ON THE NORM OF THE BEURLING–AHLFORS TRANSFORMATION

AIMO HINKKANEN

Abstract. We conjecture that functions have a hitherto unknown probabilistic structure: associated with certain combinations of first partial derivatives of functions, there are two fields of rotations, and two martingales that are martingale transforms of each other, starting from constants of equal modulus, and ending at what one obtains after rotating these combinations of the derivatives. We prove this result in certain cases of continuous piecewise affine functions in the plane depending on 13 complex parameters.

The motivation for this is that such a result would be sufficient to prove the conjectured value for the sharp $p-$norm of the Beurling–Ahlfors transformation in the plane, and its generalizations to space by T. Iwaniec and G.J. Martin. Indeed the result for the norms of these transformations would then follow from Burkholder's estimates for the norms of two martingales that are martingale transforms of each other. On the other hand, if we look for a way of obtaining the desired estimate for the norm of the Beurling–Ahlfors transformation, then we are naturally lead to considering martingales that are obtained after rotations from a function and its Beurling–Ahlfors transformation. Thus there seems to be such a strong connection between the two problems that one may be inclined to conjecture that the conjecture concerning the norm of the Beurling–Ahlfors transformation is true if, and only if, rotations and martingales of this kind exist.

In three-dimensional space, the generalized conjecture has a new physical interpretation as a relation between static electric and magnetic fields.

2000 *Mathematics Subject Classification.* Primary: 42B20; Secondary: 30E20.
This material is based upon work supported by the National Science Foundation under Grant No. 0457291.
This paper is an extended version of the talk given by the author at the conference "Lars Ahlfors Centennial Celebration" at the University of Helsinki, Finland, in August 2007. The author would like the thank the organizers of this conference, and particularly Ilkka Holopainen and Kirsi Peltonen, for their great efforts in organizing a pleasant and interesting conference.

1. Introduction: conjectured probabilistic structure of functions

We consider the problem of determining the L^p-norm of the Beurling–Ahlfors transformation. It was noticed in the 1980s that the conjectured value of the norm, $p^* - 1$, where

$$p^* = \max\{p, p/(p-1)\},$$

is the same as the constant in an inequality due to Burkholder relating L^p-norms of martingale transforms. However there has been no explanation for such a precise equality, and the known bounds, which are not sharp, have been obtained by applying martingale methods not to the function and its transform but after extending a function from the plane to a half space.

We suggest that the conjecture on the norm should be true and that the underlying reason for this should be the following: the function and its transform can be multiplied by functions of modulus one such that there *then* are martingales that are martingale transforms of each other going from constants of equal modulus to the function and its transform, multiplied by these functions of modulus one. This would then immediately yield the conclusion for the norm, in view of Burkholder's result.

The idea is based on the observation that the function and its transform do not seem to yield martingales as such, but that multiplying them by functions of modulus one would not change their norms, so that this is an admissible operation. On the basis of examples, it seems that the sigma-algebras associated with the martingales must depend on the function as well, suggesting that perhaps no approach based on some fixed structure will yield a sharp result.

We have proved that this idea works in a number of situations, making it perhaps more plausible that it might work in general.

The greater importance of this is that if these conjectures are true, this would yield a new probabilistic structure for functions (the functions in question are better viewed as certain first order partial derivatives of other functions, as we shall explain later), which generally should lead to a better understanding of the cancellation properties of partial derivatives (since martingales, by definition, are related to cancellation properties), and which might then lead to further insights and results for, for example, the cancellation properties of Jacobian determinants and other related quantities.

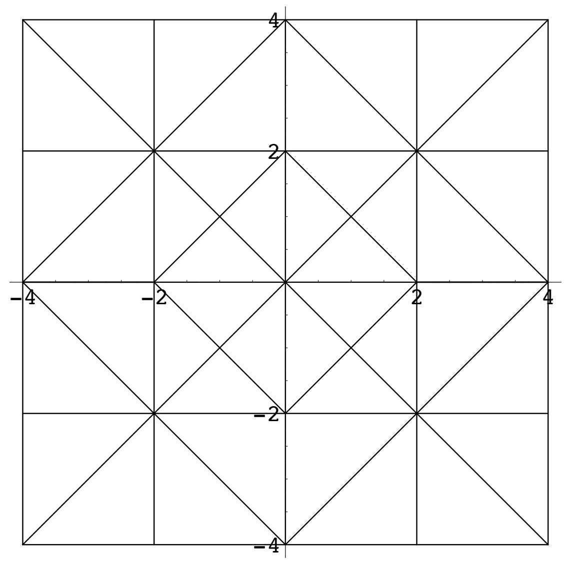

Figure 1. The case with 13 complex parameters. The big square with vertices $\pm 4 \pm 4i$ is the square Q.

