MAJORIZATION OF THE MODULUS OF CONTINUITY OF ANALYTIC FUNCTIONS

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Dedicated to Professor Walter K. Hayman, F.R.S., in admiration

Abstract. Let G be a bounded domain in the complex plane, let f be analytic in G and continuous in $\overline{G}$, and let μ be a majorant, that is, a non-negative non-decreasing function defined for $t \geq 0$ such that $\mu(2t) \leq 2\mu(t)$ for all $t \geq 0$. Suppose that $z_1 \in \partial G$ and that $|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$ for all $z_2 \in \partial G$. We show that then $|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)$ for all $z_2 \in G$ where $C = 3456$. If the assumption is made for all $z_1, z_2 \in \partial G$, then the conclusion holds for all $z_1, z_2 \in \overline{G}$. Earlier such a result, with an absolute constant C, had only been known when G is simply or doubly connected. The same result holds when G is an open set with only bounded components. We also give a survey of results on this type of problems, and explain the reductions that can be made.

1. Introduction and survey

1.1. **Hölder continuous functions.**

Let G be an open set in the complex plane $\mathbb{C}$, and let f be a complex-valued function continuous in the closure $\overline{G}$ of G and analytic in G. The maximum modulus principle states that if f is bounded in G and if $|f| \leq M$ on ∂G for some positive real number M, then $|f| \leq M$ in G. One may ask whether similar results might hold if, instead of considering $|f|$ at each point separately, we study differences of function values, and hence, effectively, the modulus of continuity of f on ∂G and in $\overline{G}$. Also, one can ask whether, if we fix one point $z_1 \in \partial G$, we may deduce from upper bounds of $|f(z) - f(z_1)|$ for all $z \in \partial G$ (presumably larger) upper bounds for $|f(z) - f(z_1)|$ for all $z \in G$.

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As the function $|f(z) - f(z_1)|$, as a function of $|z - z_1|$ for $z \in \partial G$, might, in general, have irregular growth (compare the results of Smith and Stegenga [11]), it has become customary to use a majorant whose growth is, in a suitable sense, regular. Let μ be a non-decreasing non-negative function defined for $t \geq 0$ such that

$$\mu(2t) \leq 2\mu(t) \tag{1}$$

for all $t \geq 0$. Such a function μ is called a **majorant**. We consider assumptions and results of the type that $|f(z) - f(z_1)| \leq C\mu(|z - z_1|)$ for suitable choices of z and z_1, for some majorant μ, where C denotes a constant with $C \geq 1$.

When we discuss Hölder continuity, we take μ to be a power function, that is, $\mu(t) = t^\alpha$, where $0 < \alpha \leq 1$. In this case, the known results are the most complete and most satisfactory. The following theorem was proved by Gehring, Hayman, and the author [1].

Theorem 1.1. *Let G be an open set in $\mathbb{C}$ such that ∂G contains at least two finite points, and let f be continuous in $\overline{G}$ and analytic in G. Suppose that $0 \leq \alpha \leq 1$ and $M > 0$, and that*

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2|^\alpha \tag{2}$$

whenever $z_1, z_2 \in \partial G$. If G is unbounded, suppose further that

$$f(z) = o(|z|) \tag{3}$$

as $z \to \infty$ in each unbounded component of G if $\alpha < 1$, and that

$$f(z) = o(|z|^2) \tag{4}$$

as $z \to \infty$ in each unbounded component of G if $\alpha = 1$. Then (2) holds for every pair of points $z_1, z_2 \in \overline{G}$.

Further, if (2) holds for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, then (2) also holds for this z_1 and for all $z_2 \in \overline{G}$.

As pointed out in [1], taking $G = \{z : |z| > 1\}$ and $f(z) = z$ and $f(z) = z^2$, respectively, we see that we cannot replace o by O in (3) and (4).

The assumption that ∂G contains at least two finite points is necessary, for otherwise the assumption (2) of Theorem 1.1 is vacuous while the conclusion could not possibly always hold.

When G is the unit disk, this result was proved by Sewell ([10], Theorem 1.2.7, p. 17), and a shorter proof was given by Rubel, Shields, and Taylor ([8], Theorem 2.2, p. 27). Earlier Hardy and Littlewood ([2], p. 427) had obtained a weaker result, namely, that (2) on the unit circle implies that $|f(z_1) - f(z_2)| \leq CM|z_1 - z_2|^\alpha$ in the closed unit disk, where C is an absolute constant.

1.2. **Logarithmically concave majorants.** The arguments in [1] are mostly based on the fact that the quantity

$$\log |f(z_1) - f(z_2)| - \log(M|z_1 - z_2|^\alpha),$$

for which we need an upper bound, is a subharmonic function of z_2 for each fixed z_1. Thus, if we look for an upper bound for the quantity

$$\log |f(z_1) - f(z_2)| - \log \mu(|z_1 - z_2|),$$

it should suffice to assume that $\log \mu(|z|)$ is a superharmonic function of z, or, equivalently, that $\log \mu(e^t)$ is a concave function of t for real t. The author [3] obtained the following result.

Theorem 1.2. *Let G be an open set in $\mathbb{C}$ such that ∂G contains at least two finite points, and let f be continuous in $\overline{G}$ and analytic in G. Let μ be a non-decreasing non-negative function defined for $t \geq 0$ such that $\log \mu(e^t)$ is a concave function of t for real t, and*

$$B \equiv \lim_{t \to 0+} \frac{\log \mu(t)}{\log t} \leq 1. \tag{5}$$

Set

$$A = \lim_{t \to \infty} \frac{\log \mu(t)}{\log t} \leq B \leq 1, \tag{6}$$

and assume that as $z \to \infty$ in any unbounded component of G, we have

$$f(z) = o(|z|) \tag{7}$$

if $A < 1$, and

$$f(z) = o(|z|^2) \tag{8}$$

if $A = 1$. Suppose that

$$|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|) \tag{9}$$

whenever $z_1, z_2 \in \partial G$. Then (9) holds for every pair of points $z_1, z_2 \in \overline{G}$.

Further, if (9) holds for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, then (9) also holds for this z_1 and for all $z_2 \in \overline{G}$.

If, instead of (7) or (8), we only have, for some fixed $q > 0$ that depends only on f and G, that $f(z) = o(|z|^q)$ as $z \to \infty$ in any unbounded component of G, and if (9) holds as assumed but the conclusion of Theorem 1.2 fails, then G contains a neighbourhood of infinity, and f has a pole at infinity.

For open sets G whose complement in $\overline{\mathbb{C}}$ is connected, and for bounded analytic functions f, Theorem 1.2 was proved by Tamrazov ([12], Theorem 9.2, p. 166) who in [12], Corollary, p. 167, also noted the special case $\mu(t) = t^\alpha$ corresponding to Theorem 1.1. Tamrazov ([12], Theorem 9.3, p. 167) also proved that if such a result is to hold for a non-negative non-decreasing function μ and for all simply connected domains G and all bounded analytic f, then $\log \mu(e^t)$ must be concave.

The special feature of Theorems 1.1 and 1.2 is that the assumption that $|f(z_1)-f(z_2)| \le \mu(|z_1-z_2|)$ implies, for a larger set of z_2 and possibly z_1, that the same inequality holds, rather than a weaker inequality such as $|f(z_1)-f(z_2)| \le C\mu(|z_1-z_2|)$ for some $C > 1$. This is because of the special properties of the function μ used in there theorems.

Note that $\mu(t) = Mt^\alpha$, where $0 \le \alpha \le 1$, satisfies the condition $\mu(2t) \le 2\mu(t)$.

1.3. **Upper bounds based on subharmonicity.** For more general functions μ, it is not the case that $|f(z_1)-f(z_2)| \le \mu(|z_1-z_2|)$ implies the same inequality in the closure of the domain; see, for example, [8], Proposition 4.3, p. 38, or [5], Theorem 1, p. 42, and Theorem 2, p. 42. Thus, at best, we can only hope for a conclusion of the form $|f(z_1) - f(z_2)| \le C\mu(|z_1 - z_2|)$ for some $C > 1$ when $z_2 \in G$.

Suppose that under suitable assumptions, some results have been obtained for bounded domains or open sets G. Extending these results to unbounded open sets G under a growth condition for $f(z)$ as $z \to \infty$ amounts to proving a Phragmén–Lindelöf type of theorem. In this paper, we now wish to consider the ideas required to deal with majorants; therefore we limit our attention from now on to open sets with only bounded components.

