A NEW COUNTING FUNCTION FOR THE ZEROS OF HOLOMORPHIC CURVES

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Abstract. Let $f_1, \ldots, f_p$ be entire functions that do not all vanish at any point, so that $(f_1, \ldots, f_p)$ is a holomorphic curve in $\mathbb{CP}^{p-1}$. We introduce a new and more careful notion of counting the order of the zero of a linear combination of the functions $f_1, \ldots, f_p$ at any point where such a linear combination vanishes, and, if all the $f_1, \ldots, f_p$ are polynomials, also at infinity. This enables us to formulate an inequality, which sometimes holds as an identity, that sharpens the classical results of Cartan and others.

1. Introduction

The Cartan theory of holomorphic curves in projective spaces in a generalisation of the value distribution theory of Nevanlinna. If p is an integer with $p \geq 2$, and $f_1, \ldots, f_p$ are entire functions of a complex variable z which are linearly independent over the set of complex numbers $\mathbb{C}$, and with no common zeros, then the point $(f_1(z), \ldots, f_p(z))$ may be viewed not only as a point in the affine space $\mathbb{C}^p$ but also as a well defined point in the complex projective space $\mathbb{CP}^{p-1}$. As is usual, we say that the $p-$tuple $(f_1, \ldots, f_p)$ defines a holomorphic curve in $\mathbb{CP}^{p-1}$.

We now form (non-trivial) linear combinations of the functions $f_1, \ldots, f_p$, that is, entire functions of the form

$$g(z) = \sum_{j=1}^{p} a_j f_j(z),$$

where the complex numbers a_j are not all zero. The zeros of such a funcion g play an important role in the Cartan theory. In particular, if $p = 2$ the zeros of g are precisely the points where the meromorphic function f_1/f_2 assumes the value $-a_2/a_1$. It is, of course, of fundamental importance in the Nevanlinna theory to count the number of such points and to estimate the growth of the function f_1/f_2 in some appropriate sense. For $p-$tuples of linearly independent entire functions with values in $\mathbb{CP}^n$, so that $p = n+1$, Cartan measures the growth in terms of the function

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \max\{|f_j(re^{i\theta})| : 1 \leq j \leq p\}\, d\theta.$$

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Consider now linear combinations $g_1, \ldots, g_q$, where $q > p$, of the f_j, any p of which are linearly independent and let $n_{p-1}(r, 0, g_i)$ denote the number of zeros of g_i in the disk $B(0, r)$ of radius r about the origin, where, however, a zero of multiplicity k is counted $\min\{k, p-1\}$ times. As usual we introduce the "integrated counting function"

$$N_{p-1}(r, 0, g_i) = \int_0^r (n_{p-1}(t, 0, g_i) - n_{p-1}(0, 0, g_i)) \frac{dt}{t} + n_{p-1}(r, 0, g_i) \log r.$$

The main result of the Cartan theory states that

$$(q-p)T(r) \leq \sum_{i=1}^{q} N_{p-1}(r, 0, g_i) - N(r, H) + S(r). \tag{1}$$

Here

$$H = \frac{g_1 g_2 \cdots g_q}{W}$$

where W is the Wronskian determinant of $f_1, \ldots, f_p$, $N(r, H)$ is the usual integrated counting function of the poles of H, and $S(r)$ is an error term which is small in a certain sense.

Thus the functions g_i must, on the whole, have a large number of zeros, which can then be interpreted in various ways as a large number of intersections of the holomorphic curve with hyperplanes, and so on. In the classical case when $p = 2$ and

$$F(z) = \frac{f_1(z)}{f_2(z)},$$

Cartan showed that

$$T(r, F) \leq T(r) + O(1) \text{ as } r \to \infty,$$

where $T(r, F)$ is the usual Nevanlinna characteristic, thus providing a generalisation of the Nevanlinna theory. This is explained in full detail in [4].

We show that the counting functions $n_{p-1}(r)$ and $N_{p-1}(r)$ can be replaced by "reduced" multiplicities $\nu_{p-1}(z)$ and $\mathcal{N}_{p-1}(z)$ associated with functions g_i and points $z \in \mathbb{C}$. At any point $z \in \mathbb{C}$ that is not a zero of the Wronskian determinant W of the functions $f_1, \ldots, f_p$, the possible orders of zeros of non-trivial linear combinations g of $f_1, \ldots, f_p$ form the sequence $0, 1, 2, \ldots, p-1$. Here the multiplicity of the zero of g at z is said to zero if $g(z) \neq 0$. At the zeros z of W, this sequence is replaced by $0, d_1, d_2, \ldots, d_{p-1}$ where $1 \leq d_1 < d_2 < \cdots < d_{p-1}$ and the integers $d_1, d_2, \ldots, d_{p-1}$ depend on z. This leads to the numbers

$$0 \leq \mathcal{E}_j(z) = d_j - j \leq d_{j+1} - (j+1)$$

which we call "excesses" at z. If $W(z) \neq 0$, then $\mathcal{E}_j(z) = 0$ for $1 \leq j \leq p-1$. A similar analysis applies at the point at infinity.

Suppose that $f_1, \ldots, f_p$ are polynomials of maximal degree d. The fundamental identity regarding excesses states that (compare Theorem 3.2)

$$\sum_{j=1}^{p-1} \sum_{z \in \overline{\mathbb{C}}} \mathcal{E}_j(z) = pd - p(p-1). \tag{2}$$

If $p = 2$, then a zero of the linear combination $a_1 f_1 + a_2 f_2$ corresponds to a point where the rational function $F = f_1/f_2$ of degree $d = \max\{\deg f_1, \deg f_2\}$ attains the value $-a_2/a_1 \in \overline{\mathbb{C}}$. The possible orders of zeros of non-trivial linear combinations $a_1 f_1 + a_2 f_2$ at a point z_0 are $0, m$ where m is multiplicity with which the function F takes the value $F(z_0)$ at z_0. In this case there is only one excess at each point z_0, and we have $\mathcal{E}_1(z_0) = m - 1$, which is the usual branching index of F at z_0. Now (2) reads

$$\sum_{z \in \overline{\mathbb{C}}} \mathcal{E}_1(z) = 2d - 2$$

which is the well-known Riemann–Hurwitz formula for rational functions. Hence (2) for a general p may be viewed as a generalisation of the classical Riemann–Hurwitz formula.

The concepts of reduced multiplicities and excesses also lead to a better inequality in (1). We consider polynomials $f_1, \dots, f_p$ of highest degree d and linear combinations $g_1, \dots, g_q$ any p of which are linearly independent and prove that

$$(q - p)d \leq \sum_{i=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \nu_{p-1}(g_i, z) - p(p-1).$$

The function $\nu_{p-1}(g_i, z)$ is given in Definition 2.9 and is equal to zero except at points z_0 which are the zeros of the Wronskian W. The theories of Nevanlinna and Cartan are analytic in nature with a large role being played by the zeros of the Wronskian. Our theorems clarify completely why this must be so and how the zeros of the Wronskian should be taken into account to get the best results, which in some cases amount to identities. Next, we deal with entire functions following the classical method of Cartan as given, for example in [4].

The Cartan theory has, in a way, been generalised to holomorphic curves in algebraic varieties in $\mathbb{CP}^n$ but with lower bounds obtained only for a counting function for intersections of the curve with subvarieties where the multiplicities have either not been reduced at all or, in some cases, reduced less than in the classical Cartan theory. The question of the manner in which one should or could truncate the counting function is one of the central topics in many recent works in the area. We believe that our new counting function emphasizing as it does why the zeros of the Wronskian play such a crucial role, provides the correct generalisation that should give the best results also in the more general situation of algebraic varieties.

In addition to the Nevanlinna theory, there is also the classical theory of Ahlfors which considers coverings instead of counting functions for points. Although a theory similar to that of Cartan has not been developed for this it would appear that the reduced counting function and the characteristic exponents, also defined below, should play an important part in this theory. It is not clear, however what topological invariants should take the role played in the Ahlfors theory by the Euler characteristic.

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2. Zeros of linear combinations of the functions $f_1, \dots, f_p$

Let $f_1, \dots, f_p$ be linearly independent entire functions such that for each $z \in \mathbb{C}$, at least one of the functions f_j does not vanish, and let g be a non-trivial linear combination of $f_1, \dots, f_p$. Suppose that $z_0 \in \mathbb{C}$. The condition that $g(z_0) = 0$, ignoring the multiplicity of the zero, can always be satisfied by choosing the numbers a_j, not all zero, in an appropriate way. We can ensure that not all a_j are zero since by assumption, at least one of the numbers $f_j(z_0)$ is non-zero.