The same problem arises in Euclidean $n-$space $\mathbb{R}^n$, for operators defined by T. Iwaniec and G.J. Martin that they also called Beurling–Ahlfors transformations. The same approach should work there. As we shall explain, in $3-$space this leads to the conjecture that (for a real-valued function φ and a $\mathbb{R}^3$-valued function $\mathbf{A}$)

$$||\nabla\varphi + \nabla \times \mathbf{A}||_p \leq (p^* - 1)||\nabla\varphi - \nabla \times \mathbf{A}||_p,$$

and that there are martingales related to the function $\nabla\varphi - \nabla \times \mathbf{A}$ with values in $\mathbb{R}^3$ and its "transform" $\nabla\varphi + \nabla \times \mathbf{A}$. With a physical interpretation, this reads

$$||\mathbf{E} - \mathbf{B}||_p \leq (p^* - 1)||\mathbf{E} + \mathbf{B}||_p$$

in terms of the static electric field $\mathbf{E}$ and the static magnetic field $\mathbf{B}$.

The specific theorem that we prove in this paper is the following.

Theorem 1. *Consider a square Q in $\mathbb{C}$ triangulated as in Figure 1. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous piecewise affine function that vanishes outside Q and is affine in each triangle in Figure 1. Thus f is uniquely determined by its values at the vertices of the triangulation strictly inside Q, so that f depends on 13 independent complex parameters. Then we have for $1 < p < \infty$*

$$||f_z||_p \leq (p^* - 1)||f_{\overline{z}}||_p\,. \tag{1}$$

Since the problem in invariant under translations, dilations, and rotations, the conclusion of Theorem 1 obviously applies to any square that has been triangulated in the same way as the particular square Q in Figure 1.

This paper is much longer than the minimum required to prove Theorem 1. The reason is that we will investigate the general features of approaching the problem of determining the $p-$norm of the Beurling–Ahlfors transformation by approximation using continuous piecewise affine functions of compact support in the plane, and using rotations to obtain functions that would be the end products of pairs of martingales that are martingale transforms of each other. We also discuss similar problems and results in the Euclidean $n-$space. In Section 17 we consider certain more general extremal problems, such as maximizing $||f_z||_p$ in a domain subject to prescribed boundary values for f and an upper bound for $||f_{\overline{z}}||_p$, which will lead us to the differential equation (somewhat similar to the $p-$harmonic equation)

$$\left(|f_z|^{p-2}\overline{f_z}\right)_z = \lambda \left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}$$

where λ is a positive constant.

2. Definitions and previous results

2.1. **Definitions. The Beurling–Ahlfors transformation** was introduced by A. Beurling in a talk that he gave in Uppsala in November 1949. If $f : \mathbb{C} \to \mathbb{C}$ is in $C_0^\infty(\mathbb{C})$, we define its *Beurling–Ahlfors transform* Sf at $z \in \mathbb{C}$ to be the Cauchy principal value integral

$$(Sf)(z) = \lim_{\varepsilon\to 0} \frac{-1}{\pi} \int_{|\zeta-z|>\varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta \tag{2}$$

where $\zeta = \xi + i\eta$. Beurling [7] provided the definition and proved that S extends as an isometry to $L^2(\mathbb{C} \to \mathbb{C})$. The terms "Beurling transformation" ([3], [18]) and "two-dimensional Hilbert transformation" ([1], [22]) have also been used of S.

Calderon and Zygmund ([12], 1952) proved that many singular integral operators in $\mathbb{R}^n$, where $n \geq 2$, including S in $\mathbb{R}^2$, extend to bounded linear operators in L^p for $1 < p < \infty$. Further, their analysis shows that for $f \in L^p$, (2) still holds for almost every $z \in \mathbb{C}$.

The Beurling–Ahlfors transformation soon found use in the theory of quasiconformal mappings, first independently by Ahlfors ([1], 1955) and by Vekua ([24], 1955), then particularly in the seminal paper by Bojarski ([8], 1957).

2.2. **Known facts concerning the L^p−norm of S.** We write

$$||f||_p^p = \int_{\mathbb{C}} |f(x+iy)|^p \, dx \, dy$$

and

$$||S||_p = \sup\{\, ||Sf||_p \, : \, ||f||_p \le 1 \,\}.$$

Beurling proved that $||S||_2 = 1$ and indeed $||Sf||_2 = ||f||_2$ for all $f \in L^2(\mathbb{C})$.

It is easy to see that if $1 < p < \infty$ and $1/p + 1/q = 1$ then $||S||_p = ||S||_q$.