The **modulus of continuity** of a function F on a set E is defined as

$$\omega(t, F, E) = \sup\{\, |F(z) - F(w)| \,:\, z, w \in E,\ |z - w| \le t \,\}.$$

A non-negative non-decreasing function $\varphi(t)$ defined for $t \ge 0$ is called **subadditive** if $\varphi(t_1 + t_2) \le \varphi(t_1) + \varphi(t_2)$ whenever $t_1, t_2 \ge 0$. Clearly a subadditive function is a majorant. Some results in the literature use subadditive functions rather than majorants as upper bounds. One justification for this is the fact that a modulus of continuity of a function F on a set E is subadditive if E is convex; for example, if E is a disk. On a circle, say, the unit circle, one can also consider the subadditive function of the argument given by

$$\sup\{\, |f(e^{i\alpha}) - f(e^{i\beta})| \,:\, |\alpha - \beta| \le t \,\}.$$

In the early 1970's, such results were obtained by Tamrazov [12] and by Rubel, Shields, and Taylor [7], [8]. Let us first consider the case when G is a simply connected domain.

Rubel, Shields, and Taylor proved the following result ([8], Theorem 1.1, p. 24, Theorem 2.6, p. 30, and Theorem 2.7, p. 31). Note that these theorems deal with subadditive functions, rather than majorants, but an inspection of the arguments shows that the same results hold for majorants.

Theorem 1.3. *Let G be a bounded simply connected domain in the plane. Let f be continuous in $\overline{G}$ and analytic in G. Let μ be a majorant. Suppose that*

$$|f(z_1) - f(z_2)| \le \mu(|z_1 - z_2|) \tag{10}$$

for all $z_1, z_2 \in \partial G$. Then we have

$$|f(z_1) - f(z_2)| \le C\mu(|z_1 - z_2|) \tag{11}$$

for all $z_1, z_2 \in \overline{G}$, where the constant C is chosen as follows.

If G is the unit disk, we may take $C = 3$. For an arbitrary bounded simply connected domain, we may take $C = 2 + 2e^A < 3.38 \cdot 10^7$ where $A = (4/\pi)\sum_{n=2}^{\infty}(n+1)2^{-(n-1)/2} = (4\sqrt{2}(3-2\sqrt{2})/((3-2\sqrt{2})\pi) < 16.643$.

If there are constants δ_0 and K with $\delta_0 > 0$ and $K \ge 1$ such that for all $\zeta \in \partial G$ and for all $\delta \in (0, \delta_0)$, the closed disk $\overline{B}(\zeta, \delta)$ contains a point $\zeta' \notin \overline{G}$ such that $|z - \zeta'| \ge \delta/K$ for all $z \in \overline{G}$, we may take $C = 2 + 8K$.

Note that in the expression for A, there is a misprint in [8], page 31, line 5 from the bottom, but looking at two lines earlier one obtains the correct expression.

The last condition in Theorem 1.3 means that the exterior of G (that is, the open set $\overline{\mathbb{C}} \setminus \overline{G}$) is large enough.

Tamrazov considered domains of arbitrary connectivity with a sufficiently thick complement. The thickness condition is expressed in terms of **logarithmic capacity**. We denote the logarithmic capacity of a compact set E by $\operatorname{cap} E$. For $z \in \mathbb{C}$ and $r > 0$, we write $B(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$, $\overline{B}(z, r) = \{w \in \mathbb{C} : |w - z| \le r\}$ and $S(z, r) = \{w \in \mathbb{C} : |w - z| = r\}$. Tamrazov proved the following result [12], Theorem 3.1, p. 149 and Theorem 9.1, p. 166.

Theorem 1.4. *Let G be a domain in the Riemann sphere $\overline{\mathbb{C}}$. Let f be continuous and bounded in $\overline{G}$ and analytic in G. Let $\mu(t)$ be a non-negative non-decreasing function for $t \ge 0$ such that for all $t > 0$ and* *all* $u > 1$, *we have*

$$\mu(ut) \leq \psi(u)\mu(t),$$

where $\log\psi(e^v)$ *is concave in* v *for* $v > 0$. *For* $z \in \mathbb{C}$ *and* $t > 0$, *write*

$$C^*(z,t) = \operatorname{cap}\,(\overline{B}(z,t) \setminus G)$$

and

$$u(z,t) = \frac{27t}{4C^*(z,t/2)}.$$

Suppose that

$$|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$$

for a fixed $z_1 \in \partial G$ *and for all* $z_2 \in \partial G$. *Then if* $z \in G$, *we have*

$$|f(z_1) - f(z)| \leq A,$$

where

$$A = \inf\{\, \psi(u(z_1,v))\mu(v) \,:\, v \geq |z - z_1| \,\}.$$

In particular, if G *is simply connected, we may take*

$$A = \psi(54)\mu(|z_1 - z|).$$

If, further, μ *is a majorant, so that* $\psi(u) = 2u$, *we may take* $A = 108\mu(|z_1 - z|)$. *Thus, in this case,*

$$|f(z_1) - f(z_2)| \leq 108\mu(|z_1 - z_2|)$$

for this z_1 *and for all* $z_2 \in \overline{G}$. *Hence if* G *is a simply connected domain and* μ *is a majorant, and if*

$$|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$$

whenever $z_1, z_2 \in \partial G$, *then*

$$|f(z_1) - f(z_2)| \leq 108\mu(|z_1 - z_2|)$$

for every pair of points $z_1, z_2 \in \overline{G}$.

The proof of Theorem 1.4 is based on considering each fixed $z_1 \in \partial G$ separately, so that the corresponding conclusions are also valid separately for each fixed $z_1 \in \partial G$. In contrast, the proof of Theorem 1.3 makes essential use of all $z_1 \in \partial G$ simultaneously.

The proofs by Rubel, Shields, and Taylor [8] and by Tamrazov [12] are based on the fact that $\log|f(z) - f(z_1)|$ is a subharmonic function of z. This explains why concepts from potential theory, such as logarithmic capacity, enter the argument. However, in terms of the connectivity of the domain, this amounts to making optimal use of the fact that f is analytic only when G is simply connected. Namely, such arguments only use the modulus of f but not the argument of f.

For example, if G is the annulus $\{z : 1 < |z| < R\}$, then similar potential theoretical arguments only imply that above, one can choose C in (11) to be $C = AR$ where A is an absolute constant. But one would like to show that one can take C to be an absolute constant.

For further results in the unit disk and for generalisations to quasiconformal maps, we refer to [5] and [6].

1.4. **The use of the argument of the function.**

Since potential theoretical arguments based on making use of only the modulus of the analytic function f, yield absolute constants C for (11) only in simply connected domains, it becomes essential to devise methods for using the argument of f in multiply connected domains. If, say, the domain G is bounded by finitely many Jordan curves γ, then, as z goes around each γ, the argument of $f(z)$ changes by an amount that is an integer multiple of 2π (recall that f is continuous in $\overline{G}$). This seems to be a natural piece of information that one should try to use.

In [4], the author developed a method for using the argument of f in this case, by obtaining a formula for $\log|f|$ in terms of a Green's function and the harmonic measures of the boundary curves γ, and appealing to a lemma due to Teichmüller on periodic functions on the real axis, with period 1, that are concave on $[0, 1]$. This then enabled the author to prove ([4], Theorem 4, p. 310) that if G is doubly connected, we obtain (11) with an absolute constants C, say, $C = 1.63 \cdot 10^7$.

However, as explained in [4], p. 328, Jang-Mei Wu pointed out to the author at the time that this method runs into problems already for triply connected domains. Thus for domains of higher connectivity, other methods of using the argument of f should be found.

In this paper, we make use of the fact that we may assume that f is rational, as will be explained in a moment, and use level sets of the rational function in a new way to estimate $|f(z_1) - f(z_2)|$. We will, in fact, use a combination of methods. If a certain component of the level set $\{z : |f(z)| = K\}$ for a suitable number K is large, then we use Theorem 1.9 below, a local result referring to the logarithmic capacity of the set on which we already have information about the magnitude of f, to conclude that we get the desired estimate. If such a component of the level set is small, then potential theoretic capacity arguments do not yield any information. In that case, however, the level set is localized in a small region, which allows us to use what one might call the topology of level sets to obtain a contradiction since the zeros and poles of the rational functions would then be in the wrong places. This is based on counting the number of the zeros and poles of the function as well as the number of times that $f(z)$ traverses the circle $S(0, K)$

as z traverses various components of the level set $\{z : |f(z)| = K\}$. In fact, we use this argument also for an auxiliary rational function g defined in terms of f.

1.5. **Reduction to finitely connected domains with a smooth boundary.**

When trying to obtain estimates for majorants of the modulus of continuity of an analytic function, it often simplifies the thinking if one is able to consider only domains or majorants with special regularity properties. This allows one to concentrate on the really important aspects of the problem without the burden of other technicalities. Therefore we will now discuss such reductions until we are able to conclude that it suffices to consider rational functions of a special kind in domains bounded by finitely many analytic Jordan curves, and majorants that are almost linear. We start by studying the reduction of domains.

