GENERALISATIONS OF UNIFORMLY NORMAL FAMILIES

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Abstract. We obtain separation and growth results for meromorphic functions f in the unit disk such that for some positive integer k, $f \neq 0$ and $f^{(k)} \neq 1$ in the disk, or such that $f' f^k \neq 1$ in the disk. These results are only slightly weaker than those for functions f such that $|f| > \delta$ in the disk.

1. Introduction

In this paper, we prove the following result. We denote the unit disk by $\mathbb{D}$.

Theorem 1. *Suppose that f is analytic in $\mathbb{D}$ and that there exist positive functions $\delta(r)$ and $\lambda(r)$ defined in $(0,1)$ such that $\delta(r)$ decreases, $0 < \delta(r) < 1$, $\lambda(r)$ increases, and for all r, $0 < r < 1$, the conditions*

$$|z_1| \leq r, \quad |z_2| \leq r, \tag{1}$$

and

$$|f(z_1)| \leq e^{-\lambda(r)}, \quad |f(z_2)| \geq e^{\lambda(r)} \tag{2}$$

imply that

$$\left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right| \geq \delta(r). \tag{3}$$

Then if $|z_0| < r$ and

$$\log |f(z_0)| > \frac{8\lambda(r)}{\delta(r)}, \tag{4}$$

AMS (2000) *Mathematics Subject Classification.* Primary 30D45. Secondary 30D35.
This material is based upon work supported by the National Science Foundation under Grant No. 0200752.

we have

$$|f'(z_0)| < \frac{2r|f(z_0)|}{r^2 - |z_0|^2} \left\{ \log|f(z_0)| + \frac{8\lambda(r)}{\delta(r)} \right\}. \tag{5}$$

If $\lambda(r)$ and $\delta(r)$ are constant, then the functions f satisfying the hypotheses of Theorem 1 form a uniformly normal family in the sense of [1].

In most of the applications we have

$$\frac{8\lambda(r)}{\delta(r)} = a + b \log \frac{1}{1-r}, \tag{6}$$

where a and b are non-negative constants.

Corollary 1. *Let f, $\lambda(r)$ and $\delta(r)$ satisfy the hypotheses of Theorem 1, and also (6). If $|z| = t$ and $r = 1 - \frac{1}{2}(1-t)^2$, then either $\log|f(z)| \le \lambda^*(t)$ or*

$$\frac{|f'(z)|}{|f(z)|} < \left\{ \frac{2}{1-t^2} + 8 \right\} \left\{ \log|f(z)| + \lambda^*(t) \right\}, \tag{7}$$

where $\lambda^*(t) = a + b\log 2 + 2b \log \frac{1}{1-t} = a + b \log \frac{1}{1-r}$. Further we have in all cases

$$\log|f(z)| \le e^8 \left\{ 3a + 8b + \log^+ |f(0)| \right\} \frac{1+|z|}{1-|z|}. \tag{8}$$

2. A SPECIAL CASE

In this section we deal with the case when $\delta(r)$ and $\lambda(r)$ are constant.

Lemma 1. *Suppose that $\lambda(r) = \lambda$ and $\delta(r) = \delta$, where λ and δ are constants which satisfy the hypotheses of Theorem 1, and that (4) holds so that*

$$\alpha = \log|f(z_0)| > \frac{8\lambda}{\delta}. \tag{9}$$

Then

$$|f(z)| > e^{\lambda}, \qquad \text{when } \left| \frac{z - z_0}{1 - \overline{z_0} z} \right| < r_1, \tag{10}$$

where

$$1 - r_1 = \frac{4\lambda}{\alpha\delta} < \frac{1}{2}. \tag{11}$$

Thus

$$|f'(z_0)| < \frac{2|f(z_0)|}{1-|z_0|^2}\left\{\log|f(z_0)| + \frac{8\lambda}{\delta}\right\}. \tag{12}$$

We proceed to prove (10) subject to (11). Since the hypotheses (1) to (4) are now conformally invariant, we assume without loss of generality that $z_0 = 0$. Let r_2 be maximal subject to $|f(z)| > e^{-\lambda}$ if $|z| < r_2$.