Lehto [21] proved in 1965 that if $p > 2$ then $||S||_p \ge p - 1$. So if $1 < p < 2$ then $||S||_p \ge 1/(p-1)$.

The first upper bound for $||S||_p$ for $p \ne 2$ arises from the work of Calderon and Zygmund ([12], 1952) who proved that $||S||_p = O(p)$ as $p \to \infty$.

For a real number p with $p > 1$, define

$$p^* = \max\{p, p/(p-1)\}.$$

So $p^* = p$ if $p \ge 2$.

Conjecture of T. Iwaniec ([17], 1982) We have $||S||_p = p^* - 1$.

Motivated by this and by Burkholder's results [9], [10], several people have obtained upper bounds of the form $C(p^*-1)$ for $||S||_p$, for absolute constants $C > 1$.

Bañuelos and Wang ([5], 1995) showed that $||S||_p \le 4(p^* - 1)$. Nazarov and Volberg ([25], 2003) obtained $||S||_p \le 2(p^*-1)$. Dragičević and Volberg ([14], 2005) proved that

$$||S||_p \le \sqrt{2}(p-1) \left(\int_0^{2\pi} |\cos\theta|^p \, d\theta \right)^{-1/p} \qquad \text{for} \qquad 2 \le p < \infty.$$

Further, they were able to get the better asymptotic bound

$$\limsup_{p\to\infty} ||S||_p / p \le \sqrt{2}. \tag{3}$$

The best bound known at this time is due to Bañuelos and Janakiraman ([4], 2006). It says that $||S||_p \le \sqrt{2p(p-1)}$ for $2 \le p < \infty$, and in particular,

$$||S||_p \le 1.575\,(p-1).$$

They obtained from this the same asymptotic bound (3) as Dragičević and Volberg.

There are generalizations of S to weighted L^p-spaces (e.g., [23]) and to $\mathbb{R}^n$ ([19], [20]). There is a lot of recent literature on the Beurling–Ahlfors transformation on weighted L^p-spaces, but we will not attempt to provide more references since that subject is outside the scope of this paper. We will discuss the Beurling–Ahlfors transformation in $\mathbb{R}^n$ later on in this paper.

At one point, an important application of $||S||_p = p^* - 1$ would have been a proof of Gehring's [16] conjecture on the integrability of the derivatives of a quasiconformal map, but this was settled in another way by Astala [2]. Questions on the norm of S and its generalizations are nonetheless still of interest in quasiconformal theory, c.f. [23].

3. Burkholder's auxiliary functions

Starting in the early 1980's, D.L. Burkholder obtained the sharp solutions to several extremal problems involving martingale transforms, and developed various methods of proof to study such situations, in a series of papers, the principal ones for our purposes being [9], [10].

We recall the definition of a martingale. Usually martingales are defined on probablility spaces. For our purposes, it is convenient to consider finite measure spaces. There could be two ways of proceeding. We could always multiply any measure by the reciprocal of the total finite measure, getting a probability measure. Or we could observe that practically any results proved for martingales, and certainly all those that we will use, are valid even if we were to define all concepts (suitably modified when necessary) for finite measure spaces instead of probability spaces. Since either approach would work for us, we do not need to take sides on which to choose. It should be clear in each instance that either alternative would be acceptable. With these caveats, we will now recall some definitions in their customary setting.

3.1. **Definitions.** Thus, let Ω be a probability space with a sigma-algebra $\mathcal{F}$ of measurable sets for the measure P.

A discrete-time complex–valued **martingale** on Ω is a sequence of complex-valued functions g_n in $L^1(\Omega)$ such that g_n is measurable with respect to a sigma-algebra $\mathcal{F}_n$, where $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$, such that for each $A \in \mathcal{F}_n$, we have

$$\int_A (g_{n+1} - g_n)\, dP = 0.$$

In a **finite martingale**, each $\mathcal{F}_n$ contains only finitely many sets and each g_n takes only finitely many values, and $g_n, \mathcal{F}_n$ are the same for all sufficiently large n. In particular, if, for example, some minimal $A \in \mathcal{F}_n$ is the disjoint union of only two minimal sets $A_1, A_2 \in \mathcal{F}_{n+1}$ of equal measure, we have, for some $\alpha \in \mathbb{C}$, $g_{n+1} = g_n + \alpha \quad \text{on } A_1$, $g_{n+1} = g_n - \alpha \quad \text{on } A_2$, while g_n is constant on A.