In [4], the author obtained the following reductions of the majorization problem ([4], Theorem 2, p. 309 and Theorem 3, p. 310).

Theorem 1.5. Hypothesis: *There is an absolute constant C such that whenever G is a bounded open set with at most two components, the boundary of G consists of finitely many disjoint analytic Jordan curves, f is rational with poles outside $\overline{G}$, and (10) holds for all $z_1, z_2 \in \partial G$ for some majorant μ, then (11) holds for all $z_1, z_2 \in \overline{G}$.*

Conclusion: *Whenever G is open with only bounded components, f is continuous in $\overline{G}$ and analytic in G, and (10) holds for all $z_1, z_2 \in \partial G$ for some majorant μ, then (11) holds for all $z_1, z_2 \in \overline{G}$ with this C.*

Theorem 1.6. *Let n be a positive integer.*

Hypothesis: *There is a constant C_n depending on n only such that whenever G is a bounded domain of connectivity n whose the boundary consists of n disjoint analytic Jordan curves, f is rational with poles outside $\overline{G}$, and (10) holds for all $z_1, z_2 \in \partial G$ for some majorant μ, then (11) holds for all $z_1, z_2 \in \overline{G}$ with C replaced by C_n.*

Conclusion: *Whenever G is bounded domain of connectivity n, f is continuous in $\overline{G}$ and analytic in G, and (10) holds for all $z_1, z_2 \in \partial G$ for some majorant μ, then (11) holds for all $z_1, z_2 \in \overline{G}$ with C replaced by C_n.*

1.6. **Reduction to rational functions using Runge's theorem.**

One can make the reduction to the case when f is rational with poles outside $\overline{G}$ somewhat more precise by using Runge's theorem ([9], p. 288). In fact, all we have to do is to note that in the application of Runge's theorem in the proof of [4], Theorem 2, p. 315, we merely observe that we may choose our rational function so that it has at most one pole, ignoring multiplicities, in each component of the complement of G. For future reference, we note that for each given domain of finite connectivity bounded by disjoint Jordan curves, we may even specify the locations of the poles independently of the function f, but not their multiplicities, which will depend on f. This leads to the following result.

Theorem 1.7. *Let n be a positive integer.*

Hypothesis: *There is a constant C_n depending on n only such that whenever G is a bounded domain of connectivity n whose boundary consists of n disjoint analytic Jordan curves, f is rational with poles outside $\overline{G}$ and with at most one pole, ignoring multiplicities, at a preassigned point in each component of $\overline{\mathbb{C}} \setminus \overline{G}$, and (10) holds for all $z_1, z_2 \in \partial G$ for some majorant μ, then (11) holds for all $z_1, z_2 \in \overline{G}$ with C replaced by C_n.*

Conclusion: *Whenever G is bounded domain of connectivity n, f is continuous in $\overline{G}$ and analytic in G, and (10) holds for all $z_1, z_2 \in \partial G$ for some majorant μ, then (11) holds for all $z_1, z_2 \in \overline{G}$ with C replaced by C_n.*

Of course, if one can prove that the hypothesis here holds with a constant independent of n, then it follows that the conclusion is valid with the same constant independent of n.

1.7. **Reduction to special majorants.** Let G be an open set in the complex plane. Let f be a function continuous in $\overline{G}$ and analytic in G. Let μ be a majorant. We now show that without loss of generality, we may assume that μ is of the form $\mu(t) = \max\{t_0, t\}$ for some $t_0 > 0$. Note that any μ of this form is a majorant. This reduction can be performed for each open set G separately, varying only μ and the points $z_2 \in G$, and replacing f only by its constant multiples. Hence we do not assume here that G or f is of any special form.

Lemma 1.8. *Let G be an open set in $\overline{\mathbb{C}}$. Let f be continuous in $\overline{G}$ and analytic in G. Suppose that $C \geq 1$.*

Hypothesis: *for every function μ of the form*

$$\mu(t) = \max\{t_0, t\} \tag{12}$$

for some $t_0 > 0$, and for every function F of the form $F = cf$ where c is a constant, whenever

$$|F(z_1) - F(z_2)| \leq \mu(|z_1 - z_2|)$$

for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, we have

$$|F(z_1) - F(z_2)| \leq C\mu(|z_1 - z_2|)$$

for this z_1 *and for all* $z_2 \in \overline{G}$ *such that* $|z_1 - z_2| = t_0$.

Conclusion: *for every majorant* μ*, whenever*

$$|f(z_1) - f(z_2)| \le \mu(|z_1 - z_2|) \tag{13}$$

for a fixed $z_1 \in \partial G$ *and for all* $z_2 \in \partial G$*, we have*

$$|f(z_1) - f(z_2)| \le 2C\mu(|z_1 - z_2|)$$

for this z_1 *and for all* $z_2 \in \overline{G}$.

Lemma 1.8 shows that if we do not aim at the best constant C, we may limit our consideration to special majorants of the form (12).

Proof of Lemma 1.8. Let G and f be as in the assumptions of Lemma 1.8. Suppose that μ is a majorant, that $z_1 \in \partial G$, and that (13) holds for all $z_2 \in \partial G$. Pick $z \in G$ and define $t_0 = |z - z_1| > 0$. Set $F(z) = (t_0 f(z))/(2\mu(t_0))$ and define $\mu_1(t) = \max\{\, t_0, t\,\}$. Suppose that $z_2 \in \partial G$. If $|z_1 - z_2| \le |z - z_1| = t_0$, then

$$\begin{aligned} |F(z_1) - F(z_2)| &= \frac{t_0|f(z_1) - f(z_2)|}{2\mu(t_0)} \le \frac{t_0\mu(|z_1 - z_2|)}{2\mu(t_0)} \le t_0/2 \\ &< t_0 = \max\{\, t_0, |z_1 - z_2|\,\} = \mu_1(|z_1 - z_2|). \end{aligned}$$

If $|z_1 - z_2| > |z - z_1|$, pick a positive integer n such that $2^{n-1}|z - z_1| < |z_1 - z_2| \le 2^n|z - z_1|$. Note that by (1), we have

$$\mu(|z_1 - z_2|) \le \mu(2^n|z - z_1|) \le 2^n\mu(|z - z_1|) = 2^n\mu(t_0).$$

Thus

$$\begin{aligned} |F(z_1) - F(z_2)| = \frac{t_0|f(z_1) - f(z_2)|}{2\mu(t_0)} &\le \frac{t_0\mu(|z_1 - z_2|)}{2\mu(t_0)} \le \frac{2^n\mu(t_0)t_0}{2\mu(t_0)} \\ = \quad 2^{n-1}|z - z_1| < |z_1 - z_2| &= \max\{\, t_0, |z_1 - z_2|\,\} = \mu_1(|z_1 - z_2|). \end{aligned}$$

Thus for all $z_2 \in \partial G$, we have $|F(z_1) - F(z_2)| \le \mu_1(|z_1 - z_2|)$. Since F is a constant multiple of f, our hypothesis now implies that for all $z_2 \in \overline{G}$ with $|z_1 - z_2| = t_0$, we have $|F(z_1) - F(z_2)| \le C\mu_1(|z_1 - z_2|)$, and taking here z_2 to be the point z, and noting that $\mu_1(t_0) = t_0$, we obtain

$$|f(z_1) - f(z)| = \frac{2\mu(t_0)|F(z_1) - F(z)|}{t_0} \le \frac{2\mu(t_0)C\mu_1(t_0)}{t_0} = 2C\mu(|z - z_1|).$$

Since $z \in G$ was arbitrary, we deduce that the final inequality

$$|f(z_1) - f(z)| \le 2C\mu(|z - z_1|)$$

applies to every $z \in G$, and hence we obtain the conclusion of Lemma 1.8. This completes the proof of Lemma 1.8.

1.8. **Local results based on capacity.** We will use the following result from [4], Theorem 1, p. 308. It is a local result in the spirit of Tamrazov [12], Theorem 3.1, p. 149. The important thing about it is that we obtain an estimate for $|f(w_1) - f(w_2)|$ for certain particular points $w_1 \in \partial G$ and $w_2 \in G$ provided that the complement of G is thick enough close to w_2. When we consider our problem in multiply connected domains, while the complement of the original domain need not be thick enough close to w_2, it may be that we can make the domain smaller and thereby enlarge the complement to be thick enough if we have extra information about f on a level set that comes sufficiently close to w_2. We denote the Euclidean distance of the point z from the set E by $d(z, E)$.

