore, to prove the first statement of Theorem 1, it suffices to show that U_α contains no zeros of f''.

3. PRELIMINARY RESULTS

In [5, Theorem 1.4], we showed that if f is an anti-reciprocal polynomial satisfying the assumptions of Theorem 1, then f'' has exactly one zero outside $\overline{\mathbb{D}}$, which must be a real number between 1 and α. If we know that U_α does not intersect $\overline{\mathbb{D}}$, then the number of zeros of f'' in U_α is either 0 or 1. So, our first task is to show that $\overline{\mathbb{D}} \cap U_\alpha$ is empty.

Theorem 2. *Let f be a non-constant meromorphic or entire function in $\mathbb{C}$ such that the Newton map N of f is not a rational function of degree at most 1. Suppose that $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1}{2}$ whenever $|z| = 1$ and $f(z) \neq 0, \infty$. For any zero α of f lying outside the closed unit disk, the immediate basin of α for the Newton map N for f does not intersect the closed unit disk.*

Proof. Let U_α denote the immediate basin of α under N. Suppose that $U_\alpha \cap \overline{\mathbb{D}}$ is not empty. Then there exists a point $w \in U_\alpha \cap \partial\mathbb{D}$, so that $f(w) \neq 0, \infty$. We can express N in the form

$$N(z) = z\left(1 - \frac{1}{g(z)}\right),$$

where $g(z) = \dfrac{zf'(z)}{f(z)}$. By the assumption, for any $z \in \partial\mathbb{D}$ with $f(z) \neq 0, \infty$, we have $g(z) = x + iy$ for some $x, y \in \mathbb{R}$ with $x \geq \frac{1}{2}$, and so

$$|N(z)| = |z|\left|1 - \frac{1}{x+iy}\right| = \left|\frac{x-1+iy}{x+iy}\right| \leq 1, \tag{2}$$

because $|x-1| \leq x$ for all $x \geq \frac{1}{2}$. Thus, $N(w) \in \overline{\mathbb{D}}$ and $N(w)$ still stays in U_α because U_α is forward invariant under N. Since U_α is connected, there is a closed Jordan arc γ_0 lying entirely in U_α such that γ_0 has end points at w and $N(w)$. Since $N^k(\gamma_0)$ converges uniformly to α as $k \to \infty$, the set $\cup_{k=0}^{\infty} N^k(\gamma_0) \cup \{\alpha\}$ forms a continuous curve γ joining w and α and lying inside U_α.

Let β be the last intersection point of γ with $\partial\mathbb{D}$ (that is, the part of the curve γ that connects β to α stays outside $\overline{\mathbb{D}}$ except at β). So N must map β to a point outside $\overline{\mathbb{D}}$, otherwise β is a fixed point of N, which is impossible because α is the only fixed point of N inside U_α and $|\alpha| > 1$. But, by (2), applied to $z = \beta$, the point $N(\beta)$ still belongs to $\overline{\mathbb{D}}$, hence we get a contradiction. Therefore $U_\alpha \cap \overline{\mathbb{D}}$ is empty. □

As consequences of Theorem 2, by Lemma 2.1 and Lemma 2.2 in [5], we derive the following corollaries.

Corollary 3. *Let* $f(z) = c\prod_{k=1}^{n}(z - \alpha_k)$ *be a polynomial of degree* $n \geq 2$, *where* $c \neq 0$. *Suppose that* $\alpha_k \notin \overline{\mathbb{D}}$ *for* $1 \leq k \leq m$, *and the remaining* α_k *are in* $\overline{\mathbb{D}}$. *If*

$$\sum_{k=1}^{m} \frac{1}{1-|\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1+|\alpha_k|} > \frac{1}{2}, \tag{3}$$

then, for $1 \leq k \leq m$, *the immediate basin of* α_k *under the Newton map* N *for* f *does not intersect the closed unit disk.*

Namely, by Lemma 2.1 in [5], the assumption (3) implies that $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1}{2}$ whenever $|z| = 1$ and $f(z) \neq 0$. Thus Corollary 3 now follows from Theorem 2.

Corollary 4. *Let* $f(z) = c\prod_{k=1}^{n}(z - \alpha_k)$ *be a polynomial of degree* $n \geq 2$, *where* $c \neq 0$. *Suppose that* $\alpha_1 = \alpha$, $\alpha_2 = \alpha^{-1}$, *where* α *is real and* $|\alpha| > 1$, *and all the remaining* α_k, *if any, are in* $\overline{\mathbb{D}}$. *Then the immediate basin of* α_1 *under the Newton map* N *for* f *does not intersect the closed unit disk.*

Namely, by Lemma 2.2 in [5], the assumptions of Corollary 4 imply that $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1}{2}$ whenever $|z| = 1$ and $f(z) \neq 0$. Thus Corollary 4 now follows from Theorem 2.