Let X_n and Y_n be two martingales on Ω with respect to the same sequence $\mathcal{F}_n$ of sigma-algebras and with $X_0 \equiv 0$, $Y_0 \equiv 0$. We say that $Y = \{Y_n\}$ is a **martingale transform** of X if

$$|Y_n - Y_{n-1}| \leq |X_n - X_{n-1}|$$

$P-$almost everywhere, for each $n \geq 1$. Thus X and Y are martingale transforms of each other if

$$|Y_n - Y_{n-1}| = |X_n - X_{n-1}|$$

$P-$almost everywhere, for each $n \geq 1$. In particular, $|X_1| = |Y_1|$ $P-$almost everywhere. One way for this to happen is for the functions X_1 and Y_1 to be constant functions of equal modulus.

Burkholder [9] proved that if Y is a martingale transform of X, and if

$$||X||_p = \sup_n ||X_n||_p < \infty,$$

for some $p \in (1, \infty)$, then

$$||Y||_p \leq (p^* - 1)||X||_p, \tag{4}$$

with equality if $p = 2$ and if X is also a transform of Y.

It is essential for us to discuss the method of proof introduced by Burkholder. It is based on considering certain auxiliary functions.

Following Burkholder, set

$$\alpha_p = p(1 - 1/p^*)^{p-1} \quad \text{for} \quad 1 < p < \infty,$$

and consider

$$u(z, w) = (|w| - (p^* - 1)|z|)(|z| + |w|)^{p-1}$$

for $z, w \in \mathbb{C}$. In this definition, we can replace $\mathbb{C}$ by $\mathbb{R}^n$ or a Hilbert space, which will be used later on. Burkholder proved that for all $z, w \in \mathbb{C}$

$$|w|^p - (p^* - 1)^p|z|^p \leq \alpha_p \, u(z, w). \tag{5}$$

In fact, Burkholder denoted the right hand side of (5) by $u(z, w)$. Since the constant α_p is unimportant for most of our considerations, we find it more convenient to omit it from the definition of u and to add it into any one of the few formulas where it is really needed.

Burkholder also considered the function u_0 defined by

$$u_0(z, w) = |w|^2 - |z|^2 \qquad \text{for} \qquad |z| + |w| \leq 1 \tag{6}$$

and by

$$u_0(z, w) = 1 - 2|z| \qquad \text{for} \qquad |z| + |w| > 1. \tag{7}$$

A. Baernstein and S. Montgomery–Smith ([6], 1997) proved that with

$$u_1(z,w) = u_0(z,w) - (|w|^2 - |z|^2),$$
$$\beta_p = 2/(p(2-p)), \qquad \gamma_p = 2/(p(p-1)(p-2)),$$

we have

$$\int_0^\infty t^{p-1} u_0\left(\frac{z}{t}, \frac{w}{t}\right) dt = \beta_p u(z,w) \quad \text{if} \quad 1 < p < 2$$

and

$$\int_0^\infty t^{p-1} u_1\left(\frac{z}{t}, \frac{w}{t}\right) dt = \gamma_p u(w,z) \quad \text{if} \quad 2 < p < \infty,$$

so that to prove suitable inequalities for the functions u, it would suffice to consider u_0. However the definition of u_0 in two parts may make it complicated.

3.2. **Properties of Burkholder's functions.** Burkholder showed that u (and u_0) has the following **concavity property**. Suppose that $z, w, h, k \in \mathbb{C}$ with $zw \neq 0$ and $|k| \leq |h|$ (the same properties are valid if $\mathbb{C}$ here is replaced by a real or complex Hilbert space). Then

$$u(z+h, w+k) \leq u(z,w) + \operatorname{Re}\{u_z(z,w)\overline{h} + u_w(z,w)\overline{k}\}.$$

Here we have used the complex partial derivatives of u given by

$$u_z = (z/|z|)[(p-p^*)|w| - p(p^*-1)|z|](|z|+|w|)^{p-2},$$
$$u_w = (w/|w|)[p|w| + (p+p^*-pp^*)|z|](|z|+|w|)^{p-2}.$$

These same expressions can be used also when z, w, h, k lie in a general Hilbert space. Note that since u takes only real values, we have $u_{\overline{z}} = \overline{u_z}$ and $u_{\overline{w}} = \overline{u_w}$.

This concavity property has the following important consequence. Suppose that $A_i > 0$ where the index i runs over finitely many values, and that the following **martingale conditions** are satisfied:

$$\sum_i A_i h_i = \sum_i A_i k_i = 0. \tag{8}$$

Suppose further that

$$|k_i| \leq |h_i| \qquad \text{for all } i. \tag{9}$$

Then

$$\sum_i A_i u(z+h_i, w+k_i) \leq \left(\sum_i A_i\right) u(z,w). \tag{10}$$

This is the property that Burkholder used to prove that

$$||Y||_p \leq (p^*-1)||X||_p$$

for martingales X and Y (with values, say, in a Hilbert space) and for $1 < p < \infty$ if Y is a martingale transform of X.