More generally, suppose that we look for a linear combination g that has a zero of order (exactly) k at z_0, where $1 \le k \le p-1$. In the sequel, when we say that a function has a zero of order k at a point, we mean that the order of the zero is exactly k. This amounts to requiring that the following k linear equations are satisfied by the variables $a_1, \ldots, a_p$, not all equal to zero:

$$\sum_{j=1}^{p} a_j f_j^{(m)}(z_0) = 0, \qquad 0 \le m \le k-1. \tag{3}$$

In addition, we require that

$$\sum_{j=1}^{p} a_j f_j^{(k)}(z_0) \neq 0. \tag{4}$$

We now analyse whether or not this can be achieved.

Recall that the Wronskian determinant $W = W(z_0)$ at the point z_0 is defined to be the determinant of the $p \times p$ matrix $A = A(z_0)$ whose $i,j-$entry, for $1 \le i,j \le p$ is given by

$$f_j^{(i-1)}(z_0).$$

Suppose first that $W(z_0) \neq 0$. Then the rank of the matrix A is p, so that any k of its rows are linearly independent, if $1 \le k \le p$. In particular, the first k rows are linearly independent, which implies that the linear system (3) has a non-zero solution $a_1, \ldots, a_p$ if $1 \le k \le p-1$. Since row $k+1$ of A is linearly independent of the first k rows, it is possible to choose this solution so that the inequality (4) holds.

We conclude that at each point z_0 at which $W(z_0) \neq 0$, for each k with $1 \le k \le p-1$, there is a non-trivial linear combination g of the functions $f_1, \ldots, f_p$ that has a zero of order k at z_0.

2.1. **Characteristic exponents.** In order to discuss the situation at points z_0 where the Wronskian W vanishes, it is convenient to make some preliminary observations that apply to every point.

At each point z_0, by assumption, there is j with $1 \le j \le p$ such that $f_j(z_0) \neq 0$. Let us choose one such j and, if necessary, multiply f_j by a non-zero complex constant depending on z_0, to obtain a function, say g_0, such that $g_0(z_0) = 1$.

Suppose that there are other functions f_i, where $i \neq j$, that do not vanish at z_0. We replace each $f_i(z)$ by the function $f_i(z) - f_i(z_0)g_0(z)$, then multiply this new function by a suitable non-zero complex constant depending on z_0, and obtain a function of the form

$$(z-z_0)^m + O((z-z_0)^{m+1})$$

at z_0. Here $m \ge 1$, and m depends on the function f_i. The effect of the last multiplication of the function by a suitable non-zero complex constant has been to arrange so that the coefficient of $(z-z_0)^m$ is equal to 1.

At this point, ignoring the initial f_j and the resulting function g_0, we have $p-1$ functions, each being either one of the original functions f_k or one of the new linear combinations of the functions $f_1, \ldots, f_p$. It is clear from the process used that these $p-1$ functions together with g_0 form a set of p linearly independent functions, since the original functions $f_1, \ldots, f_p$ were assumed to be linearly independent.

Among these $p-1$ functions, each of which vanishes at z_0, there is a function whose order of zero at z_0 is minimal. We multiply this function by a suitable non-zero complex constant, if necessary (this will be necessary perhaps only if the function is one of the original functions that has not yet been modified in any way), so that the resulting function, say g_1, can be written as

$$(z-z_0)^{d_1} + O((z-z_0)^{d_1+1})$$

at z_0. Here $d_1 \geq 1$. If among the other $p-2$ functions h there are any that also have a zero of order d_1 at z_0, we replace each of them by $h - \lambda g_1$ for a suitable $\lambda \in \mathbb{C} \setminus \{0\}$ depending on h, so that each of these new functions has a zero of order at least $d_1 + 1$ at z_0.

We now continue in this way, obtaining distinct non-negative integers d_j for $0 \leq j \leq p-1$ such that

$$d_0 = 0 < 1 \leq d_1 < d_2 < \cdots < d_{p-1},$$

and linearly independent functions g_j for $0 \leq j \leq p-1$ such that each g_j is a linear combination of the functions $f_1, \ldots, f_p$ and such that

$$g_j(z) = (z-z_0)^{d_j} + O((z-z_0)^{d_j+1})$$

at z_0. In particular, none of the functions g_j vanishes identically.

Definition 2.1. We call the integers d_j for $0 \leq j \leq p-1$, as obtained above, the *characteristic exponents* of $(f_1, \ldots, f_p)$ at z_0.

If we wish to emphazise dependence on the point z_0, we may also denote d_j by $d_j(z_0)$.

Remark 1. *We note that the concept that we call characteristic exponents is by no means new. For example, the same exponents are pointed out in the book by Griffiths and Harris* [3]*, p. 266, in the context of writing down the normal form at a point of a curve whose components are all polynomials. An examination of the fundamental paper of Ahlfors* [1]*, pp. 5–6, shows that these exponents are mentioned, at least implicitly, there also. In both cases, the concept is obtained by discussing the so-called associated curves of* $f = (f_1, \ldots, f_p)$*, defined in terms of the derivatives* $(f_1^{(i)}, \ldots, f_p^{(i)})$ *for* $i \geq 1$.

Now let g be any non-trivial linear combination of $f_1, \ldots, f_p$. Then g can also be expressed as a linear combination of the functions g_j for $0 \leq j \leq p-1$, say as

$$g = \sum_{j=0}^{p-1} a_j g_j.$$

Let m be the smallest integer j such that $a_j \neq 0$. Then g has a zero of order d_m at z_0 (which we interpret to mean that $g(z_0) \neq 0$ if $m = 0$). Since some a_j must be non-zero, this implies that if $g(z_0) = 0$, then the order of the zero of g at z_0 must be one of the numbers $d_1, \ldots, d_{p-1}$.

Our previous analysis at a point z_0 where $W(z_0) \neq 0$ shows that then $d_j = j$ for all j with $1 \leq j \leq p-1$. Thus, then, $d_j(z_0) = j$ for $0 \leq j \leq p-1$.

Consider now an arbitrary point z_0. If we replace the $p-$tuple $(f_1, \ldots, f_p)$ by $(g_0, \ldots, g_{p-1})$, then the Wronskian determinant is only multiplied by a non-zero constant, which does not affect the location and order of the zeros of W. Therefore, for our purposes, we may now consider the Wronskian in the form

$$\det |g_j^{(i)}| = \det |d_j(d_j-1)\cdots(d_j-(i-1))(z-z_0)^{d_j-i} + O((z-z_0)^{d_j-i+1})|,$$

where $0 \leq i, j \leq p-1$.

If we perform a formal expansion of this determinant according to the powers of $z - z_0$, we see that there will be a term of the form $\alpha(z-z_0)^b$, where the integer b is minimal for formal reasons, while α is a complex number of which it is yet to be determined whether or not it vanishes.

Since for each term in the expansion of the determinant we are taking exactly one term from each row and from each column, we see that

$$b = \sum_{j=0}^{p-1} d_j - \sum_{i=0}^{p-1} i = \sum_{j=1}^{p-1} d_j - \frac{p(p-1)}{2}.$$

The number α is equal to the determinant

$$\det |d_j(d_j-1)\cdots(d_j-(i-1))|_{i,j=0}^{p-1}$$

This is true even if some of the entries of this determinant were equal to zero. In the first row of this determinant, each entry is equal to 1. In the second row, the general entry is d_j, and in the third row, the general entry is $d_j(d_j-1) = d_j^2 - d_j$. Without changing the value of the determinant, we may replace the third row by the sum of the second and third rows, causing the new third row to have the general entry d_j^2. In the same way, we may cause the general entry of row k to be d_j^{k-1}. We see that the determinant is equal to the Vandermonde determinant of the distinct numbers $d_0, d_1, \ldots, d_{p-1}$, and hence not equal to zero. Thus $\alpha \neq 0$, so that $\alpha(z-z_0)^b$ is indeed the lowest order term in W at z_0. We conclude that the order of the zero of W at z_0 is b. Since $1 \leq d_1 < d_2 < \cdots < d_{p-1}$, we have $d_j \geq j$ for all j. Thus $b = 0$ if, and only if, we have $d_j = j$ for all j. This recaptures the result that we obtained before, to the effect that if $W(z_0) \neq 0$, then it is possible for a linear combination g of $f_1, \ldots, f_p$ to have a zero of order j at z_0, for each j with $1 \leq j \leq p-1$. We now see that if $W(z_0) \neq 0$, then it is not possible for such a g to have a zero of order $\geq p$ at z_0.

In contrast, when $W(z_0) = 0$, so that $b > 0$, then we necessarily have $d_j > j$ for at least one value of j with $1 \leq j \leq p-1$ and hence, in particular, $d_{p-1} \geq p$, so that then there are non-trivial linear combinations g of $f_1, \ldots, f_p$ with a zero of order $\geq p$ at z_0.

It is convenient to record some of these observations as a lemma.