We suppose first that $r_2 < 1$. Let r_1 be the largest number such that

$$|f(z)| > e^{\lambda} \quad \text{for } |z| < r_1. \tag{13}$$

Then $0 < r_1 < r_2 < 1$ and there exists $z_2 = r_2 e^{i\theta}$ such that

$$|f(z_2)| = e^{-\lambda}.$$

We set $z_1 = r_1 e^{i\theta}$. Then (3) yields

$$\left|\frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right| = \frac{r_2 - r_1}{1 - r_1 r_2} \geq \delta,$$

i.e.,

$$r_2 \geq \frac{r_1 + \delta}{1 + \delta r_1}. \tag{14}$$

We now apply Harnack's inequality to $\log|\phi|$ where

$$\phi(z) = e^{\lambda} f(r_2 z)$$

and choose z so that $|z| = r_1/r_2$ and $|f(r_2 z)| = e^{\lambda}$. This is possible since r_1 is maximal subject to (13). Then $|\phi(z)| > 1$ for $|z| < 1$, and so

$$\log|\phi(z)| \geq \frac{1-|z|}{1+|z|}\log|\phi(0)|, \tag{15}$$

i.e.,

$$2\lambda \geq \frac{r_2 - r_1}{r_2 + r_1}(\lambda + \alpha). \tag{16}$$

If $r_2 = 1$ and r_1 is maximal subject to (13), then either $r_1 = 1$, in which case (14) and (16) are trivial, or we can choose z, such that $|z| = r_1$ and $|f(z_1)| = e^{\lambda}$. Now (15) still holds and we obtain (16) as before with $r_2 = 1$. Thus (16) is always true. Also (14) is true if $r_2 = 1$ and $r_1 < 1$, since $\delta < 1$. Thus (14) and (16) always hold.

We substitute r_2 from (14) in (16) and obtain

$$\begin{aligned}\frac{2\lambda}{\lambda+\alpha} &\geq \left(\frac{r_1+\delta}{1+\delta r_1} - r_1\right)\Big/\left(\frac{r_1+\delta}{1+\delta r_1} + r_1\right) = \frac{\delta(1-r_1^2)}{\delta(1+r_1^2)+2r_1} \\ &\geq \frac{\delta(1-r_1^2)}{(1+r_1)^2} = \delta\frac{1-r_1}{1+r_1} \geq \frac{\delta}{2}(1-r_1).\end{aligned}$$

Thus

$$1 - r_1 \leq \frac{4\lambda}{\delta(\lambda + \alpha)} \leq \frac{4\lambda}{\delta\alpha} < \frac{1}{2}$$

by (9). This proves (10) subject to (11).

The function

$$\Psi(z) = e^{-\lambda} f(r_1 z)$$

satisfies $|\Psi(z)| > 1$ if $|z| < 1$. Thus by Borel's inequality

$$|\Psi'(0)| \leq 2|\Psi(0)| \log |\Psi(0)|,$$

i.e.,

$$\begin{aligned} \frac{|f'(0)|}{|f(0)|} &\leq \frac{2}{r_1}(\log |f(0)| - \lambda) = \frac{2}{r_1}(\alpha - \lambda) \leq 2(\alpha - \lambda)\frac{1}{1 - \frac{4\lambda}{\delta\alpha}} \\ &= 2(\alpha - \lambda) + 2(\alpha - \lambda)\frac{4\lambda}{\delta\alpha - 4\lambda} \\ &< 2\alpha + \frac{16\alpha\lambda}{\delta\alpha} = 2\left(\alpha + \frac{8\lambda}{\delta}\right) \end{aligned}$$

since $\delta\alpha - 4\lambda > \delta\alpha/2$ by (11). This proves (12) if $z_0 = 0$. If $z_0 \neq 0$, we apply the above result to F instead of f at the origin, where $F(z) = f((z + z_0)/(1 + \overline{z_0} z))$. This yields (12) in general.

3. Proof of Theorem 1

To prove Theorem 1, we fix r with $0 < r < 1$, and apply Lemma 1 with $F(z) = f(rz)$ instead of $f(z)$ and with $\lambda = \lambda(r)$ and $\delta = \delta(r)$.