We note that

$$u(z, w) = u(|z|, |w|) = u(|z|, |z|) \le 0 \quad \text{if} \quad |z| = |w|. \tag{11}$$

In spite of its simplicity and triviality, this is an essential property. We expect to have a symmetry between the two functions we will consider, so we will only use the assumption that

$$|k| = |h|. \tag{12}$$

4. Considering S in a discrete setting

We do not discretize the integral transformation S, but consider the application of S to functions that can be described by finitely many parameters. In terms of the usual Wirtinger derivatives

$$f_z = \frac{\partial f}{\partial z} = (1/2)\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$

and

$$f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = (1/2)\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

the transformation S has the basic property that

$$S\left(\frac{\partial f}{\partial \overline{z}}\right) = \frac{\partial f}{\partial z}.$$

Thus the question associated with the Iwaniec conjecture becomes the following. Find the smallest C_p such that for all $f : \mathbb{C} \to \mathbb{C}$ with L^p-derivatives, we have

$$\left|\left|\frac{\partial f}{\partial z}\right|\right|_p \le C_p \left|\left|\frac{\partial f}{\partial \overline{z}}\right|\right|_p. \tag{13}$$

Indeed, the conjecture is that

$$\left|\left|\frac{\partial f}{\partial z}\right|\right|_p \le (p^* - 1) \left|\left|\frac{\partial f}{\partial \overline{z}}\right|\right|_p. \tag{14}$$

4.1. **Approximation by continuous piecewise affine mappings of compact support.** It is routine to see that it would suffice to consider questions like (13) for functions in special classes. We could certainly assume that f in (13) is in the class $C_0^\infty(\mathbb{C})$. But we may go further and assume that $f : \mathbb{C} \to \mathbb{C}$ is continuous, piecewise affine, affine on certain triangles, and of compact support in the plane. A rigorous justification for such approximation can be obtained by, for example, quoting the results of Ciarlet and Raviart [13]. Suppose that we use triangles T such that there is a constant $C_0 > 1$ for which

$$\operatorname{diam} T \leq C_0 R(T) \tag{15}$$

where $R(T)$ is the radius of the largest disk contained in T. Pick $f \in C_0^\infty(\mathbb{C})$ and cover the support of f by finitely many triangles T, each with diameter not exceeding $\delta > 0$, for which the above holds for a fixed C_0. Suppose that for all $z \in \mathbb{C}$, the modulus of each derivative of f of order 2 does not exceed C_1. Then, by Theorem 2 on page 184 of [13], there is a continuous piecewise affine mapping $f_0 : \mathbb{C} \to \mathbb{C}$ that vanishes outside this cover of the support of f such that for all $z \in \mathbb{C}$, we have

$$|f_0(z) - f(z)| \leq C_1 C_2 \delta^2$$

and

$$|Df_0(z) - Df(z)| \leq C_1 C_2 \delta$$

where $|Df|$ denotes the matrix norm of the derivative matrix Df of f. Here C_2 depends only on C_0. It follows that $||f_z - (f_0)_z||_p \leq AC_1C_2\delta$ and $||f_{\overline{z}} - (f_0)_{\overline{z}}||_p \leq AC_1C_2\delta$ where A is the area of the union of those triangles T needed to cover the support of f. Thus, even if we were to vary the triangulations that we use and decrease δ, we may choose an upper bound for A to be a fixed constant only depending on f but not on f_0. Since we may take δ to be arbitrarily small, we can approximate f_z and $f_{\overline{z}}$ as closely as we like in L^p−norm by the derivatives of continuous piecewise affine mappings of compact support.

We denote by

$$\mathcal{F}(X, Y)$$

the set of all continuous piecewise affine mappings of compact support defined on X and taking values in Y, where Y is a real or complex Hilbert space and X is $\mathbb{R}^n$ for some $n \geq 2$, where we identify $\mathbb{R}^2$ with $\mathbb{C}$.

4.2. **A reformulation of the problem of estimating norms.** Thus our situation is now as follows. We assume that $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$. Each of

$$\frac{\partial f}{\partial z} \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}}$$

is piecewise constant, constant on each triangle used, and of compact support. It does not matter exactly what kinds of triangulations are used as long as we allow triangles of arbitrarily small diameter and all angles are bounded away from zero, this latter condition being what we need to satisfy (15) with a fixed constant C_0.