Lemma 2.2. *If $W(z_0) = 0$, then the order of the zero of W at z_0 is equal to*

$$\sum_{j=1}^{p-1} d_j(z_0) - \frac{p(p-1)}{2}.$$

Definition 2.3. Suppose that a non-trivial linear combination g of $f_1, \ldots, f_p$ has a zero of order $m \geq 1$ at z_0, so that $m = d_j$ for a unique j with $1 \leq j \leq p-1$. We say that j is the *reduced multiplicity* of the zero of g at z_0, and that

$$\mathcal{E}(g, z_0) \equiv d_j - j \geq 0 \tag{5}$$

is the *excess* of g at z_0.

Lemma 2.4. *At any point z_0, the excesses form a (finite) non-decreasing sequence, that is,*

$$\mathcal{E}_j(z_0) = d_j - j \leq d_{j+1} - (j+1) \tag{6}$$

for all j with $0 \leq j \leq p-2$.

For the proof of Lemma 2.4, we note that (6) is equivalent to $d_{j+1} - d_j \geq 1$, which is true since $d_j < d_{j+1}$.

The content of Lemma 2.4 is clear also from the notation used in [3], p. 266, for polynomials, and in [1], pp. 5–6.

Definition 2.5. Let $d_j \geq 1$ be one of the characteristic exponents of $(f_1, \dots, f_p)$ at z_0. We denote by Y_j or $Y_j(z_0)$ the space consisting of the constant function zero together with all non-trivial linear combinations of $f_1, \dots, f_p$ that have a zero of order at least d_j at z_0. Thus the dimension of Y_j as a complex vector space is equal to $p - j$, and $1 \leq p - j \leq p - 1$.

In view of Lemma 2.4, for all $g \in Y_j(z_0) \setminus \{0\}$, the excess is at least $d_j - j$ at z_0.

By the above analysis, the following result is now clear.

Lemma 2.6. *If $W(z_0) \neq 0$, then the excess of every non-trivial linear combination of the functions $f_1, \dots, f_p$ at z_0 is equal to zero.*

If the Wronskian W has a zero of order $m \geq 1$ at z_0, then m is equal to the sum of all the excesses of the non-trivial linear combinations of the functions $f_1, \dots, f_p$ at z_0 in any set of $p - 1$ such linear combinations corresponding to distinct characteristic exponents d_j.

2.2. **Polynomials at the point at infinity.** We proceed to perform a similar analysis at infinity under the assumption that all the functions f_j are polynomials. As pointed out by Gundersen and Hayman in [4], (3.3), p. 436, if we were to start merely with the assumption that each ratio f_i/f_j is a rational function, we would find an entire function h such that every $f_i = e^h F_i$, where each F_i is a polynomial. Since replacing $(f_1, \dots, f_p)$ by $(F_1, \dots, F_p)$ does not change the problems that we are considering, we may assume that such a replacement has been made automatically whenever possible. Thus our considerations also cover the case when each ratio f_i/f_j is a rational function.

This time we begin with the polynomials f_j of highest possible degree. We set

$$d = \max\{\deg f_j \,:\, 1 \leq j \leq p\} \tag{7}$$

and choose j so that $\deg f_j = d$. We let g_1 be equal to λf_j where $\lambda \in \mathbb{C} \setminus \{0\}$ is so chosen that the leading coefficient of g_1 is equal to 1.

Analogously to the previous operation for a finite point z_0, if there are polynomials f_i with $i \neq j$ such that $\deg f_i = d$, we subtract from each of them a suitable multiple of g_1 and then multiply the resulting function by a non-zero complex constant so that the final function has leading coefficient equal to 1 and is of degree $< d$.

Among the $p - 1$ remaining functions, obtained by ignoring g_1, choose one of highest possible degree and multiply it by a non-zero complex number so as to get a function g_2 with leading coefficient equal to 1. If there are other functions remaining of the same degree, add a suitable multiple of g_2 to each of them to get a polynomial of lower degree and then normalize it so that it will have leading coefficient equal to 1.

All functions obtained in this way are non-trivial linear combinations of the original functions $f_1, \dots, f_p$ and the p functions we have at each time are linearly independent. In particular, when we continue this process, we never encounter the constant function zero.

Continuing in this way we find linearly independent polynomials g_j of degree δ_j, say, for $1 \leq j \leq p$, each of leading coefficient 1:

$$g_j(z) = z^{\delta_j} + O(z^{\delta_j - 1}) \tag{8}$$

as $z \to \infty$.

The Wronskian W of $f_1, \dots, f_p$ is equal to the Wronskian of $g_1, \dots, g_p$ multiplied by a non-zero complex constant.

We clearly have

$$d = \delta_1 > \delta_2 > \cdots > \delta_p \geq 0. \tag{9}$$

We get two consequences from these concepts: the definition of characteristic exponents at infinity, and a formula for the degree of the Wronskian W, which is a polynomial when each f_j is a polynomial.

2.2.1. *Characteristic exponents at infinity.* Firstly, we note that since, as points in the projective space $\mathbb{CP}^{p-1}$, the points $(f_1(z),\dots,f_p(z))$ and $(f_1(z)/z^n,\dots,f_p(z)/z^n)$ are equal, for any integer n, whenever $z \neq 0$, we may apply this with $n = d = \delta_1$. Assuming that s is chosen so that $\deg f_s = d$, we find that as $z \to \infty$, the point $(f_1(z)/z^d,\dots,f_p(z)/z^d)$ tends to a point all of whose coordinates are finite and whose $s-$th coordinate (at least) is non-zero, so that it corresponds to a well defined point in $\mathbb{CP}^{p-1}$. We define this point to be the image of $(f_1(z),\dots,f_p(z))$ at $z = \infty$.

The same linear combinations that lead us from $(f_1,\dots,f_p)$ to $(g_1,\dots,g_p)$, will lead us from $(f_1/z^d,\dots,f_p/z^d)$ to $(g_1/z^d,\dots,g_p/z^d)$. In $(g_1/z^d,\dots,g_p/z^d)$, the $j-$th component, for $1 \le j \le p$, is of the form $z^{\delta_j-\delta_1}+O(z^{\delta_j-\delta_1-1})$ as $z \to \infty$. Thus, for $2 \le j \le p$, this component has a zero of order $\delta_1 - \delta_j$ at infinity.

Since any non-trivial linear combination of $f_1,\dots,f_p$ can also be written as a non-trivial linear combination of $g_1,\dots,g_p$, it is clear that every non-trivial linear combination of $f_1,\dots,f_p$ that vanishes at infinity, has a zero at infinity whose order is one of the numbers $\delta_1 - \delta_j$ for $2 \le j \le p$.

Definition 2.7. We call the numbers $d_{j-1}(\infty) = \delta_1 - \delta_j$ for $1 \le j \le p$ the *characteristic exponents* of $(f_1,\dots,f_p)$ at infinity.

Clearly

$$0 = d_0(\infty) < d_1(\infty) < \cdots < d_{p-1}(\infty).$$

Definition 2.8. Let g be a non-trivial linear combination of $f_1,\dots,f_p$. If $g(z)/z^d = az^{-m} + O(z^{-m-1})$ as $z \to \infty$, where $a \neq 0$ and $m \ge 1$, we say that g has a zero of order m at infinity.

Definition 2.9. Let g be a non-trivial linear combination of $f_1,\dots,f_p$. If g has a zero of order $m \ge 1$ at infinity, so that $m = \delta_1 - \delta_j$ for a unique j with $2 \le j \le p$, we call

$$\nu_{p-1}(g,\infty) = j-1 \in \{\,1,2,\dots,p-1\,\}$$

the *reduced multiplicity* of the zero of g at infinity, and we call

$$\mathcal{E}\,(g,\infty) = \delta_1 - \delta_j - (j-1) \tag{10}$$

the *excess* of g at infinity. We also write

$$\mathcal{E}_{\,j}(\infty) = d_j(\infty) - j.$$

If g has an excess $t \ge 1$ at a point $z_0 \in \overline{\mathbb{C}}$, we say that g *realizes* the excess t at z_0, or that the excess t at z_0 is *covered by* g.

We note that the sum of the excesses at infinity, corresponding to the different values of j for $2 \le j \le p$, is equal to

$$(p-1)d - \sum_{j=2}^{p} \delta_j - \frac{p(p-1)}{2}, \tag{11}$$

where, as before, $d = \max\{\deg f_j : 1 \le j \le p\}$.

2.2.2. *The degree of the Wronskian for a polynomial holomorphic curve.* We continue to assume that each f_j is a polynomial. To study the degree of W, we may assume that W has been formed as the Wronskian of the functions $g_1, \ldots, g_p$ as defined above making use of the point at infinity.

By (8), we may express W as a $p \times p$ matrix A whose rows and columns are indexed by i and j with $0 \leq i \leq p-1$, $1 \leq j \leq p$, such that the $ij-$element of A is equal to

$$\delta_j(\delta_j - 1) \cdots (\delta_j - (i-1)) z^{\delta_j - i} + O(z^{\delta_j - i - 1}).$$

As in the case of a finite point z_0, we deduce that the determinant W can be written as

$$W(z) = \alpha z^b + O(z^{b-1})$$

as $z \to \infty$, where $\alpha \neq 0$ (since the numbers δ_j are all distinct) and

$$\deg W = b = \sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2}.$$

For future reference, we formalize this result as a lemma.