Theorem 1.9. *Let f be analytic in the open set G, and continuous and bounded in $\overline{G}$. Suppose that $w_1 \in \partial G$, $w_2 \in G$, that there is a compact set Ω such that w_2 belongs to the unbounded component of $G \cup (\mathbb{C} \setminus \Omega)$, and that*

$$|z - w_2| \le A_1 |w_1 - w_2| \tag{14}$$

for all $z \in \Omega$ and

$$\operatorname{cap}((\mathbb{C} \setminus G) \cap \Omega) \ge A_2 |w_1 - w_2| \tag{15}$$

for some positive constants A_1 and A_2. If

$$|f(w_1) - f(z_2)| \le \mu(|w_1 - z_2|) \tag{16}$$

for all $z_2 \in \partial G$, where μ is a majorant, then

$$|f(w_1) - f(w_2)| \le C\mu(|w_1 - w_2|) \tag{17}$$

where

$$C = 2A_1^2 A_2^{-1} \left(1 + \frac{|w_1 - w_2|}{d}\right) \tag{18}$$

and

$$d = \max\{d(w_2, \partial G), d(w_2, \Omega)\}.$$

In [4], Theorem A, p. 309, the author used Theorem 1.9 to obtain (11) with $C = 74$ when G is a bounded simply connected domain.

1.9. **Results in multiply connected domains.** For the case when G is a doubly connected domain, the author obtained the following result in [4], Theorem 4, p. 310.

Theorem 1.10. *If G is a bounded doubly connected domain, if f is analytic in G and continuous in $\overline{G}$, if μ is a majorant, and if (10) holds for all $z_1, z_2 \in \partial G$, then (11) holds for all $z_1, z_2 \in \overline{G}$ with $C = 1.63 \cdot 10^7$.*

If (10) holds for a fixed $z_1 \in \partial G$ *and for all* $z_2 \in \partial G$*, then (11) holds for this* z_1 *and for all* $z_2 \in \overline{G}$ *with* $C = 1.63 \cdot 10^7$.

We are able to extend Theorem 1.10 to domains of any finite connectivity, hence to arbitrary bounded domains, and to reduce the constant C. We prove the following result.

Theorem 1.11. *If* G *is an open set with only bounded components, if* f *is analytic in* G *and continuous in* $\overline{G}$*, if* μ *is a majorant, and if (10) holds for all* $z_1, z_2 \in \partial G$*, then (11) holds for all* $z_1, z_2 \in \overline{G}$ *with* $C = 3456$.

If (10) holds for a fixed $z_1 \in \partial G$ *and for all* $z_2 \in \partial G$*, then (11) holds for this* z_1 *and for all* $z_2 \in \overline{G}$ *with* $C = 3456$.

The second statement of Theorem 1.11 implies the first, so we only prove the second statement. The reason is that, as has been noted in all earlier literature, for each fixed $t > 0$, the maximum of $|f(z_1) - f(z_2)|$ for $z_1, z_2 \in \overline{G}$ subject to $|z_1 - z_2| \le t$ occurs when at least one of z_1 and z_2 lies on ∂G. This can be seen by maximizing $|f(z+h) - f(z)|$ as z varies in the closure of the open set $\{w : w \in G, w + h \in G\}$ for any fixed h with $|h| \le t$.

2. Proof of Theorem 1.11.

If Γ is a Jordan curve in $\mathbb{C}$, we denote by $\operatorname{int}\Gamma$ the bounded component of $\mathbb{C} \setminus \Gamma$ and call it the interior of Γ. Thus $\operatorname{int}\Gamma$ is a Jordan domain, and $\partial(\operatorname{int}\Gamma) = \Gamma$.

2.1. **Preliminary results.** To prove Theorem 1.11, we may assume, in view of Theorem 1.5, that G is bounded by finitely many disjoint analytic Jordan curves and that G has at most two components. To bring out the essential ideas of the proof, we first consider the case when G is a doubly connected domain, and discuss later how to modify the proof in the general situation.

Let G be a bounded doubly connected domain with boundary components Γ_0 and Γ_1. We assume that Γ_0 and Γ_1 are analytic Jordan curves and that Γ_1 is the outer boundary component. To prove Theorem 1.11, we may further assume, in view of Theorem 1.7 and Lemma 1.8, that f is a rational function with at most one pole, ignoring multiplicities, in each component of $\overline{\mathbb{C}} \setminus \overline{G}$. We may pick the two points at which f may have a pole, and we assume that one of them is the point at infinity.

We assume that $z_1 \in \partial G$. We may normalize the situation by assuming that $f(z_1) = 0$. In view of Lemma 1.8, we may limit ourselves to the following situation. We pick $w_2 \in G$ and set $t_0 = |z_1 - w_2|$.

Since our problem is invariant under scale changes, we may assume that $t_0 = 1$. We assume that

$$|f(z)| = |f(z) - f(z_1)| \leq \max\{1, |z - z_1|\}$$

for all $z \in \partial G$. In the doubly connected case we are able to prove that now

$$|f(w_2)| \leq 768.$$

Then this together with Theorem 1.7 and Lemma 1.8 implies the conclusion of Theorem 1.11 since $2 \cdot 768 = 1536 < 3456$.

We first deal with the simpler case when $z_1 \in \Gamma_1$ by an argument that is independent of the connectivity of G. If $z_1 \in \Gamma_1$, let Ω be a proper subarc of Γ_1, to be chosen more precisely in a moment, so that then w_2 lies in the unbounded component of $G \cup (\mathbb{C} \setminus \Omega) = \mathbb{C} \setminus \Omega$. We assume that $z_1 = 0$ and $w_2 = 1$. Our intention is to apply Theorem 1.9 to our G and μ, taking there $w_1 = z_1 = 0$ and $w_2 = 1$. Since Γ_1 surrounds the point 1 and contains the origin, it follows that Γ_1 contains a subarc Ω joining the origin to a point of $S(0, 1/2)$ in $\overline{B}(0, 1/2)$. Then $d(1, \Omega) \geq 1/2$, $\operatorname{diam} \Omega \geq 1/2$, and

$$\operatorname{cap}((\mathbb{C} \setminus G) \cap \Omega) = \operatorname{cap} \Omega \geq (1/4) \operatorname{diam} \Omega \geq 1/8.$$

Thus we may take $A_1 = 3/2$ and $A_2 = 1/8$ in Theorem 1.9. Now (17) and (18) imply that

$$|f(w_2)| \leq 36\,(1 + 1/(1/2)) = 108.$$

From now on, we may assume that $z_1 \in \Gamma_0$. The normalization $z_1 = 0$ used above will no longer be valid. Since our problem is translation and rotation invariant, we may assume that $w_2 = 1$, that z_1 is arbitrarily close to the origin, and that the possible finite pole of f can only be at the origin. To achieve this, it may be necessary to change the value of t_0 from 1 by an amount that can be made arbitrarily small, so we simplify our notation by going through the argument without considering this effect. Note that now $0 \in \operatorname{int} \Gamma_0$. This may give the impression that we are free to move the location of the finite pole of f depending on the boundary point z_1. This is correct. By Runge's theorem, once z_1 has been specified, we can approximate our initial f as closely as we like in $\overline{G}$ using a new rational function, now with a pole at the newly chosen point, say the origin. See the proof of [4], Theorem 2, pp. 314–315 for the details. Since this procedure will not change the eventual constants obtained, we ignore the details.

2.2. **Location of the boundary components of G.** We first use Theorem 1.9 to show that we may assume that neither one of the boundary components of G gets too close to w_2, in the sense that if one of them does, then we obtain our desired conclusion already. Thus there is any need for further work only when the boundary components stay far away from w_2.

Suppose that $\overline{\operatorname{int}\Gamma_0 \cap S(0,1/2)} \neq \emptyset$. We apply Theorem 1.9 taking Ω to be the component of $\overline{\operatorname{int}\Gamma_0} \cap \overline{B}(0,1/2)$ containing the origin, $w_1 = z_1$, $w_2 = 1$, and $\mu(t) = \max\{1,t\}$. Then we may take $A_1 = 3/2$, and $A_2 = 1/8$ since

$$\operatorname{cap}((\mathbb{C}\setminus G)\cap\Omega) = \operatorname{cap}\Omega \geq (1/4)\operatorname{diam}\Omega \geq (1/4)\,(1/2) = 1/8.$$

Now (17) and (18) give

$$|f(w_2)| \leq 36\,(1+1/(1/2)) = 108$$

since $\max\{d(w_2,\partial G), d(w_2,\Omega)\} \geq d(1,\Omega) \geq 1/2$. So in this case (11) holds with $C = 108$. Thus we may assume from now on that $\Gamma_0 \subset B(0,1/2)$.