We write $z_1 = rZ_1$, $z_2 = rZ_2$ and suppose that (1) and (2) imply (3). Thus if $|Z_1| < 1$, $|Z_2| < 1$ and

$$|F(Z_1)| \leq e^{-\lambda}, \quad |F(Z_2)| \geq e^{\lambda}, \tag{17}$$

we deduce that

$$\left| \frac{r(Z_2 - Z_1)}{1 - r^2 \overline{Z_1} Z_2} \right| \geq \delta, \tag{18}$$

with $\delta = \delta(r)$ and $\lambda = \lambda(r)$.

We next prove that

$$\frac{|Z_2 - Z_1|}{|1 - \overline{Z_1} Z_2|} > \frac{r|Z_2 - Z_1|}{|1 - r^2 \overline{Z_1} Z_2|}. \tag{19}$$

To see this, note that $Z_1 \neq Z_2$ and

$$|1 - r^2 \overline{Z_1} Z_2|^2 - r^2 |1 - \overline{Z_1} Z_2|^2 = (1 - r^2)(1 - r^2 |Z_1|^2 |Z_2|^2) > 0,$$

which yields (19). Thus (18) implies that

$$\left|\frac{Z_2 - Z_1}{1 - \overline{Z_1} Z_2}\right| > \delta. \tag{20}$$

So we can apply Lemma 1 to $F(z) = f(rz)$ instead of $f(z)$, and with $Z_0 = z_0/r$ instead of z_0. This yields (5).

We next prove Corollary 1. We have, with the notation of Corollary 1,

$$\frac{1}{r^2 - t^2} - \frac{1}{1 - t^2} = \frac{(1-r)(1+r)}{(r-t)(r+t)(1-t)(1+t)} \leq \frac{(1-t)^2}{\frac{1}{2}(1-t)^2 \cdot \frac{1}{2}} = 4,$$

since $r = 1 - \frac{1}{2}(1-t)^2 \geq \frac{1}{2}$ and $r - t \geq \frac{1}{2}(1-t)$.

Thus (5) yields, with $\alpha = \log|f(z_0)|$,

$$|f'(z_0)| < \frac{2|f(z_0)|}{r^2 - t^2}\{\alpha + \lambda^*(t)\} < 2|f(z_0)|\left\{\frac{1}{1-t^2} + 4\right\}\{\alpha + \lambda^*(t)\}.$$

This proves (7).

To deduce (8), we fix θ with $\theta \in [0, 2\pi)$ and write $y(t) = \log|f(te^{i\theta})|$ for $0 \leq t < 1$. If $y(t) \leq \lambda^*(t)$ or if $t = 0$, (8) clearly holds for $z = te^{i\theta}$. So we suppose that for some t with $0 < t < 1$, we have

$$y(t) > \lambda^*(t).$$

We choose t_0 to be maximal, subject to $0 \leq t_0 < t$ and $y(t_0) \leq \lambda^*(t_0)$. If $y(\tau) > \lambda^*(\tau)$ for all τ with $0 \leq \tau < t$, we set $t_0 = 0$. Then

$$y(\tau) > \lambda^*(\tau) \quad \text{for } t_0 < \tau < t.$$

Thus we can apply (7) with τ instead of t in this range and obtain

$$y'(\tau) - \left\{\frac{2}{1-\tau^2} + 8\right\} y(\tau) < \left\{\frac{2}{1-\tau^2} + 8\right\} \lambda^*(\tau), \qquad \text{for } t_0 < \tau < t.$$

Multiplying by

$$P(\tau) = e^{-8\tau}\frac{1-\tau}{1+\tau}$$

and integrating with respect to τ from t_0 to t we obtain

$$y(t) \leq \frac{1}{P(t)}\left\{y(t_0)P(t_0) + \int_{t_0}^{t}\left(\frac{2}{1-\tau^2} + 8\right)P(\tau)\lambda^*(\tau)\,d\tau\right\}. \tag{21}$$

If $t_0 = 0$, we get $y(t_0)P(t_0) = y(t_0) \leq \log^+|f(0)|$.

If $t_0 > 0$, we have $y(t_0)P(t_0) = \lambda^*(t_0)P(t_0)$.