Lemma 2.10. *If all the functions f_j, for $1 \leq j \leq p$, are polynomials, then their Wronskian determinant W is a polynomial of degree*

$$\sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2} \tag{12}$$

where the numbers δ_j are as defined above.

2.3. **Counting functions for zeros of linear combinations.** Let $f_1, \ldots, f_p$ be linearly independent entire functions, not all of them vanishing at any point. Let g be a non-trivial linear combination of $f_1, \ldots, f_p$, and suppose that g has a zero of order $d_j \geq 1$ at a point $z_0 \in \mathbb{C}$. Thus j is the reduced multiplicity of g at z_0.

Definition 2.11. We denote the order of the zero of g at $z_0 \in \overline{\mathbb{C}}$ by $n(g, z_0)$; when $z_0 = \infty$, we use this notation only if the functions $f_1, \ldots, f_p$ are all polynomials.

We denote the reduced multiplicity of g at $z_0 \in \mathbb{C}$ by $\nu_{p-1}(g, z_0)$, that is,

$$\nu_{p-1}(g, z_0) = j.$$

If the functions $f_1, \ldots, f_p$ are all polynomials, we denote the reduced multiplicity of g at infinity by $\nu_{p-1}(g, \infty)$, that is,

$$\nu_{p-1}(g, \infty) = j - 1$$

if $n(g, \infty) = \delta_1 - \delta_j$, where $2 \leq j \leq p$.

Remark 2. *It follows from the definitions that whenever g is a non-trivial linear combination of $f_1, \ldots, f_p$ and $z_0 \in \overline{\mathbb{C}}$, we have*

$$n(g, z_0) = \nu_{p-1}(g, z_0) + \mathcal{E}(g, z_0). \tag{13}$$

The counting function ν_{p-1} is our new counting function for the zeros of g. We note that in order to evaluate $\nu_{p-1}(g, z_0)$, it is necessary to know more than just the actual order of the zero of g at z_0. It is, on the other hand, not necessary to know anything about any other specific linear combination at z_0. Of course, this is distinct from the fact that to know the entire set of the numbers d_k, which is necessary for knowing the location of the number d_j in this set, one must know something about the behaviour of *the set of all linear combinations* of the functions $f_1, \dots, f_p$ at z_0.

If $g(z_0) \neq 0$, we write $\nu_{p-1}(g, z_0) = 0$.

For $r > 0$, we denote the new non-integrated counting function of the zeros of g by

$$\nu_{p-1}(r, g) = \sum_{|z| \leq r} \nu_{p-1}(g, z),$$

where the sum is finite since g has only finitely many zeros of modulus $\leq r$.

Even though the term "reduced counting function" has been used before in the literature in various meanings, and often to mean a quantity obtained by considering $\min\{n(g, z_0), p-1\}$, we shall, for lack of a better term, call $\nu_{p-1}(r, g)$ the *(non-integrated) reduced counting function* of the zeros of g.

Definition 2.12. We write

$$\mathcal{N}_{p-1}(r, g) = \int_0^r \frac{\nu_{p-1}(t, g) - \nu_{p-1}(0, g)}{t} \, dt + \nu_{p-1}(0, g) \log r$$

and call $\mathcal{N}_{p-1}(r, g)$ the *integrated reduced counting function* of the zeros of g.

It could be argued that the notations $\nu_{p-1}(g, z_0)$, $\nu_{p-1}(r, g)$, and $\mathcal{N}_{p-1}(r, g)$ do not adequately indicate the fact that the values of these functions do not depend on g, p, and z_0 alone but also on the properties of the entire set $f_1, \dots, f_p$ at z_0. However, in spite of this, we do not want to adopt an even longer notation.

If g has a zero of order $k \geq 1$ at $z_0 \in \overline{\mathbb{C}}$, we clearly have

$$\nu_{p-1}(g, z_0) \leq \min\{k, p-1\},$$

but it is possible to have $\nu_{p-1}(g, z_0) < p - 1$ even if we were to have $k \geq p$. It depends on the details of the situation whether this occurs or not.

It seems that in earlier literature, the authors have used exclusively the counting function

$$n_{p-1}(g, z_0) = \min\{k, p-1\},$$

which counts a zero at least as many times as $\nu_{p-1}(g, z_0)$, and sometimes a strictly greater number of times; and the corresponding non-integrated counting function

$$n_{p-1}(r, g) = \sum_{|z| \leq r} n_{p-1}(g, z)$$

and, by analogy with the Nevanlinna theory, the integrated counting function

$$N_{p-1}(r, g) = \int_0^r \frac{n_{p-1}(t, g) - n_{p-1}(0, g)}{t} \, dt + n_{p-1}(0, g) \log r.$$

One of our contributions is to show that one can obtain the same inequalities as in the classical Cartan theory by replacing the counting functions $n_{p-1}(r, g)$ and $N_{p-1}(r, g)$ used there by $\nu_{p-1}(r, g)$ and $\mathcal{N}_{p-1}(r, g)$.

3. The inequality for the new counting function

When at least one of the functions $f_1, \dots, f_p$ is transcendental, and this cannot be changed by replacing $(f_1, \dots, f_p)$ by $(hf_1, \dots, hf_p)$ for some zero-free entire function, it makes sense to consider $(f_1, \dots, f_p)$ only in a disk of radius r at any one time, so that the number of zeros that one counts would be finite. We will do this when we improve Cartan's theory in Section 7.

Suppose for the time being that each f_j is a polynomial. Thus we may consider $f_1, \dots, f_p$ on the entire Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ at once. In this case we obtain the following inequality.

Theorem 3.1. Inequality for excesses realized by functions satisfying a linear independence condition. *Suppose that $p \geq 2$, and let $f_1, \dots, f_p$ be polynomials in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z \in \mathbb{C}$, at least one f_j assumes a non-zero value. Set $d = \max\{\deg f_j : 1 \leq j \leq p\}$.*

Suppose that $q \geq 1$, and let $g_1, g_2, \dots, g_q$ be non-trivial linear combinations of $f_1, \dots, f_p$ such that any $\min\{p, q\}$ of $g_1, g_2, \dots, g_q$ are linearly independent.

Then

$$(q-p)d \leq \sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \nu_{p-1}(g_j, z) - p(p-1). \tag{14}$$

Equivalently, we have

$$\sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \mathcal{E}\,(g_j, z) \leq pd - p(p-1), \tag{15}$$

where the excesses are defined by (5) and (10), respectively.

Remark 3. *For results such as Theorem 3.1, it has been customary to assume that $q > p$ so that the left hand side of (14) would be positive. Due to the presence of the term $-p(p-1)$, the inequality (14) might be non-trivial in some situations even if we were to have $q \leq p$. The inequality (15) does not involve q on the right hand side and hence may be worth stating for any value of q.*

Theorem 3.1 has its basis in the following identity, which refers to the individual excesses without a reference to how they might be realized by functions.

Theorem 3.2. Identity for excesses. *Suppose that $p \geq 2$, and let $f_1, \dots, f_p$ be polynomials in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z \in \mathbb{C}$, at least one f_j assumes a non-zero value. Set $d = \max\{\deg f_j : 1 \leq j \leq p\}$.*

Then

$$\sum_{z \in \overline{\mathbb{C}}} \sum_{j=1}^{p-1} \mathcal{E}_j(z) = \sum_{z \in \overline{\mathbb{C}}} \sum_{j=1}^{p-1} (d_j(z) - j) = pd - p(p-1). \tag{16}$$

As mentioned in the Introduction, when $p = 2$, the identity (16) reduces to the Riemann-Hurwitz formula for rational functions.

3.1. **Proof of Theorem 3.2.** Let the assumptions of Theorem 3.2 be satisfied. We have shown earlier that at each finite z_0, the sum of the excesses is equal to the order of the zero of the Wronskian W at z_0 (and that in particular, we have $\sum_{j=1}^{p-1}(d_j(z_0) - j) = 0$ if, and only if, $W(z_0) \neq 0$). Thus

$$\sum_{z \in \mathbb{C}} \sum_{j=1}^{p-1} (d_j(z) - j) = \deg W = \sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2}$$

by (12) in Lemma 2.10.

By (11), we have

$$\sum_{j=1}^{p-1}(d_j(\infty)-j)=(p-1)d-\sum_{j=2}^{p}\delta_j-\frac{p(p-1)}{2}.$$

Hence

$$\begin{aligned}\sum_{z\in\overline{\mathbb{C}}}\sum_{j=1}^{p-1}(d_j(z)-j)&=\sum_{j=1}^{p}\delta_j-\frac{p(p-1)}{2}+(p-1)d-\sum_{j=2}^{p}\delta_j-\frac{p(p-1)}{2}\\&=\delta_1+(p-1)d-p(p-1)=pd-p(p-1)\end{aligned}$$

since $\delta_1=d$.