Suppose that Γ_1 contains a point in $B(0,2)$. We apply again Theorem 1.9 this time tentatively taking Ω to be $\Gamma_1 \setminus \gamma$ where γ is a short arc of Γ_1. Then it will still be the case that $w_2 = 1$ lies in the unbounded component of $G \cup (\mathbb{C}\setminus\Omega) = \mathbb{C}\setminus\Omega$. Since 0 and 1 lie in the interior of Γ_1, we have $\operatorname{diam}\Gamma_1 > 1$ so that $\operatorname{cap}\Gamma_1 > 1/4$. Taking γ to be short enough, we can guarantee that $\operatorname{cap}\Omega > 1/4$ and that $\Omega \cap B(0,2) \neq \emptyset$.

Next, suppose that the set $\Gamma_1 \subset \overline{B}(0,3)$. If $\Omega \cap B(1,1/2) = \emptyset$, we do not modify Ω further, and then we may take $A_1 = 4$ and $A_2 = 1/4$, and we have $d(1,\Omega) \geq 1/2$. If $\Omega \cap B(1,1/2) \neq \emptyset$, we replace Ω by a closed subarc of Γ_1 that does not intersect $B(1,1/2)$ but whose end points lie on $S(1,1/2)$, such that the Jordan curve consisting of Ω and the line segment in $B(1,1/2)$ joining the two end points of Ω surrounds the origin. Then we still have $\operatorname{diam}\Omega \geq 1/2$ and hence $\operatorname{cap}\Omega \geq 1/8$. Now we may take $A_1 = 4$ and $A_2 = 1/8$, and $d(1,\Omega) \geq 1/2$.

If, however, Γ_1 contains a point with modulus greater than 3, then we take Ω to be a subarc of Γ_1 that lies in $\overline{B}(0,3)\setminus B(0,2)$ and joins a point of modulus 2 to a point of modulus 3. Then we may still take $A_1 = 4$, and since then $\operatorname{diam}\Omega \geq 1$, we still have $\operatorname{cap}\Omega \geq 1/4$ so that we may take $A_2 = 1/8$. In all cases, (17) and (18) give

$$|f(w_2)| \leq 256\,(1+1/(1/2)) = 768$$

since in each case $\max\{d(w_2,\partial G), d(w_2,\Omega)\} \geq d(1,\Omega) \geq 1/2$.

Thus we may assume from now on that $\Gamma_1 \cap B(0,2) = \emptyset$. This means that $\mu(|z-z_1|) = |z-z_1|$ for all $z \in \Gamma_1$. Since we may take z_1

arbitrarily close to the origin, we will assume that $\mu(|z-z_1|) = |z|$ and hence $|f(z)| \leq |z|$ for all $z \in \Gamma_1$ to simplify notation.

2.3. **An auxiliary rational function.** Next we define an auxiliary rational function and go through the rest of the proof by analysing level sets of rational functions.

We write $g(z) = f(z)/z$. Then g can have a pole at most at the origin and at infinity, and we have

$$|g(z)| \leq 1$$

for all $z \in \Gamma_1$. The reason for introducing g is the need to obtain this inequality.

Suppose that $f(0) \neq \infty$. Choose $\varepsilon > 0$. Then there is $\varepsilon_1 > 0$ such that for all $z \in \mathbb{C} \setminus \operatorname{int} \Gamma_0$, we have $|\varepsilon_1/z| < \varepsilon$ and hence $|f(z)+\varepsilon_1/z| < |f(z)| + \varepsilon$. Thus we may replace the function f by $f + \varepsilon_1/z$ in our arguments, changing any resulting constant by an amount comparable to ε. Since ε is arbitrary, the results are then valid without any change. Hence we may and will assume that f has a pole at the origin. Similarly, adding a term of the form $\varepsilon_1 z^2$ to f, if necessary, we may assume that both f and g have a pole at infinity.

Choose $K > 1$. We will specify later how large K should be. We may assume that f is locally homeomorphic at each point z at which $|f(z)| = K$ or $|g(z)| = K$, by changing K slightly, if necessary.

Suppose that $|f(1)| > K$. We explore the consequences of this assumption.

2.4. **Components of level sets close to the origin.** Since $|f(z)| \leq 1 < K$ when $z \in \Gamma_0 \subset B(0, 1/2)$, but since f has a pole at the origin, there is at least one component γ_1 of the level set $L(f, K) = \{z : |f(z)| = K\}$ surrounding the origin and contained in $B(0, 1/2)$. Note that each component of $L(f, K)$ is a Jordan curve. We may choose γ_1 to be the outer boundary component of the component D_0 of the set $\{z : |f(z)| > K\}$ containing the origin, so that $D_0 \subset D_1 = \operatorname{int} \gamma_1 \subset \operatorname{int} \Gamma_0 \subset B(0, 1/2)$. We have $|f(z)| < K$ for each z outside $\overline{D_1}$ but close enough to γ_1. Hence there is a component D_2 of $\{z : |f(z)| < K\}$ such that $\gamma_1 \subset \partial D_2$. Since $f(\infty) = \infty$, the set D_2 is bounded. Let γ_2 be the outer boundary component of D_2. Then γ_2 is a Jordan curve. Clearly $0 \in \operatorname{int} \gamma_2$.

If ∂D_2 has components other than γ_1 and γ_2, then they are bounded Jordan curves whose interior must contain components of $\{z : |f(z)| > K\}$ other than D_1. Since f is rational, each component of $\{z : |f(z)| > K\}$ contains a pole of f. Since the only poles of f are at the origin and at infinity, there are no such components. Hence $\partial D_2 = \gamma_1 \cup \gamma_2$ and $\operatorname{int} \gamma_2 = D_2 \cup \overline{D_1}$.

Since $|f(1)| > K$, we have $1 \notin \overline{D_2}$.

2.5. **Components of level sets close to the origin do not stretch too far.** Suppose that D_2 intersects $S(0, 1/2)$. Now $\Gamma_0 \subset D_2$ since Γ_0 surrounds the origin, is contained in $B(0, 1/2) \setminus \overline{D_1}$, and is contained in a component of $\{z : |f(z)| < K\}$. We may pick any point $z_0 \in \Gamma_0$ and choose $z_2 \in S(0, 1/2) \cap D_2$ in such a way that we can join z_0 to z_2 by an open Jordan arc γ_3 contained in $D_2 \cap \overline{B}(0, 1/2)$. We set $\Omega = \overline{\gamma_3} \cup \Gamma_0$ and $G_1 = G \setminus \Omega$. We apply Theorem 1.9 to G_1 instead of G and to $\mu_1 = K\mu$ instead of μ. Then $\operatorname{diam} \Omega \geq 1/2$ since Γ_0 surrounds the origin, so that $\operatorname{cap} \Omega \geq 1/8$, and we may take $A_1 = 3/2$. We take $w_1 = z_1$ and $w_2 = 1$ in Theorem 1.9. Then (17) and (18) imply that $|f(1)| \leq 108K$. Hence (11) holds with $C = 216$ if $K \leq 2$, and so we may assume that $D_2 \subset B(0, 1/2)$.

2.6. **Number of zeros of g in D_1 and D_2.** Recall that D_2 is a component of the set $\{z : |f(z)| < K\}$, so that D_2 contains at least one zero z_3 of f. Now $z_3 \neq 0$ since f has a pole at the origin. Thus also $g(z_3) = 0$. Hence z_3 lies in a component D_3 of the set $\{z : |g(z)| < K\}$. If we move from z_3 outwards along any arc joining z_3 to a point of ∂D_2 in D_2, then before we hit the boundary of D_2, on which $|f(z)| = K$ and hence $|g(z)| = |f(z)|/|z| \geq 2K$ (since $D_2 \subset B(0, 1/2)$), we find points where $|g(z)| = K$. Thus $D_3 \subset D_2 \subset B(0, 1/2)$. This applies to every zero of g in D_2. Of course, several zeros of g in D_2 may lie in a single component V of $\{z : |g(z)| < K\}$. Each such component V must be simply connected, for otherwise the interior of the outer boundary component of V would contain a component of the set $\{z : |g(z)| > K\}$ and hence a pole of g, which is impossible since such a pole could not be the origin or infinity.

Similarly, if g, and hence f, has $\nu_1 \geq 0$ zeros in D_1, each such zero lies in a component Y_i of $\{z : |f(z)| < K\}$ with $\overline{Y_i} \subset D_1$, where $1 \leq i \leq J_1$, say, and in a component W_j of $\{z : |g(z)| < K\}$, where $1 \leq j \leq J_2$. All such components Y_i and W_j are simply connected, and each W_j is contained in some Y_i. Several sets W_j may be contained in the same Y_i, and a set of the form Y_i or W_j may contain several zeros of g.