We write

$$\lambda^*(t) = a^* + b^* \log\frac{1}{1-t}$$

with $a^* = a + b\log 2$ and $b^* = 2b$. Then

$$\begin{aligned}\lambda^*(t)P(t) &= \left(a^* + b^* \log\frac{1}{1-t}\right)\frac{1-t}{1+t}e^{-8t}\\ &\le a^* + b^* \sup_{0\le t<1}(1-t)\log\frac{1}{1-t} = a^* + \frac{b^*}{e}.\end{aligned}$$

Again

$$\int_0^1 \left(\frac{2}{1-t^2}+8\right)\frac{1-t}{1+t}e^{-8t}\,dt < \int_0^1 10e^{-8t}\,dt < \frac{5}{4}$$

while

$$\begin{aligned}&\int_0^1 \left(\frac{2}{1-t^2}+8\right)\left(\frac{1-t}{1+t}\log\frac{1}{1-t}\right)e^{-8t}\,dt\\ &< 2\int_0^1 \log\frac{1}{1-t}\,dt + \frac{8}{e}\int_0^1 e^{-8t}\,dt < 2 + \frac{1}{e}.\end{aligned}$$

Thus

$$\int_0^1 \left(\frac{2}{1-t^2}+8\right)P(t)\lambda^*(t)\,dt \le \frac{5}{4}a^* + \left(2+\frac{1}{e}\right)b^*.$$

Hence (21) yields finally

$$\begin{aligned}y(t) &\le \frac{1}{P(t)}\left\{a^* + \frac{b^*}{e} + \frac{5}{4}a^* + \left(2+\frac{1}{e}\right)b^* + \log^+|f(0)|\right\}\\ &\le \frac{1}{P(t)}\left\{3a + 8b + \log^+|f(0)|\right\}\end{aligned}$$

which implies (8).

This proves Corollary 1.

4. Applications

Suppose that M_k and A_k are respectively the families of meromorphic functions f in $\mathbb{D}$ and analytic functions f in $\mathbb{D}$, such that $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for all $z \in \mathbb{D}$. Then M_k and in particular A_k are normal families in $\mathbb{D}$ when $k \ge 1$, see, e.g., [3, Corollary 4.5.9, p. 150]. It follows that there exist positive constants λ_k and δ_k depending only on k such that if $f \in M_k$ and $|z_j| \le \delta_k$, for $j = 1, 2$, then we cannot have $|f(z_1)| \le e^{-\lambda_k}$ and $|f(z_2)| \ge e^{\lambda_k}$. In fact the conclusion must hold for every sufficiently small δ_k, and λ_k depending on δ_k and k only.

We apply the above conclusion with $|z_1| < 1$ and

$$F(z) = (1-|z_1|)^{-k} f(z_1 + (1-|z_1|)z) \tag{22}$$

instead of $f(z)$. Then clearly F belongs to M_k or A_k if f does.

We deduce the following result.

Corollary 2. *If k is a positive integer and $f \in M_k$, then $1/f$ satisfies the hypotheses and hence the conclusion of Theorem 1 and Corollary 1 with $\delta(r) = \frac{1}{3}\delta_k$ for $0 < r < 1$, and*

$$\frac{8\lambda(r)}{\delta(r)} = \frac{24}{\delta_k}\left\{\lambda_k + k\log\frac{1}{1-r}\right\}, \tag{23}$$

$$\lambda^*(t) = \frac{24}{\delta_k}\left\{\lambda_k + k\log 2 + 2k\log\frac{1}{1-t}\right\}, \tag{24}$$

where δ_k and λ_k are positive constants depending only on k. If f is also analytic then f satisfies the same conclusions.

We suppose that $|z_1| \le |z_2| = r$ and that $f \in M_k$. We consider

$$F(Z) = \frac{1}{(1-|z_1|)^k} f(z_1 + (1-|z_1|)Z).$$

Then clearly $F(Z) \neq 0$ and $F^{(k)}(Z) = f^{(k)}(z_1 + (1-|z_1|)Z) \neq 1$ when $Z \in \mathbb{D}$, so that $F \in M_k$. In particular if $Z_1 = 0$ and $Z_2 = (z_2 - z_1)/(1-|z_1|)$ and

$$|Z_2| < \delta_k \tag{25}$$

we cannot have

$$|F(Z_j)| \le e^{-\lambda_k}, \qquad |F(Z_{j'})| \ge e^{\lambda_k}, \tag{26}$$

where (j, j') is a permutation of $(1,2)$.