This completes the proof of Theorem 3.2.

3.2. **Lemmas for the proof of Theorem 3.1.** To structure the proof of Theorem 3.1 better, it is convenient to formulate the following lemmas.

Lemma 3.3. *Suppose that $p\ge 2$, and let $f_1,\dots,f_p$ be polynomials in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z\in\mathbb{C}$, at least one f_j assumes a non-zero value. Set $d=\max\{\deg f_j : 1\le j\le p\}$.*

Suppose that $q\ge 1$, and let $g_1,g_2,\dots,g_q$ be non-trivial linear combinations of $f_1,\dots,f_p$ such that any $\min\{p,q\}$ of $g_1,g_2,\dots,g_q$ are linearly independent.

Suppose that the Wronskian determinant W of $f_1,\dots,f_p$ has a zero of order $m\ge 1$ at $z_0\in\mathbb{C}$.

Then

$$\sum_{j=1}^{q}\mathcal{E}(g_j,z_0)\le m. \tag{17}$$

At infinity, we have

$$\sum_{j=1}^{q}\mathcal{E}(g_j,\infty)\le(p-1)d-\sum_{j=2}^{p}\delta_j-\frac{p(p-1)}{2}. \tag{18}$$

Lemma 3.4. *Suppose that $p\ge 2$, and let $f_1,\dots,f_p$ be polynomials in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z\in\mathbb{C}$, at least one f_j assumes a non-zero value. Set $d=\max\{\deg f_j : 1\le j\le p\}$.*

Let g be a non-trivial linear combinations of $f_1,\dots,f_p$.

Then

$$d-\sum_{z\in\overline{\mathbb{C}}}\nu_{p-1}(g,z)=\sum_{z\in\overline{\mathbb{C}}}\mathcal{E}(g,z). \tag{19}$$

3.3. **Proof of Lemma 3.3.** Let the assumptions of Lemma 3.3 be satisfied.

Recall that for $0\le j\le p-1$, the space $Y_j=Y_j(z_0)$ consists of the constant function zero together with those non-trivial linear combinations of $f_1,\dots,f_p$ that have a zero of order at least $d_j(z_0)$ at z_0. We have $\dim Y_j(z_0)=p-j\le p-1$. Thus, for each j with $1\le j\le p-1$, at most $p-j$ of the functions g_k can lie in Y_j. This is clear if $q\le p-j$. If more than $p-j$ of the functions g_k lie in Y_j while $q>p-j$, then certain $p-j+1$ functions g_k with

$p-j+1 \le \min\{p,q\}$ lie in Y_j, and they are thus linearly dependent. This contradicts our assumption that any $\min\{p,q\}$ of the functions $g_1,\dots,g_q$ are linearly independent.

The union of Y_j for $1 \le j \le p-1$ is Y_1, which can therefore contain at most $p-1$ of the functions g_k. Hence at most $p-1$ of the functions g_k can vanish at z_0, and the remaining functions g_k cannot contribute to the excesses at z_0.

Suppose that actually $\ell \le p-1$ of the functions g_k lie in Y_1, and label them as h_s, for $1 \le s \le \ell$, and assume that $h_1,\dots,h_{r_{p-1}}$ lie in Y_{p-1}, $h_{r_{p-1}+1},\dots,h_{r_{p-2}}$ lie in $Y_{p-2} \setminus Y_{p-1}$, and so on. If $r_{j+1}=r_j$, this means that no function g_k lies in $Y_j \setminus Y_{j+1}$. The functions that lie in a certain Y_j are precisely the functions $h_1,\dots,h_{r_j}$, so that $r_j \le p-j$. Also $r_1=\ell$. Set $r_p=0$. Thus

$$\begin{aligned}
&\sum_{j=1}^{q} \mathcal{E}\,(g_j,z_0) = \sum_{j=1}^{p-1}(r_j-r_{j+1})(d_j-j) = \sum_{j=1}^{p-1} r_j(d_j-d_{j-1}-1)\\
&\le \sum_{j=1}^{p-1}(p-j)(d_j-d_{j-1}-1) = p\sum_{j=1}^{p-1}(d_j-d_{j-1}) - p(p-1) - \sum_{j=1}^{p-1} j(d_j-d_{j-1}) + \frac{p(p-1)}{2}\\
&= pd_{p-1} - \frac{p(p-1)}{2} - \sum_{j=1}^{p-1} j(d_j-d_{j-1})\\
&= pd_{p-1} - \frac{p(p-1)}{2} - \left((p-1)d_{p-1} - \sum_{j=1}^{p-2} d_j - d_0\right)\\
&= \sum_{j=1}^{p-1} d_j - \frac{p(p-1)}{2} = m
\end{aligned}$$

by Lemma 2.2 and the fact that $d_0=0$. This proves (17).

At infinity, the same proof as above applies, except that each number d_j must be replaced by $\delta_1-\delta_{j+1}$, for $0 \le j \le p-1$. Thus the upper bound

$$\sum_{j=1}^{p-1} d_j - \frac{p(p-1)}{2}$$

above is replaced by

$$\sum_{j=1}^{p-1}(\delta_1-\delta_{j+1}) - \frac{p(p-1)}{2} = (p-1)\delta_1 - \sum_{j=2}^{p}\delta_j - \frac{p(p-1)}{2},$$

which yields (18) since $\delta_1 = d$.

This completes the proof of Lemma 3.3.

3.4. **Proof of Lemma 3.4.** By (13), we have

$$\sum_{z\in\overline{\mathbb{C}}} n(g,z) = \sum_{z\in\overline{\mathbb{C}}} \nu_{p-1}(g,z) + \sum_{z\in\overline{\mathbb{C}}} \mathcal{E}\,(g,z). \tag{20}$$

Set $\deg g = \delta$, so that $0 \le \delta \le d$. Then

$$\delta = \sum_{z\in\mathbb{C}} n(g,z).$$

Furthermore, since $g(z)/z^d = az^{d-\delta} + O(z^{d-\delta-1})$ for some $a \in \mathbb{C} \setminus \{0\}$ as $z \to \infty$, it follows that $n(g, \infty) = d - \delta \geq 0$. Thus

$$d = \sum_{z \in \overline{\mathbb{C}}} n(g, z). \tag{21}$$

Combining (20) and (21) we obtain (19). This completes the proof of Lemma 3.4.

3.5. **Proof of Theorem 3.1.** Let the assumptions of Theorem 3.1 be satisfied. Let g_j have degree $t_j \geq 0$. The assumptions imply that no g_j vanishes identically and at most one g_j can be a (non-zero) constant function.

By Lemma 3.4 we have

$$qd - \sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \nu_{p-1}(g_j, z) = \sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \mathcal{E}\,(g_j, z).$$

Consider any finite point z_0 such that some g_j vanishes at z_0. This can happen only if the Wronskian W of $f_1, \ldots, f_p$ vanishes at z_0. Thus, also interchaging the order of summation, we get

$$\sum_{j=1}^{q} \sum_{z \in \mathbb{C}} \mathcal{E}\,(g_j, z) \leq \sum_{W(z)=0} \sum_{j=1}^{q} \mathcal{E}\,(g_j, z).$$

Since any $\min\{p, q\}$ of the functions $g_1, \ldots, g_q$ are linearly independent, it follows from Lemma 3.3 that whenever $W(z) = 0$ and W has a zero of order m at z, we have

$$\sum_{j=1}^{q} \mathcal{E}\,(g_j, z) \leq m.$$

Summing over all zeros z of W we see that the sum of the numbers m does not exceed

$$\deg W = \sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2},$$

where we have used (12).

By (18), we have

$$\sum_{j=1}^{q} \mathcal{E}\,(g_j, \infty) \leq (p-1)d - \sum_{j=2}^{p} \delta_j - \frac{p(p-1)}{2}.$$

Combining these results, we see that

$$\begin{aligned} &\sum_{W(z)=0} \sum_{j=1}^{q} \mathcal{E}\,(g_j, z) + \sum_{j=1}^{q} \mathcal{E}\,(g_j, \infty) \leq \sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2} + (p-1)d - \sum_{j=2}^{p} \delta_j - \frac{p(p-1)}{2} \\ &= \delta_1 + (p-1)d - p(p-1) = pd - p(p-1) \end{aligned}$$

since $\delta_1 = d$.

These inequalities prove both (14) and (15), so that the proof of Theorem 3.1 is now complete.

4. The identity for the new counting function

We consider briefly the cases when equality is attained in the inequality (14).