In order to prove (13) with $C_p = p^* - 1$ at least for certain functions $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$, we follow the ideas that Burkholder used to prove a similar inequality in the setting of martingale transforms. Thus our goal becomes to prove for such a function f the stronger condition

$$\begin{aligned} (16) \qquad & \frac{1}{\alpha_p} \int_{\mathbb{C}} \left(\left| \frac{\partial f}{\partial z} \right|^p - (p^* - 1)^p \left| \frac{\partial f}{\partial \overline{z}} \right|^p \right) \, dx\, dy \\ & \leq \int_{\mathbb{C}} u \left(\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial z} \right) \, dx\, dy \leq 0. \end{aligned}$$

Suppose that the bounded region in the plane where f need not be identically 0 is the closure of the union of disjoint open triangles T_j, and suppose that we write

$$f(z) = a_j z + b_j \overline{z} + c_j \tag{17}$$

for all $z \in T_j$. For simplicity, suppose that we choose the triangles so that all the T_j have equal areas. Then we need to prove that

$$\sum_j u(b_j, a_j) \leq 0 \tag{18}$$

since the right hand side of (16) is equal to a positive constant (the common area of the triangles T_j) times the left hand side of (18). If, instead, the triangle T_j has area A_j, then (16) becomes

$$\sum_j A_j u(b_j, a_j) \leq 0. \tag{19}$$

The question is now how to get upper bounds for the left hand side of (18). The strategy of Burkholder was to find a few terms at a time and show that their sum does not exceed a positive constant times a single $u-$term, e.g., $\sum_{j=1}^{N} u(b_j, a_j) \leq N u(b, a)$ for some suitable a and b. For such an inequality to be true, there needs to be a relationship between the a_j and b_j, such as something corresponding to (8) and (12). We will now pursue a more careful analysis of how one might combine terms in this way.

5. Relations between Wirtinger derivatives in neighboring triangles

In order to start implementing the strategy outlined above, we need to clarify under what circumstances it is possible to combine two or more terms of the form $u(b_j, a_j)$ and obtain a valid inequality of the form $\sum_{j=1}^{k} A_j u(b_j, a_j) \leq A u(b, a)$ for some suitable a and b and for positive constants A_j and A and for a suitable k (possibly only $k = 2$)

that may be much smaller than the total number of triangles to start with. Of course, the discussion in connection with (8) and (10) has told us what we should require of the **numbers** a_j, b_j, A_j, but since we are now dealing with a **function** whose derivatives determine these numbers, we have to ask what properties the function must have. After answering that question, we will be in a position to consider the question **geometrically**, and the experience of the author in studying this question indicates that this geometric approach will be much quicker than relying on **algebraic identities**, and will lead to properties that will be quicker to verify. Since it is necessary to get started somewhere, and since, before any combinations have yet been performed, the only terms available to be combined are the terms $u(b_j, a_j)$ corresponding to individual triangles, we must first ask how we may combine the terms for two or more triangles. We start by considering only two triangles.

5.1. **Two triangles.** Suppose that on two adjacent triangles, our continuous piecewise mapping is defined by the expressions $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$. Here we ignore translations as we may, by assuming that the origin is a common vertex of the two triangles and that the origin is mapped to itself. These assumptions do not affect the derivatives of the function. Suppose that f and g agree on a line segment joining the common vertex at 0 to another common vertex $\zeta \neq 0$. Then $f(\zeta) = g(\zeta)$, so that

$$a\zeta + b\overline{\zeta} = c\zeta + d\overline{\zeta}$$

and hence

$$(a - c)\zeta = -(b - d)\overline{\zeta}.$$

Thus

$$a - c = \alpha(b - d) \qquad \text{where} \qquad \alpha = -\overline{\zeta}/\zeta$$

so that $|\alpha| = 1$, that is,

$$|a - c| = |b - d|. \tag{20}$$

This relationship that refers to moduli of the changes we have in the $z-$ and $\overline{z}-$derivatives of the function, is of the same type as (12). Thus this is the kind of relation that we are looking for. Having an equality is needed so as to get two martingales that are transforms of each other, rather than only one being a transform of the other.

It seems to the author that this is the first time that anyone has come up with a precise identity of this nature in connection with this problem. Even now such an identity is known only in the discrete situation just discussed; it has no known obvious continuous counterpart, even though there should hopefully be one in the future.

6. Matching lengths

6.1. **Search for martingales.** Once one starts combining terms arising from different triangles, even if one were to begin with neighboring triangles, one will already at the next stage get to a situation that is no longer equally simple. Then it is typically not possible to identify pairs of affine mappings for which the moduli of the differences of the $z-$ and $\overline{z}-$derivatives are equal. Thus something more needs to be done. This fact is essentially equivalent to the following fact. In various earlier treatments of this problem, the given functions were used to define pairs of martingales, usually only after extending a function from the complex plane to the upper half space using a fixed kernel such as the Poisson kernel or the heat kernel. This shows that it is not easy to come up with martingales associated directly with the $z-$ and $\overline{z}-$derivatives of a function. Those derivatives themselves certainly do not define martingales as such, even if we were to look at a process by which a function (say a continuous piecewise affine mapping) is built up from similar functions, making the triangulation finer at each time.