Returning to f, we see that

$$|f(z_j)| < e^{-\lambda_k}(1-|z_1|)^k, \quad \text{ and } |f(z_{j'})| > e^{\lambda_k}(1-|z_1|)^k \tag{27}$$

imply that

$$|z_{j'} - z_j| > \delta_k(1-|z_1|), \tag{28}$$

i.e.,

$$\frac{|z_2 - z_1|}{1-|z_1|} > \delta_k. \tag{29}$$

Now if $\eta = |z_2 - z_1|/(1-|z_1|) \le 1$, we have

$$\begin{aligned} |1 - \overline{z_1}z_2| &= |1 - \overline{z_1}z_1 + \overline{z_1}z_1 - \overline{z_1}z_2| \le 1 - |z_1|^2 + |z_2 - z_1| \\ &= (1-|z_1|)(1 + |z_1| + \eta) < 3(1-|z_1|). \end{aligned}$$

Thus if $|z_2 - z_1| \le 1 - |z_1|$, we have

$$\left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right| > \frac{|z_2 - z_1|}{3(1 - |z_1|)} > \frac{\delta_k}{3} \tag{30}$$

if (29) holds. We have assumed that $|z_2 - z_1| \le 1 - |z_1|$. But if this is false, (30) is still true by the maximum principle. Thus (28) always implies (30), and so does

$$|f(z_j)| < (1-r)^k e^{-\lambda_k}, \qquad |f(z_{j'})| > (1-r)^{-k} e^{\lambda_k}$$

since this implies (27) and hence (28), because $r = \max\{|z_j|, |z_{j'}|\}$. This proves Corollary 2.

We note that the hypotheses (1) to (3) in Theorem 1 are the same for f and $1/f$. Hence if f satisfies (1) to (3) and $1/f$ is analytic, then $1/f$ satisfies the conclusions of Theorem 1. In particular if $f \in M_k$, so that $f \neq 0$ in $\mathbb{D}$, we can apply the conclusion of Theorem 1 to $1/f$ instead of f, with $\lambda(r)$ given by (23) and hence $\lambda^*(r)$ by (24).

As another example we have the following result. We denote by $\mathcal{M}_k$ and $\mathcal{A}_k$ respectively the family of meromorphic functions f and analytic functions f in $\mathbb{D}$ such that $f'(z)f(z)^k \neq 1$ in $\mathbb{D}$. By [3, Theorem 4.4.18, p. 132], the family $\mathcal{A}_k$ is normal. For the result that also the family $\mathcal{M}_k$ is normal we refer the reader to [4, p. 226] and the references given therein.

Corollary 3. *If k is a positive integer and $f \in \mathcal{A}_k$, then f satisfies the hypotheses and hence the conclusion of Theorem 1 and Corollary 1 with $\delta(r) = \eta_k$, and*

$$\lambda(r) = \mu_k + \frac{1}{k+1} \log \frac{1}{1-r},$$

where η_k and μ_k are positive constants depending only on k.

We write $F(z) = (1 - |z_1|)^{-1/(k+1)} f(z_1 + (1 - |z_1|)z)$ and proceed as in the proof of Corollary 2.

The functions in M_k and those in $\mathcal{M}_k$ also have locally bounded characteristic. This follows directly from Theorem 1 and Theorem 3 of [2]. In fact if

$$\mu(r) = \sup_{|z| \le r} f^\sharp(z) = \sup_{|z| \le r} \frac{|f'(z)|}{1 + |f(z)|^2},$$

then even for meromorphic f we have under the hypotheses of Corollaries 2 and 3 that

$$\mu(r) = O\left(\frac{1}{1-r}\right) \log \frac{1}{1-r},$$

so that

$$\int_0^1 (1-r)^{1/2}\mu(r)\,dr < \infty$$

which is the hypothesis of Theorem 3 in [2].

For the definition of locally bounded characteristic and other applications we refer the reader to [2].

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