Theorem 4.1. Identity for distinct excesses realized by functions. *Suppose that $p \geq 2$, and let $f_1, \dots, f_p$ be polynomials in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z \in \mathbb{C}$, at least one f_j assumes a non-zero value. Set $d = \max\{\deg f_j : 1 \leq j \leq p\}$.*

Suppose that $q \geq 1$, and let $g_1, g_2, \dots, g_q$ be non-trivial linear combinations of $f_1, \dots, f_p$. Suppose that for each (finite) zero z_0 of the Wronskian determinant W of $f_1, \dots, f_p$, and for each characteristic exponent $d_j(z_0) \geq 1$, $1 \leq j \leq p-1$, there is exactly one function among $g_1, \dots, g_q$ that has a zero of order $d_j(z_0)$ at z_0. Suppose, in addition, that for each characteristic exponent $d_j(\infty) \geq 1$ at infinity, $1 \leq j \leq p-1$, there is exactly one function among $g_1, \dots, g_q$ that has a zero of order $d_j(z_0)$ at infinity.

Then

$$(q-p)d = \sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \nu_{p-1}(g_j, z) - p(p-1), \tag{22}$$

and, equivalently, we have

$$\sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \mathcal{E}\,(g_j, z) = pd - p(p-1). \tag{23}$$

Considering Theorem 3.1 and its proof we see that Theorem 4.1 is obvious, since each excess is used exactly once. Note that in Theorem 4.1, there is no assumption concerning the linear independence of the functions g_j. Obviously there has to be a sufficient number of the functions g_j so that each excess would be exactly realized by some function g_j. Also there cannot be too many functions g_j of the same nature, since each excess is realized only once.

The question now arises as to whether for a given set of polynomials $f_1, \dots, f_p$ it is possible to realize this identity. One can certainly find a set of functions g_j such that every excess is realized by at least one of them. A problem arises if it is not possible to cover every excess in such a way that then no excess is covered more than once. In Section 5, we will see that there are situations where such precise coverage is not possible.

On the other hand, even in such a situation, the equalities (22) and (23) may hold coincidentally. For example, a certain excess may not be covered at all, and another excess of the same size may be covered exactly twice, while all excesses other than these two are covered exactly once. An example in Section 5 will illustrate this possibility also.

5. Examples

To see how the concepts and results discussed above work in specific cases, we consider several examples based on the same set-up.

We set $p = 3$, $f_1 = 1$, $f_2 = z^3(z-1)^3$, and $f_3 = z^3(z+1)^3$. Then $d = 6$. The Wronskian determinant of f_1, f_2, f_3 is equal to

$$W(z) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = -36z^4(z-1)(z+1)(5z^2-2).$$

Thus W has a zero of order 4 at the origin, and a simple zero at each of the points ± 1 and $\pm\sqrt{2/5}$. We have $\deg W = 8$.

The function

$$f_2 + f_3 = 2z^4(z^2+3)$$

has a zero of order 4 at the origin. Each of the functions f_2 and f_3 has a zero of order 3 at the origin. The sequence of all possible orders of a zero at the origin, attainable by any non-trivial linear combination of f_1, f_2, f_3 and including the number zero (corresponding to a linear combination not having a zero at the origin) must be of the form $(0, d_1, d_2)$ where $1 \le d_1 < d_2$. We see that we must have $d_1 = 3$ and $d_2 = 4$. This gives rise to the excesses $d_1 - 1 = 2$ and $d_2 - 2 = 2$, making the total excess equal to 4. Thus the total excess is equal to the order of the zero of W at the origin, in accordance with the theory.

At the point $z = 1$, the total excess is 1. This can only be if $(d_0, d_1, d_2) = (0, 1, 3)$. The excess is realized by the function f_2, which has a zero of order 3 at $z = 1$.

Similarly, at the point $z = -1$, the only excess arises from the function f_3 (and its non-zero constant multiples), which has a zero of order 3 at $z = -1$.

At each of the points $z = \pm\sqrt{2/5}$, the total excess is 1, so that $(d_0, d_1, d_2) = (0, 1, 3)$. A zero of order 3 is realized by the function

$$F_1(z) = 24(8\sqrt{10} + 25)f_1 + 125(80\sqrt{10} + 253)f_2 - 375f_3$$

at $z = \sqrt{2/5}$, and by the function

$$F_2(z) = 24(8\sqrt{10} - 25)f_1 + 125(80\sqrt{10} - 253)f_2 + 375f_3$$

at the point $z = -\sqrt{2/5}$.

At infinity, the function f_1 has a zero of order 6. The function $f_2 - f_3$ is a polynomial of degree 5 and therefore has a simple zero at infinity. Thus $(\delta_1, \delta_2, \delta_3) = (6, 5, 0)$, and $(\delta_1 - \delta_1, \delta_1 - \delta_2, \delta_1 - \delta_3) = (0, 1, 6)$. There is only one excess at infinity, realized by the function f_1, and it is equal to $6 - 2 = 4$.

The sum of the distinct excesses is therefore

$$2 + 2 + 1 + 1 + 1 + 1 + 4 = 12,$$

in accordance with Theorem 3.2.

For any q non-trivial linear combinations $g_1, \ldots, g_q$ of f_1, f_2, f_3 such that any $\min\{p, q\}$ of them are linearly independent, Theorem 3.1 gives

$$\sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \mathcal{E}\,(g_j, z) \le pd - p(p-1) = 18 - 6 = 12.$$

We use this basic set-up in all the examples that now follow.

Example 5.1. We choose $q = 5$ and take the functions g_j to be the functions f_1, f_2, f_3, F_1, F_2. Then the sum of all possible excesses for these functions is $4 + 2 + 2 + 1 + 1 + 1 + 1 = 12$. These five functions have the property that any three of them are linearly independent, so that by Theorem 3.1, the total excess is at most 12. That the total excess is equal to 12 is due to a coincidence. The excess equal to 2 at the origin is counted twice, for both f_2 and f_3. On the other hand, the excess equal to 2 for functions that have a zero of order 4 at the origin is not taken into account at all, since that can happen only if among the functions we consider, we have a non-zero constant multiple of $f_2 + f_3$. Each other excess is taken into account exactly once. Counting a quantity equal to 2 twice but omitting a quantity equal to 2 have effects that cancel out each other. Hence the total excess is still equal to 12.

Example 5.2. We choose $q = 6$ and take the functions g_j to be the functions $f_1, f_2, f_3, f_2 + f_3, F_1, F_2$. Then the sum of all possible excesses for these functions is $4+2+2+2+1+1+1+1 = 14 > 12$. These six functions do not have the property that any three of them are linearly independent (since $f_2, f_3, f_2 + f_3$ are linearly dependent), so that Theorem 3.1 cannot be applied, and one can certainly expect it to be the case that the total excess is > 12. This indeed happens since each excess is counted at least once, and the excess, equal to 2, for functions that have a zero of order 3 at the origin is counted twice, due to the presence of both f_2 and f_3.

This choice of functions is not unreasonable since both f_2 and f_3 must be included if the excesses at the points 1 and -1 are to be counted. On the other hand, it is then unavoidable that the excess, equal to 2, for functions that have a zero of order 3 at the origin is counted twice. Next, it is not possible to include the excess for functions that have a zero of order 4 at the origin without including (a non-zero constant multiple of) $f_2 + f_3$. Hence if all excesses are to be included, we are forced to give up the property that any three of the functions chosen are linearly independent. Giving up this property can understandably lead to certain excesses being counted more than once, and while that need not happen in every such situation, it does happen here.

Example 5.3. We choose $q = 4$ and take the functions g_j to be the functions f_1, f_2, f_3, f_2+f_3. Then the sum of all possible excesses for these functions is $4 + 2 + 2 + 2 + 1 + 1 = 12$. These four functions do not have the property that any three of them are linearly independent. Yet the sum of all the excesses realized by these functions is 12 since excesses at the points $\pm\sqrt{2/5}$ are not taken into account.

Example 5.4. We choose $q = 3$ and take the functions g_j to be the functions $f_2, f_3, f_2 + f_3$. Then the sum of all possible excesses for these functions is $2 + 2 + 2 + 1 + 1 = 8$. These three functions are not linearly independent. Yet the sum of all the excesses realized by these functions is < 12 since excesses at the points $\pm\sqrt{2/5}$ and ∞ are not taken into account.

6. The counting function of unrealized excesses

We start by observing that Lemma 3.3 remains valid, with the same proof, at a finite point z_0 also for entire functions $f_1, \ldots, f_p$ that are not necessarily polynomials.

Lemma 6.1. *Suppose that $p \geq 2$, and let $f_1, \ldots, f_p$ be entire functions in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z \in \mathbb{C}$, at least one f_j assumes a non-zero value.*

Suppose that $q \geq 1$, and let $g_1, g_2, \ldots, g_q$ be non-trivial linear combinations of $f_1, \ldots, f_p$ such that any $\min\{p, q\}$ of $g_1, g_2, \ldots, g_q$ are linearly independent.