Next, since $|f(1)| > K$ and $1 \notin \operatorname{int} \gamma_2$, the set $\mathbb{C} \setminus D_2$ must contain a component D_4 of $\{z : |f(z)| > K\}$, and this component must contain a pole of f. This pole cannot be the origin, so it must be the point at infinity. Thus the set $\{z : |f(z)| > K\}$ has no components other than D_4 and D_0, and $1 \in D_4$. Since we must have $|f(z)| > K$ for all z outside

γ_2 but close enough to γ_2, it follows that $\gamma_2 \subset \partial D_4$. There may be other components of ∂D_4 corresponding to components of $\{z : |f(z)| < K\}$, which contain further zeros of f (and of g).

2.7. **Traversing components of level sets.** As z traverses ∂D_1 in either direction, $f(z)$ traverses $S(0, K)$ $m_1 \geq 1$ times, say. Here and later, we ignore the direction in which $f(z)$ or $g(z)$ traverses a circle, and only count the number of times that the circle is traversed. Thus if $|\zeta| < K$ and $|\zeta|$ is close enough to K, then in D_2, close to ∂D_1, the function f takes the value ζ m_1 times.

As z traverses γ_2 in either direction, $f(z)$ traverses $S(0, K)$ k times, say, where $k \geq 1$. If $|\zeta| < K$ and $|\zeta|$ is close enough to K, then f takes the value ζ k times in D_2 close to γ_2. Thus f takes the value ζ $m_1 + k$ times in D_2.

It follows that f, and hence also g, has $m_1 + k$ zeros in D_2.

As z traverses the sets ∂Y_i, the point $f(z)$ covers $S(0, K)$ a total of ν_1 times. As z traverses all the components of ∂D_0, the point $f(z)$ covers $S(0, K)$ a total of $m_1 + \nu_1$ times. On the other hand, this covering number must be equal to the number of poles of f in D_0, which is m if $m \geq 1$ is the order of the pole of f at the origin. Thus $m = m_1 + \nu_1$.

As z traverses the sets ∂W_j, the point $g(z)$ covers $S(0, K)$ also a total of ν_1 times.

2.8. **The component of $\{z : |g(z)| > K\}$ containing the point 1.** We are assuming that $|f(1)| > K$, hence $|g(1)| = |f(1)| > K$. Let Ω_1 be the component of $\{z : |g(z)| > K\}$ containing the point $z = 1$. Since $|g(z)| \leq 1$ when $z \in \Gamma_1$, the set $\partial\Omega_1$ has an outer component Γ_5, and the set Ω_1 is bounded. There must be at least one pole of g in Ω_1, and hence in the bounded domain $D_5 = \operatorname{int} \Gamma_5$ since $\Omega_1 \subset D_5$, so this pole must be the origin.

For all $z \in \gamma_1 \cup \gamma_2$, we have $|g(z)| = |f(z)|/|z| = K/|z| > K$. Thus the $m_1 + k$ zeros of g in D_2 lie in simply connected components, say V_j for $1 \leq j \leq J$, of the set $\{z : |g(z)| < K\}$, such that each $\overline{V_j} \subset D_2$. Since there are these $m_1 + k$ zeros of g inside the sets V_j, it follows that $g(z)$ traverses $S(0, K)$ a total of $m_1 + k$ times as z traverses the sets ∂V_j. The sets ∂V_j are the components of the set $L(g, K)$ in D_2.

Suppose that g has ν zeros in $D_5 \setminus \overline{\operatorname{int} \gamma_2}$. Each such zero lies in a component U of $\{z : |g(z)| < K\}$, and each such component U must be contained in $D_5 \setminus \overline{\operatorname{int} \gamma_2}$ (since $|g| > |f| = K$ on γ_2).

If some such component U is multiply connected, then a bounded component X of $\mathbb{C} \setminus \overline{U}$ contains a pole of g, which must be the origin. The domain X is necessarily simply connected. Since there is an arc in

Ω_1 joining the points 0 and 1, so that $|g| > K$ on this arc, it follows that also $1 \in X$. But then we must have $\partial X = \Gamma_5$, so that $U \cap D_5 = \emptyset$, which is a contradiction. Thus any such component U is simply connected.

When z traverses the boundaries of these sets U, which are components of $L(g, K)$, the point $g(z)$ traverses $S(0, K)$ a total of ν times.

Suppose that $g(z)$ traverses $S(0, K)$ $\ell \geq 1$ times when z traverses $\Gamma_5 = \partial D_5$.

Then on these components of $L(g, K)$ (namely, the sets ∂U as above, the sets ∂V_j for $1 \leq j \leq J$, the sets ∂W_j and Γ_5), the point $g(z)$ traverses $S(0, K)$ altogether $\nu + m_1 + k + \nu_1 + \ell = m + k + \nu + \ell$ times, that is, at least $m + 2$ times, since $k, \ell \geq 1$ and $\nu \geq 0$. Since there is an arc in Ω_1 joining the points 0 and 1, we see that these components of $L(g, K)$ form the boundary of Ω_1, and the origin is the only pole of g in Ω_1. Since for each $\zeta \in \mathbb{C} \cup \{\infty\}$ with $|\zeta| > K$ there is the same number of points z in Ω_1 with $g(z) = \zeta$, and since for each ζ with $|\zeta|$ close enough to K this number is equal to the total number of times that g covers $S(0, K)$ on $\partial\Omega_1$, it follows that the order of the pole of g at the origin is

$$m + k + \nu + \ell \geq m + 2.$$

But $g(z) = f(z)/z$, so the order of the pole of g at the origin is $m + 1$, which is a contradiction. (This contradiction, derived from the assumption that $|f(1)| > K$, really means that we have $|f(1)| \leq K \leq 2$, or one of the cases excluded above by appealing to Theorem 1.9 must occur, in which case (11) holds with $C = 768$.)

This argument is valid for any $K > 1$ satisfying the condition that neither f nor g has any critical value of modulus K, so that K can be arbitrarily close to 1. Thus we may take $K \leq 2$, as required. This contradiction shows that (11) holds with $C = 768$ for these special majorants, as claimed. This completes the proof of Theorem 1.11 for doubly connected domains.

Remark. We saw above that certain assumptions lead to a contradiction. One can ask what the level sets of g must actually look like. The reality of the situation is that we must have $D_2 \cap \{z : |z| > 1\} \neq \emptyset$, so that on part of ∂D_2 outside $B(0, 1)$, we can have $|g| < K$. Thus ∂V_j can go outside D_2 for some j with $1 \leq j \leq J$, so that the argument given above does not apply.

2.9. **Domains of higher connectivity.**

Suppose now that G is a domain of finite connectivity at least 3. We indicate how the above argument needs to be modified. The same argument still shows that we may assume that z_1 does not lie on the outer boundary component Γ_1 of G. Let Γ_0 denote the inner boundary component of G containing

z_1. We may assume that $t_0 = |z_1 - w_2| = 1$, $w_2 = 1$, z_1 is close to 0, and $0 \in \operatorname{int}\Gamma_0$ is the only possible pole of f in $\operatorname{int}\Gamma_0$, and $g(z) = f(z)/z$. Again this may require a small change in the value of t_0, but we ignore that. We may further assume that $\Gamma_1 \cap B(0,2) = \emptyset$.

We need to be more careful when considering level curves of f and g and hence a variation is required. Therefore this time we take $K \geq 4$ but may take K arbitrarily close to 4, so that neither f nor g has any critical value of modulus K or $K/2$. We assume that $|f(1)| = |g(1)| > K > K/2$ and explore the consequences of this assumption.

We define Ω_1 as the component of $\{z : |g(z)| > K/2\}$ containing the point 1, let Γ_5 be the outer boundary component of Ω_1, and set $D_5 = \operatorname{int}\Gamma_5$ so that $\Omega_1 \subset D_5 \subset \operatorname{int}\Gamma_1$. We may assume that 0 and ∞ are among the poles of f.

2.10. **On the finite poles of f.** Now f may have finite poles $p \neq 0$. We need to consider only certain poles of f. Let p be a pole of f in D_5, possibly $p = 0$. Then $p \in D_p = \operatorname{int}\Gamma_p$, say, where Γ_p is a boundary component of G other than Γ_1.

If $|p| \leq 1$ and if Γ_p is not contained in $B(0,2)$, we apply Theorem 1.9 taking Ω to be a subarc of Γ_p joining the boundary components of $D = \overline{B}(0,2) \setminus B(0,3/2)$ in $\overline{D}$. Then $d(1,\Omega) \geq 1/2$, $\operatorname{cap}\Omega \geq 1/8$, and we may take $A_1 = 3$. Now (17) and (18) yield $|f(1)| \leq 18 \cdot 8 \cdot 3 = 432$. So we may assume that if $|p| \leq 1$ then $\Gamma_p \subset B(0,2)$.