6.2. **Effect of rotations.** Thus, since there turn out to be rather few situations where equations such as (20) are satisfied, we need to do something more to get closer to **repeatedly** having circumstances under which (8) and (10) are satisfied. Since $u(z,w)$ depends only on $|z|$ and $|w|$, we may independently replace z and w by αz and βw, where $|\alpha| = |\beta| = 1$. This is a *key observation* that in spite of its simplicity was not used in earlier approaches to this problem. We may use Burkholder's technique to compare values of $u(z,w)$ for several pairs (z,w) simultaneously. Consider the case of 2 pairs. If we want to compare $u(a,b)$ and $u(c,d)$ using Burkholder's results, we need to have $|a-c| = |b-d|$.

Matching lengths. If it is not initially true that $|a-c| = |b-d|$, we may consider the effect of replacing

$$(a,b) \qquad \text{by} \qquad (\alpha a, \beta b), \qquad \text{where } |\alpha| = |\beta| = 1.$$

Now $|\alpha a - c| = |\beta b - d|$ means that there is a complex number γ with $|\gamma| = 1$ such that

$$\alpha a - c = \gamma(\beta b - d).$$

This can be written in the more useful form

$$aA + b\overline{A} = E(cB + d\overline{B}) \tag{21}$$

where

$$A^2 = -\alpha\overline{\beta\gamma},\ B^2 = -\overline{\gamma},\ E = -\overline{\beta\gamma AB},$$

so that

$$|A| = |B| = |E| = 1.$$

With the notation $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$, (21) states that

$$|f(A)| = |g(B)|, \qquad \text{where} \qquad |A| = |B| = 1.$$

The conclusion is that for us to be able to compare the derivatives of f and g after using appropriate rotations, then $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$ must map *line segments of equal length* in **some** directions to *line segments of equal length*. When this occurs, let us say that f and g have **matching lengths**, or that f and g are **comparable affine maps**.

This now raises the question of how we may determine whether given affine f and g actually have matching lengths. Of course, one could use the definition and perform some computations, but that might be too clumsy in many cases, so we look for methods for seeing that this is the case.

6.3. **Intervals of lengths.** To be able to detect whether there are matching lengths, we need to characterize the possible lengths that can occur for any particular affine mapping.

For every affine map F, there are numbers m, M with

$$0 \leq m \leq M < \infty$$

such that the interval $[m, M]$ is precisely the set of lengths of the images of segments of unit length under the map normalized to $F(0) = 0$. More prescisely, for any affine mapping F, the image of the unit circle under $F - F(0)$ is an ellipse (possibly a degenerate ellipse) with semi-axes m and M, say. Here we have $m = 0 < M$ when $F(\mathbb{C})$ is a line, and $m = M = 0$ when $F(\mathbb{C})$ is a point.

Two affine mappings with intervals $[m_1, M_1]$ and $[m_2, M_2]$ are obviously comparable if, and only if,

$$[m_1, M_1] \cap [m_2, M_2] \neq \emptyset,$$

that is,

$$\max\{m_1, m_2\} \leq \min\{M_1, M_2\},$$

and in that case the common length that we would make use of would be one of the lengths in $[m_1, M_1] \cap [m_2, M_2] \neq \emptyset$. In the same way we may settle the question of whether a single affine mapping is comparable to each of a larger number of affine mappings. This question arises when trying to apply (10) with more than two terms on the left hand side.

Given finitely many affine maps F_j with intervals $[m_j, M_j]$ for $1 \leq j \leq n$, it is easy to find affine maps F with interval $[m, M]$ that are comparable to every given map F_j. For example, choose m and M so that

$$0 \le m \le \min\{m_j : 1 \le j \le n\}, \qquad M \ge \max\{M_j : 1 \le j \le n\}.$$

However, usually the martingale conditions (8) are not satisfied for these comparisons. Indeed, some thought shows that much greater care is required to obtain the necessary cancellations.

7. Directions labeled as "horizontal" and "vertical"

A calculation shows that since we may use rotations, we may designate, for each triangle independently, two orthogonal directions given by c and ic, where $|c| = 1$, as "horizontal" and "vertical" directions, and denote their images under the affine map F in the triangle by H and V, so $H = F(c) - F(0)$, $V = F(ic) - F(0)$.