Suppose that the Wronskian determinant W of $f_1, \ldots, f_p$ has a zero of order $m \geq 1$ at $z_0 \in \mathbb{C}$.

Then

$$\sum_{j=1}^{q} \mathcal{E}\,(g_j, z_0) \leq m. \tag{24}$$

Theorem 3.1 gives a lower bound for the new counting function of the functions g_j where zeros are taken into account according to their reduced multiplicities. However, in any application one may wonder how sharp that result is. As the theory we have developed makes clear, there is still space in this inequality if not all excesses are realized. Thus, in any specific application it might be worth taking this factor into account, if possible at all. In a theoretical consideration this may sometimes be difficult, while when considering a special case it might be easier to incorporate such an effect.

Therefore it may be worth defining the concept of the counting function of unrealized excesses. The idea is that if any functions g_j are given as in Lemma 6.1 above, then at a point z_0 where the Wronskian W of $f_1, \ldots, f_p$ vanishes, some of the excesses might not be covered by $g_1, \ldots, g_q$, that is, the quantity

$$m - \sum_{j=1}^{q} \mathcal{E}\left(g_j, z_0\right)$$

arising from (24) may be positive. This difference represents the unrealized excesses. It seems simplest to discuss this total amount of unrealized excesses at a point instead of attempting to divide the total among several functions.

Definition 6.2. Suppose that $p \geq 2$, and let $f_1, \ldots, f_p$ be entire functions in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z \in \mathbb{C}$, at least one f_j assumes a non-zero value.

Suppose that $q \geq 1$, and let $g_1, g_2, \ldots, g_q$ be non-trivial linear combinations of $f_1, \ldots, f_p$ such that any $\min\{p, q\}$ of $g_1, g_2, \ldots, g_q$ are linearly independent. Write $\mathcal{G} = (g_1, \ldots, g_q)$.

Suppose that the Wronskian determinant W of $f_1, \ldots, f_p$ has a zero of order $m \geq 1$ at $z_0 \in \mathbb{C}$.

Then we call the number

$$\mathcal{V}\left(\mathcal{G}, z_0\right) = m - \sum_{j=1}^{q} \mathcal{E}\left(g_j, z_0\right) \geq 0 \tag{25}$$

the *total excess at z_0 unrealized by the functions $g_1, \ldots, g_q$.*

We call

$$\mathcal{V}(r, \mathcal{G}) = \sum_{|z| \leq r} \mathcal{V}\left(\mathcal{G}, z\right)$$

the *non-integrated counting function of the unrealized excesses*, and call

$$\mathcal{U}(r, \mathcal{G}) = \int_0^r \frac{\mathcal{V}(t, \mathcal{G}) - \mathcal{V}(0, \mathcal{G})}{t}\, dt + \mathcal{V}(0, \mathcal{G}) \log r$$

the *integrated counting function of the unrealized excesses* for $\mathcal{G}$.

Even though the unrealized excesses depend also on $(f_1, \ldots, f_p)$, we suppress this in the notation.

In view of our earlier results and their proofs, the following statement is immediately obvious.

Theorem 6.3. *Suppose that $p \geq 2$, and let $f_1, \ldots, f_p$ be polynomials in $\mathbb{C}$, linearly independent over the set of complex numbers, such that for each $z \in \mathbb{C}$, at least one f_j assumes a non-zero value.*

Suppose that $q \geq 1$, and let $g_1, g_2, \ldots, g_q$ be non-trivial linear combinations of $f_1, \ldots, f_p$ such that any $\min\{p,q\}$ *of $g_1, g_2, \ldots, g_q$ are linearly independent. With $\mathcal{G} = (g_1, \ldots, g_q)$, we have*

$$(q-p)d = \sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \nu_{p-1}(g_j, z) - \sum_{z \in \overline{\mathbb{C}}} \mathcal{V}(\mathcal{G}, z) - p(p-1) \tag{26}$$

and

$$\sum_{j=1}^{q} \sum_{z \in \overline{\mathbb{C}}} \mathcal{E}\,(g_j, z) + \sum_{z \in \overline{\mathbb{C}}} \mathcal{V}(\mathcal{G}, z) = pd - p(p-1). \tag{27}$$

7. An improved version of the Cartan theory

We now explain how a sharper form of the Cartan theory for the value distribution of holomorphic curves, itself a generalisation of the Nevanlinna theory, can be obtained by using our new counting function for the zeros of linear combination of the given functions. Even though the changes in the proofs from the original form of Cartan's argument are minor, it seems best for the sake of clarity to produce a complete proof. In doing so, we mainly follow the exposition of the Cartan theory given by Gundersen and Hayman in [4] without further reference. We indicate the differences when we come to them.

We make use of the standard notation of the Nevanlinna theory as given, for example, in [5].

Suppose that $p \geq 2$ and that $f_1, \ldots, f_p$ are linearly independent entire functions in $\mathbb{C}$ such that for each $z_0 \in \mathbb{C}$, there is an index j such that $f_j(z_0) \neq 0$.

We define, for $z \in \mathbb{C}$, the real number

$$u(z) = \max\{\log|f(re^{i\theta})| : 1 \leq j \leq p\}.$$

For $r > 0$, we define the Cartan characteristic (function) $T(r)$ of $f = (f_1, \ldots, f_p)$ by

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta - u(0).$$

Suppose that $q > p$, and let $g_1, \ldots, g_p$ be linear combinations of $f_1, \ldots, f_p$, any p of which are linearly independent.

We consider the auxiliary function

$$H = \frac{g_1 g_2 \cdots g_q}{W} \tag{28}$$

where W is the Wronskian determinant of $f_1, \ldots, f_p$.

Now we perform a more careful analysis on the orders of the zeros of H than has been done in previous literature. We write again $\mathcal{G} = (g_1, \ldots, g_q)$.

Both the numerator and denominator of H are entire functions. Thus, if $z_0 \in \mathbb{C}$ is a zero of H, then at least one g_j must have a zero at z_0.

Suppose first that $W(z_0) \neq 0$. Then there are no excesses and if a function g_j vanishes at z_0, then its zero is taken into account in $\nu_{p-1}(r, g_j)$ according to its actual multiplicity. Thus

$$n(H, z_0) = \sum_{j=1}^{q} n(g_j, z_0) = \sum_{j=1}^{q} \nu_{p-1}(g_j, z_0).$$

Suppose then that $W(z_0) = 0$ and let W have a zero of order m at z_0. By (25), we have

$$m = \sum_{j=1}^{q} \mathcal{E}\left(g_j, z_0\right) + \mathcal{V}(\mathcal{G}, z_0) = \sum_{j=1}^{q} n(g_j, z_0) - \sum_{j=1}^{q} \nu_{p-1}(g_j, z_0) + \mathcal{V}(\mathcal{G}, z_0).$$

It follows that if $r > 0$, then

$$N(r, 0, H) = \sum_{j=1}^{q} \mathcal{N}_{p-1}\left(r, g_j\right) - \mathcal{U}(r, \mathcal{G}). \tag{29}$$

In order to estimate the size of an error term, we note, following [2] and [4], Lemma 8.2, p. 449, that for each $z \in \mathbb{C}$ there is a permutation $m_1, \ldots, m_q$ of the integers $1, 2, \ldots, q$ such that

$$|g_{m_1}(z)| \geq |g_{m_2}(z)| \geq \cdots \geq |g_{m_q}(z)|. \tag{30}$$

Moreover there is a fixed positive constant A depending only on the coefficients of the linear combinations that the g_j form in terms of the f_i such that whenever $1 \leq j \leq p$ and $1 \leq \ell \leq q - p + 1$, we have

$$|f_j(z)| \leq A|g_{m_\ell}(z)|. \tag{31}$$

In particular, at least $q - p + 1$ of the functions g_j do not vanish at z_0, which fact we have already noted in the proof of Lemma 3.3.

For each $z \in \mathbb{C}$, we define

$$v(z) = \max\{\, \log|g_{k_1}(z) g_{k_2}(z) \cdots g_{k_{q-p}}(z)| \,:\, 1 \leq k_1 < k_2 < \cdots < k_{q-p} \leq q \,\}$$

so that $v(z)$ is a real number.