Similarly, if $|p| \leq 2$ and if Γ_p is not contained in $B(0,4)$, we apply Theorem 1.9 taking Ω to be a subarc of Γ_p joining the boundary components of $D = \overline{B}(0,4) \setminus B(0,2)$ in $\overline{D}$. Then $d(1,\Omega) \geq 1$, $\operatorname{cap}\Omega \geq 1/2$, and we may take $A_1 = 5$. Now (17) and (18) yield $|f(1)| \leq 25 \cdot 2 \cdot 2 = 100$. So we may assume that if $|p| \leq 2$ then $\Gamma_p \subset B(0,4)$.

If $|p| \geq 1$ and if $\Gamma_p \cap \overline{B}(0,1/2) \neq \emptyset$ we apply Theorem 1.9 taking Ω to be a subarc of Γ_p joining the boundary components of $D = \overline{B}(0,3/4) \setminus B(0,1/2)$ in $\overline{D}$. Then $d(1,\Omega) \geq 1/4$, $\operatorname{cap}\Omega \geq 1/16$, and we may take $A_1 = 7/4$. Now (17) and (18) yield

$$|f(1)| \leq 2 \cdot (49/16) \cdot 16 \cdot (1 + 1/(1/4)) = 490.$$

So we may assume that if $|p| \geq 1$ then $\Gamma_p \cap \overline{B}(0,1/2) = \emptyset$.

If $|p| \geq 1$ and $z \in \Gamma_p$, then $|g(z)| = |f(z)|/|z| \leq 1$ if $|z| \geq 1$, and $|g(z)| \leq 1/|z| < 2$ if $|z| < 1$. So if $K \geq 4$ then $|g(z)| < K/2$ for all $z \in \Gamma_p$. Thus $p \notin \Omega_1$. Indeed, $\partial\Omega_1$ must have as a component a Jordan curve γ_p such that $p \in \overline{\operatorname{int}\Gamma_p} \subset \operatorname{int}\gamma_p$. Of course, the curve γ_p might be the same for several such poles p.

If $|p| < 2$, then $|f(z)| \leq \max\{1, |z|\} < 4 \leq K$ for all $z \in \Gamma_p$, so the component Ω_p of $\{z : |f(z)| > K\}$ containing p lies in $\operatorname{int}\Gamma_p$.

This also shows that all these sets $\overline{\Omega_p}$ are disjoint from each other, that is, no two distinct poles p_1 and p_2 of f with $|p_1| < 2$ and $|p_2| < 2$ can lie in the same component of $\{z : |f(z)| > K\}$, and indeed $\overline{\Omega_{p_1}}$ cannot separate p_2 from infinity since each Γ_p is a Jordan curve and $\overline{\operatorname{int}\Gamma_{p_1}} \cap \overline{\operatorname{int}\Gamma_{p_2}} = \emptyset$.

When $|p| < 1$, possibly $p = 0$, let T_p denote the outer boundary component of the component of $\{z : |f(z)| < K\}$ containing the component Γ_p of ∂G. Then $\Omega_p \subset \operatorname{int} T_p$. If T is one of the sets T_p, then $\operatorname{int} T$ may contain several such poles p (that is, we may have the same set T_p for several poles p).

2.11. **The poles of g in Ω_1.** Now Ω_1 must contain at least one pole of g, and by the above, all such poles must have modulus < 1. We do not know whether $0 \in \Omega_1$.

Let $p_1, \ldots, p_\kappa$ be the distinct poles of g (and hence of f) in Ω_1. For any pole p of g, let $s(p)$ denote the order of p as a pole of g.

Pick p_j and set $T = T_{p_j}$. Now $\operatorname{int} T$ may contain several points p_i, say q of them, and we call them $\alpha_1, \ldots, \alpha_q$. If $\operatorname{int} T$ contains poles of g that are not in Ω_1, denote those distinct poles by β_j for $1 \le j \le J$. Let Ω_{α_i} or Ω_{β_j} denote the component of $\{z : |f(z)| > K\}$ containing α_i or β_j, respectively.

We may assume that $T \subset B(0,2)$ and hence $\overline{\operatorname{int} T} \subset B(0,2)$. Namely, we have $p_j \in \operatorname{int} T$ and $|p_j| < 1$. If it were not true that $T \subset B(0,2)$, we could apply Theorem 1.9 as above, taking Ω to be a subarc of T joining the boundary components of $D = \overline{B}(0,2) \setminus B(0,3/2)$ in $\overline{D}$, replacing G by $G_1 = G \setminus \Omega$, and replacing the function $\mu(t) = \max\{1,t\}$ by $\max\{K,t\}$. If K is about 4, we get from (17) and (18) that

$$|f(1)| \le 432K \approx 1728.$$

It is this condition that via $2 \cdot 1728 = 3456$ (arising from the application of Lemma 1.8) gives rise to the constant in Theorem 1.11. Hence we may assume that $\overline{\operatorname{int} T} \subset B(0,2)$. It follows that $|\beta_j| < 2$ for $1 \le j \le J$.

Recall that T is a component of $L(f,K)$. Also $|f(z)| < K$ for all $z \in \operatorname{int} T$ close enough to T. The only components of $\{z : |f(z)| > K\}$ inside $\operatorname{int} T$ are the sets Ω_{α_i} for $1 \le i \le q$ and the sets Ω_{β_j} for $1 \le j \le J$. Hence

$$W = \operatorname{int} T \setminus \left(\left(\bigcup_{i=1}^{q} \overline{\Omega_{\alpha_i}} \right) \cup \left(\bigcup_{j=1}^{J} \overline{\Omega_{\beta_j}} \right) \right)$$

is the union of components of $\{z : |f(z)| < K\}$. One component of W, say W_0, satisfies $T \subset \partial W_0$. Any other component of W is simply connected and is contained in a bounded component of $\mathbb{C} \setminus \Omega_c$ where

$c \in \{\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_J\}$. For if a component of W other than W_0 were multiply connected, then a bounded component of $\mathbb{C} \setminus \Omega_c$ would have to contain a pole of f, which is impossible since for any distinct $c_1, c_2 \in \{\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_J\}$, we have $|c_1|, |c_2| < 2$, so that the set $\overline{\Omega_{c_1}}$ cannot separate c_2 from infinity, as we saw above.

Since $|f(1)| > K$, we have $1 \notin \overline{\operatorname{int} T}$. Namely, the point 1 must lie in a component of $\{z : |f(z)| > K\}$, and this component must contain a pole c of f. If $1 \in \overline{\operatorname{int} T}$, then $c \in \{\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_J\}$, so $|c| < 2$. But we have seen that a component of $\{z : |f(z)| > K\}$ containing a pole c of f with $|c| < 2$ lies completely outside G and thus cannot contain the point $1 \in G$.

2.12. **Zeros of g in $\operatorname{int} T$.** Let $f(z)$ traverse $S(0, K)$ ℓ_T times as z traverses T. Note that $f(z)$ traverses $S(0, K)$ a total number of $s(\alpha_i)$ times as z traverses all the components of $\partial\Omega_{\alpha_i}$ if $\alpha_i \neq 0$, and $s(\alpha_i) - 1$ times if $\alpha_i = 0$. Similar observations apply to the β_j. Thus f, and hence g, has

$$\ell_T - \delta + \sum_{i=1}^{q} s(\alpha_i) + \sum_{j=1}^{J} s(\beta_j)$$

zeros in W, where $\delta = 1$ if some $\alpha_i = 0$ or some $\beta_j = 0$, and $\delta = 0$ otherwise. This is true whether W is connected or not. Regardless of the value of J, each zero of g in W lies in a component W' of the set $\{z : |g(z)| < K/2\}$.

Suppose that we move along an arc from a zero of g in a set W' towards any component of $L(f, K)$ in $\overline{\operatorname{int} T}$. Once we come to a point $\zeta \in L(f, K)$ for the first time in this way, we have $|g(\zeta)| = |f(\zeta)|/|\zeta| = K/|\zeta| > K/2$ since $|\zeta| < 2$ for all $\zeta \in \overline{\operatorname{int} T}$. Thus we will have left the set W' by then. This implies that each set W' satisfies $\overline{W'} \subset W$. Also this means that $|g(\zeta)| > K/2$ for all $\zeta \in T$, so T lies in a single component of $\{z : |g(z)| > K/2\}$.

Since $1 \notin \operatorname{int} T$, we can join the point 1 by an arc in Ω_1 to p_j, and this arc must pass through T. Thus $T \cap \Omega_1 \neq \emptyset$, and hence $T \subset \Omega_1$.

No set W' can separate any two points of the form α_i from each other or from T, since these points α_i belong to the same component Ω_1 of $\{z : |g(z)| > K/2\}$ and $T \subset \Omega_1$.