Note that the function f in Theorem 1 satisfies the assumptions of Corollary 4. So $U_\alpha \cap \overline{\mathbb{D}} = \emptyset$ in this case. Figure 1 displays the Fatou set $\mathcal{F}(N)$, where

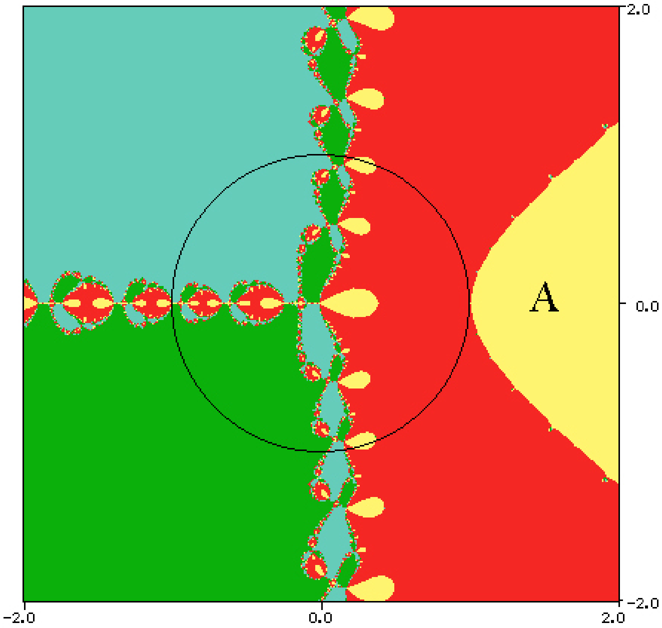

FIGURE 1

$$f(z) = (z - 11/10)(z - 10/11)(z^2 + z + 1/2)$$

satisfies the assumptions of Corollary 4. We see that the component A of $\mathcal{F}(N)$ which is the immediate basin of the zero 1.1 of f does not intersect the unit circle.

Lemma 5. *If f is a polynomial with real coefficients and α is a real zero of f, then the immediate basin U_α of α is symmetric with respect to the real axis.*

Proof. This result is well known but we recall the proof for completeness. Of course, this is merely a special case of the fact that if $g(z) = \overline{h(\overline{z})}$, then $\mathcal{F}(g) = \{\overline{z} : z \in \mathcal{F}(h)\}$; if here $h = N_f$, the Newton map for f, is a quotient of polynomials with real coefficients, then $g = h$, so that $\mathcal{F}(h)$ is symmetric about $\mathbb{R}$. Hence any component of $\mathcal{F}(h) = \mathcal{F}(N_f)$ that intersects $\mathbb{R}$ is symmetric about $\mathbb{R}$. ☐

4. PROOF OF THEOREM 1

Let f be a function satisfying the assumptions of Theorem 1. Note that $f(1) = 0$. By Theorem 1.4 in [5], each of f' and f'' has exactly one zero outside $\overline{\mathbb{D}}$, say ρ and β, respectively. By Rolle's Theorem, we have $1 < \beta < \rho < \alpha$. Since ρ is a pole of the Newton map N for f

and all poles of N are in the Julia set $\mathcal{J}(N) = \overline{\mathbb{C}} \setminus \mathcal{F}(N)$, ρ is not in U_α. Since $\rho \in \mathcal{J}(N)$, and U_α is simply connected and symmetric with respect to the real axis, we have $U_\alpha \cap (-\infty, \rho) = \emptyset$. So β does not lie in U_α. By Theorem 2, U_α does not intersect $\overline{\mathbb{D}}$, so f'' has no zeros in U_α. This means that U_α contains only one critical point of N, which is α. Therefore, by (1) and the Riemann Mapping Theorem, there exists a conformal mapping φ of U_α onto $\mathbb{D}$ such that $\varphi(\alpha) = 0$ and

$$g(z) := (\varphi \circ N \circ \varphi^{-1})(z) = cz\frac{z-a}{1-\overline{a}z},$$

for some $a, c \in \mathbb{C}$ with $|c| = 1$ and $|a| < 1$. Since $g'(w) = 0$ for $|w| < 1$ if, and only if, $N'(\varphi^{-1}(w)) = 0$, and $N'(z) = 0$ for $z \in U_\alpha$ if, and only if, $z = \alpha$, it follows that $g'(w) = 0$ for $|w| < 1$ if, and only if, $\varphi^{-1}(w) = \alpha$, that is, $w = 0$. This implies that $g(z) = cz^2$ on $\mathbb{D}$. The proof of Theorem 1 is now complete.

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