If $1 \leq a_1 < a_2 < \cdots < a_{q-p} \leq q$ and $\{b_1, b_2, \ldots b_p\} = \{1, 2, \ldots, q\} \setminus \{a_1, a_2, \ldots, a_{q-p}\}$, then the Wronskian

$$CW(g_{b_1}, \ldots, g_{b_p}) \equiv W = W(f_1, \ldots, f_p)$$

for some non-zero complex constant $C = C(b_1, \ldots, b_p)$ belonging to a certain finite set of numbers since there are only finitely many possibilities for the set $\{b_1, b_2, \ldots b_p\}$. We may thus write, at the point z,

$$H = \frac{g_1 \cdots g_q}{CW(g_{b_1}, \ldots, g_{b_p})} = \frac{g_{a_1} \cdots g_{a_{q-p}}}{CG},$$

where

$$G = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ g'_{b_1}/g_{b_1} & g'_{b_2}/g_{b_2} & \cdots & g'_{b_p}/g_{b_p} \\ \vdots & \vdots & \ddots & \vdots \\ g_{b_1}^{(p-1)}/g_{b_1} & g_{b_2}^{(p-1)}/g_{b_2} & \cdots & g_{b_p}^{(p-1)}/g_{b_p} \end{vmatrix}.$$

In the sequel, note that G depends on the choice of $b_1, \ldots, b_p$, even though we do not indicate this in the notation.

For $z \in \mathbb{C}$, we define

$$w(z) = \max\{\, \log|G(z) C(b_1, \ldots, b_p)| \,:\, 1 \leq b_1 < b_2 < \ldots b_p \leq q \,\}.$$

As in [4], p. 451, it follows that for $r > 0$,

$$\int_0^{2\pi} v(re^{i\theta})\, d\theta = \int_0^{2\pi} \log|H(re^{i\theta})|\, d\theta + \int_0^{2\pi} w(re^{i\theta})\, d\theta.$$

By Jensen's formula,

$$\frac{1}{2\pi}\int_0^{2\pi}\log|H(re^{i\theta})|\,d\theta = N(r,0,H) - N(r,H) + B,$$

where B is a real constant.

For each $z \in \mathbb{C}$, choose for $\{a_1,\ldots,a_{q-p}\}$ the particular set of distinct integers in $\{1,2,\ldots,q\}$ satisfying

$$v(z) = \log|g_{a_1}(z)g_{a_2}(z)\cdots g_{a_{q-p}}(z)|$$

so that, in particular, $g_{a_1}(z)g_{a_2}(z)\cdots g_{a_{q-p}}(z) \neq 0$. Obviously, $\{a_1,\ldots,a_{q-p}\} = \{m_1,\ldots,m_{q-p}\}$, where the numbers m_j were defined in (30).

By (31), we have

$$u(z) \leq \log A + \log|g_{a_s}(z)|$$

whenever $1 \leq s \leq q-p$. Adding up and integrating, we get

$$(q-p)T(r) \leq \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta + O(1).$$

Further, there is a fixed positive constant D, depending only on the coefficients of the linear combinations used to form the g_j in terms of the f_i, such that for each $z \in \mathbb{C}$, we have

$$w(z) \leq D + \max\{\log|G(z)| : 1 \leq b_1 < b_2 < \ldots b_p \leq q\}.$$

As in [4], p. 452, we note that (given the integers $b_1,\ldots,b_p$) the function G does not change if we replace each function $g_{b_j}^{(k)}/g_{b_j}$ in the definition of G by $(g_{b_j}/g_1)^{(k)}/(g_{b_j}/g_1)$, and that this then implies that

$$\int_0^{2\pi} w(re^{i\theta})\,d\theta \leq O(\log r) + \sum_{j=2}^{q} O(\log T(r, g_j/g_1))$$

as $r \to \infty$ outside a set of finite linear measure. Here $T(r,F)$ is the Nevanlinna characteristic of the meromorphic function F.

Combining there results, we find that

$$\begin{aligned}(q-p)T(r) &\leq N(r,0,H) - N(r,H) + \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta})\,d\theta \\ &\leq N(r,0,H) - N(r,H) + O(\log r) + \sum_{j=2}^{q} O(\log T(r,g_j/g_1)) \\ &= \sum_{j=1}^{q} \mathcal{N}_{p-1}(r,g_j) - \mathcal{U}(r,\mathcal{G}) - N(r,H) + O(\log r) + \sum_{j=2}^{q} O(\log T(r,g_j/g_1)).\end{aligned}$$

As in [4], p. 452, we now obtain the following theorem, which encapsulates our sharper form of Cartan's theory.

Theorem 7.1. *Suppose that $p \geq 2$ and that $f_1,\ldots,f_p$ are linearly independent entire functions in $\mathbb{C}$ such that for each $z_0 \in \mathbb{C}$, there is an index j such that $f_j(z_0) \neq 0$.*

Suppose that $q > p$, and let $g_1,\ldots,g_p$ be linear combinations of $f_1,\ldots,f_p$, any p of which are linearly independent. Write $\mathcal{G} = (g_1,\ldots,g_q)$. Let H be as in (28).

Then

$$(q-p)T(r) \le \sum_{j=1}^{q} \mathcal{N}_{p-1}(r, g_j) - \mathcal{U}(r, \mathcal{G}) - N(r, H) + O(\log r) + O(\log T(r)) \tag{32}$$

as $r \to \infty$ outside a set of finite linear measure.

If the ratios f_j/f_1 are all rational functions, in which case, as we have mentioned, we may assume that each f_j is a polynomial, we can obtain a sharper result. In this case, for each choice of the integers $b_1, \ldots, b_p$, the function G is a rational function.

For a more careful analysis of G, it is now useful to write G as

$$G = (g_{b_1} \cdots g_{b_p})^{-1} W(g_{b_1}, \ldots, g_{b_p}).$$

Since $W(g_{b_1}, \ldots, g_{b_p})$ is a constant multiple of $W = W(f_1, \ldots, f_p)$, its analysis at infinity is the same as we have already performed. As in (12), we see that $W(g_{b_1}, \ldots, g_{b_p})$ is a polynomial of degree

$$\sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2}$$

where the δ_j have the same meaning as before for $(f_1, \ldots, f_p)$.

We next ask how low the degrees of the g_{b_j} can be. Low degrees imply a zero at infinity, and the lowest possible result is obtained if the g_{b_j} realize all the excesses at infinity. Thus, by the proof of (18) rather than by (18) itself, but still keeping track of the possibly unrealized excess at infinity, we see that

$$\begin{aligned}
\sum_{j=1}^{p} \deg g_{b_j} &\ge pd - \frac{p(p-1)}{2} - \sum_{j=1}^{p} \mathcal{E}(g_{b_j}, \infty) \\
&\ge pd - \frac{p(p-1)}{2} - \left((p-1)d - \sum_{j=2}^{p} \delta_j - \frac{p(p-1)}{2} \right) + \mathcal{V}(\mathcal{G}, \infty) \\
&= d + \sum_{j=2}^{p} \delta_j + \mathcal{V}(\mathcal{G}, \infty).
\end{aligned}$$

Hence $G(z) = O(z^{-a})$ as $z \to \infty$, where, since $\delta_1 = d$,

$$a = \sum_{j=1}^{p} \delta_j - \frac{p(p-1)}{2} - \left(d + \sum_{j=2}^{p} \delta_j + \mathcal{V}(\mathcal{G}, \infty) \right) = -\left(\frac{p(p-1)}{2} + \mathcal{V}(\mathcal{G}, \infty) \right).$$

Thus we recover the same estimate as Gundersen and Hayman in [4], (7.8), p. 445. We may sharpen it by taking into account the unrealized excess at infinity, if any.

Theorem 7.2. *Suppose that $p \ge 2$ and that $f_1, \ldots, f_p$ are linearly independent polynomials in $\mathbb{C}$ such that for each $z_0 \in \mathbb{C}$, there is an index j such that $f_j(z_0) \ne 0$.*

Suppose that $q > p$, and let $g_1, \ldots, g_q$ be linear combinations of $f_1, \ldots, f_p$, any p of which are linearly independent. Write $\mathcal{G} = (g_1, \ldots, g_q)$. Let H be as in (28).

Then for all $r > 0$, we have

$$(q-p)T(r) \le \sum_{j=1}^{q} \mathcal{N}_{p-1}(r, g_j) - \mathcal{U}(r, \mathcal{G}) - N(r, H) + S(r), \tag{33}$$

where

$$S(r) \leq -\left(\frac{1}{2}p(p-1) + \mathcal{V}(\mathcal{G}, \infty)\right) \log r + O(1)$$

as $r \to \infty$.

7.1. **Waring's problem for functions.** Hayman [6] (see also [4] for a survey) used the classical Cartan theory to study Waring's problem

$$\sum_{j=1}^{p} f_j^k(z) = 1 \qquad \text{or} \qquad \sum_{j=1}^{p} f_j^k(z) = z$$

when $f_1, \ldots, f_p$ belong to a particular class of functions: polynomials, rational functions, entire functions, or functions meromorphic in the plane. An important goal is to obtain a lower bound for p in terms of k.

We have not been able to improve Hayman's results by using our more refined form of Cartan's theory. The reason may be that by adding multiplicities of zeros, in whatever form, one is not making sufficiently effective use of the assumption that the order of each zero of every f_j^k is exactly an integral multiple of k. This suggests that new and different methods may need to be developed to make further progress on Waring's problem for functions.

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