Those sets W' not contained in W_0 are simply connected. Any such set W' is contained in a bounded component U of $\mathbb{C} \setminus \overline{\Omega_c}$ for some $c \in \{\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_J\}$. If such a set W' were multiply connected, then U would have to contain a component of $\{z : |g(z)| > K/2\}$, and hence a pole of g (hence of f), which is impossible, as we have seen.

Let us generically denote by W'' those sets W' whose boundary intersects $\partial\Omega_1$. By the above two paragraphs, any set W' is among the W'' if $W' \subset U$ where U is a bounded component of $\mathbb{C} \setminus \overline{\Omega_c}$ for some $c \in \{\alpha_1, \dots, \alpha_q\}$.

Since $T \subset \Omega_1$, the outer boundary component ω of each W'' is part of the boundary of Ω_1. Note that for any W'', any other W' whose closure is a proper subset of $\operatorname{int}\omega$ is not among the sets W''. Thus any Jordan curve ω like this corresponds to a unique set W''. The argument principle implies that no matter how many "layers" of components of $\{z : |g(z)| < K/2\}$ and $\{z : |g(z)| > K/2\}$ may lie in $\operatorname{int}\omega$, the number of times that g covers $S(0, K/2)$ on ω is equal to $|Z - P|$ where Z is the number of zeros of g in $\operatorname{int}\omega$ and P is the number of poles of g in $\operatorname{int}\omega$, counting multiplicities. The total number of poles counted in P over all W'' is at most $\sum_{j=1}^{J} s(\beta_j)$.

In fact, $|Z - P| = Z - P$ here since $Z > P$. We prove by induction over the number of layers that $Z > P$ in a setting like this if the outermost component considered is a component of $\{z : |g(z)| < K/2\}$, and that $Z < P$ if the outermost component considered is a component of $\{z : |g(z)| > K/2\}$. This is clear if $\operatorname{int}\omega$ coincides with a component of $\{z : |g(z)| < K/2\}$ or of $\{z : |g(z)| > K/2\}$ since in the first case $Z > 0 = P$, and in the second case $Z = 0 < P$. For the induction step in the number of layers, consider a situation where ω is the outer component of ∂Y for a possibly multiply connected component of $\{z : |g(z)| < K/2\}$, and let ω_j, for $1 \le j \le J'$, be the inner components of ∂Y. Then each ω_j is the outer boundary component of a component of $\{z : |g(z)| > K/2\}$. By the induction assumption, if Z_j and P_j denote the number of zeros and poles of g inside the interior of ω_j, and if g covers $S(0, K/2)$ $\lambda_j \ge 1$ times on ω_j, then $\lambda_j = P_j - Z_j > 0$. If g covers $S(0, K/2)$ $\lambda \ge 1$ times on ω, then in the set Y, the function g has no poles and $Z_0 = \lambda + \sum_{j=1}^{J'} \lambda_j$ zeros. Hence for the set $\operatorname{int}\omega$, we have

$$Z - P = Z_0 + \sum_{j=1}^{J'} (Z_j - P_j) = \lambda + \sum_{j=1}^{J'} \lambda_j - \sum_{j=1}^{J'} \lambda_j = \lambda \ge 1,$$

as required for the induction step. If, instead, Y is a component of $\{z : |g(z)| > K/2\}$ containing P_0 poles of g, then the same argument gives $Z_j - P_j = \lambda_j \ge 1$ for each j, and $P_0 = \lambda + \sum_{j=1}^{J'} \lambda_j$, so that

$$P - Z = P_0 + \sum_{j=1}^{J'} (P_j - Z_j) = \lambda + \sum_{j=1}^{J'} \lambda_j - \sum_{j=1}^{J'} \lambda_j = \lambda \ge 1,$$

as required.

As z traverses the outer boundary components of the sets W'', the point $g(z)$ traverses $S(0, K/2)$ a total of at least

$$\left(\ell_T - \delta + \sum_{i=1}^{q} s(\alpha_i) + \sum_{j=1}^{J} s(\beta_j)\right) - \sum_{j=1}^{J} s(\beta_j) = \ell_T - \delta + \sum_{i=1}^{q} s(\alpha_i)$$

times.

Hence each set of points $\alpha_1, \ldots, \alpha_q$ like this contributes $\sum_{i=1}^{q} s(\alpha_i)$ poles of g into Ω_1, and at least $\ell_T - \delta + \sum_{i=1}^{q} s(\alpha_i)$ times to the coverage of $S(0, K/2)$ by $g(z)$ on $\partial\Omega_1$.

All the poles of g in Ω_1 arise in this way. It follows that the total number of poles of g in Ω_1 is

$$M_1 = \sum_T \sum_{i=1}^{q} s(\alpha_i)$$

where the sum is taken over all distinct curves T like this and each q depends on T. Further, the number of times that $g(z)$ covers $S(0, K/2)$ on $\partial\Omega_1$ is M_2, say, where

$$M_2 \geq M_3 = \ell - \delta + \sum_T \left(\ell_T + \sum_{i=1}^{q} s(\alpha_i)\right).$$

Here $\delta = 1$ if $0 \in \Omega_1$, and $\delta = 0$ otherwise. Recall that ℓ is the number of times that g covers $S(0, K/2)$ on Γ_5, the outer boundary component of Ω_1. In this calculation, we have taken into account Γ_5 and the outer boundary components of the sets W''. These are certainly components of $\partial\Omega_1$, but possibly not all the components of $\partial\Omega_1$.

Note that we may have $M_2 > M_3$ since $\partial\Omega_1$ may have further components arising from poles p of g with $|p| \geq 1$ as explained above, or from zeros of g in D_5 not accounted for above (corresponding to the number ν in the argument for doubly connected domains).

We must have $M_1 = M_2$. We have

$$M_2 - M_1 \geq M_3 - M_1 = \ell - \delta + \sum_T \ell_T \geq 1$$

since $\ell \geq 1$, $\delta \leq 1$, each $\ell_T \geq 1$, and since there is at least one curve T involved.

This is a contradiction (essentially the same contradiction as in the doubly connected case), and the proof of Theorem 1.11 is complete when G is connected.

2.13. **Disconnected open sets.** If G is an open set with distinct bounded components G_1 and G_2 with disjoint closures, if $w_1 = 0 \in \partial G_1$ and $w_2 = 1 \in G_2$, and if we wish to apply the argument above to estimate $|f(w_1) - f(w_2)|$, we proceed as follows. If G_1 is contained in a bounded component of $\mathbb{C} \setminus G_2$, the situation is the same as for a domain G_2, so we follow the proof above throughout. If G_1 is contained in the unbounded component of $\mathbb{C} \setminus G_2$, we apply Theorem 1.9, taking Ω to be an arc of the outer boundary component ω of G_2 containing a point $r \in (0, 1)$ (there must be such a point) and of diameter $(1 - r)/2$; such an arc is obtained as a subarc of an arc of ω joining r to a point on $S(0, (1 + r)/2)$ (note that $1 \in \operatorname{int} \omega$). Then in Theorem 1.9, we may take $A_1 = (3/2)(1 - r)$, $A_2 = (1 - r)/8$, and $d = (1 - r)/2$, so that $C = 36(3-r) < 108$ in (18). This completes the proof of Theorem 1.11.

2.14. **Discussion.** We compared the number of poles of g in Ω_1 to the number of times that g covers $S(0, K/2)$ on $\partial\Omega_1$. These numbers must be equal. We show that the covering number is greater than the number of poles, obtaining a contradiction. The contradiction is only obtained if certain level curves do not stretch too far. But if they do, then Theorem 1.9 gives an upper bound for $|f(1)|$ anyway.

The condition that f is small enough on ∂G caused at least the poles of g of modulus ≥ 1 not to be in Ω_1. We thus find an upper bound for the number of poles of g in Ω_1. We had to define g to get a function that is bounded, and indeed small enough, on the outer boundary Γ_1 of G. Then g will have one more finite pole than f. This gives rise to the number δ above. This may increase the number of poles of g in Ω_1 by 1, which is the price to be paid for introducing g.

However, the price was not too great, since the coverage of $S(0, K/2)$ under g on $\partial\Omega_1$ is increased by at least 1 due to the presence of the outer boundary component of Ω_1. Other than that, zeros of g (that is, zeros of f) in D_5 force there to be more coverage of $S(0, K/2)$ under g on $\partial\Omega_1$. The way to see that there have to be a lot of zeros of f in D_5 was to show that there were components of $\{z : |f(z)| < K\}$ and of $\{z : |f(z)| > K\}$ in D_5 in such a relative position that the presence of many zeros of f was forced to exist. Also the sets W' were kept localized, and forced to lie inside definite sets of the form $\operatorname{int} T$ in this way, to make the counting clear. If it were the case that the components of $L(f, K)$ could extend further, then this argument concerning the relative position of certain sets would fail. But then Theorem 1.9 would give the desired upper bound for $|f(1)|$.

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