The partial derivatives of F can be expressed in terms of H and V. For simplicity, suppose that $F(0) = 0$. Write $F(z) = az + b\overline{z}$. Then

$$H = ac + b\overline{c}, \qquad V = i(ac - b\overline{c}).$$

Thus

$$a = \frac{1}{2c}\left(H - iV\right), \qquad b = \frac{1}{2\overline{c}}\left(H + iV\right), \tag{22}$$

so that

$$\begin{aligned} u\left(\frac{\partial F}{\partial \overline{z}}, \frac{\partial F}{\partial z}\right) &= u\left(\frac{1}{2\overline{c}}\left(H + iV\right), \frac{1}{2c}\left(H - iV\right)\right) \\ &= u\left(\frac{1}{2}\left(H + iV\right), \frac{1}{2}\left(H - iV\right)\right) \end{aligned} \tag{23}$$

If we have two such triangles, then it turns out that we may combine their $u-$terms in (18) into a single term if $H_1 = H_2$ or if $V_1 = V_2$. (These are special cases of the most general sufficient condition.)

To see this, let us first verify that we have matching lengths in those cases. Suppose that the horizontal directions used are c_1 and c_2. Thus $|c_1| = |c_2| = 1$, and if the two mappings are written as $a_1 z + b_1\overline{z}$ and $a_2 z + b_2 \overline{z}$, then $H_1 = H_2$ means that

$$a_1 c_1 + b_1 \overline{c_1} = a_2 c_2 + b_2 \overline{c_2}, \tag{24}$$

and in particular $|a_1 c_1 + b_1\overline{c_1}| = |a_2 c_2 + b_2 \overline{c_2}|$, while $V_1 = V_2$ means that

$$a_1(ic_1) + b_1 \overline{ic_1} = a_2(ic_2) + b_2 \overline{ic_2}, \tag{25}$$

and hence $|a_1(ic_1) + b_1\overline{ic_1}| = |a_2(ic_2) + b_2\overline{ic_2}|$. In both cases it is obvious that we have matching lengths in a particularly simple way.

7.1. **Rewriting (10) when $H_1 = H_2$ or $V_1 = V_2$.** Let us see what (10) implies when $H_1 = H_2$ and notation is as above. By (10) and (24) we have

$$u(a_1, b_1) + u(a_2, b_2) = u(c_1 a_1, \overline{c_1} b_1) + u(c_2 a_2, \overline{c_2} b_2) \leq 2u\left(\frac{1}{2}(c_1 a_1 + c_2 a_2), \frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2)\right) \tag{26}$$

since

$$\left(\frac{1}{2}(c_1 a_1 + c_2 a_2) - c_1 a_1\right) + \left(\frac{1}{2}(c_1 a_1 + c_2 a_2) - c_2 a_2\right) = 0,$$

$$\left(\frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2) - \overline{c_1} b_1\right) + \left(\frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2) - \overline{c_2} b_2\right) = 0,$$

$$\left|\frac{1}{2}(c_1 a_1 + c_2 a_2) - c_1 a_1\right| = \left|\frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2) - \overline{c_1} b_1\right|,$$

and

$$\left|\frac{1}{2}(c_1 a_1 + c_2 a_2) - c_2 a_2\right| = \left|\frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2) - \overline{c_2} b_2\right|.$$

Similarly, if $V_1 = V_2$, we obtain by (10) and (25) that

$$\begin{aligned} u(a_1, b_1) + u(a_2, b_2) &= u(c_1 a_1, -\overline{c_1} b_1) + u(c_2 a_2, -\overline{c_2} b_2) \\ &\leq 2u\left(\frac{1}{2}(c_1 a_1 + c_2 a_2), -\frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2)\right) \\ &= 2u\left(\frac{1}{2}(c_1 a_1 + c_2 a_2), \frac{1}{2}(\overline{c_1} b_1 + \overline{c_2} b_2)\right). \end{aligned} \tag{27}$$

7.2. **The function $\tilde{u}$.** When discussing H and V, it turns out to be convenient to introduce new notation and write

$$\tilde{u}(a, b) = u\left(\frac{1}{2}(a + ib), \frac{1}{2}(a - ib)\right). \tag{28}$$

Some useful properties of $\tilde{u}$ for any $a, b \in \mathbb{C}$ are

$$\tilde{u}(a, 0) = u(a/2, a/2) \leq 0, \quad \tilde{u}(0, b) = u(ib/2, -ib/2) \leq 0, \tag{29}$$

and

$$\tilde{u}(a, b) = \tilde{u}(-a, -b) = \tilde{u}(ca, cb) \quad \text{whenever} \quad |c| = 1. \tag{30}$$

Furthermore, we have

$$\tilde{u}(a, b) = \tilde{u}(b, -a) = \tilde{u}(-b, a) \tag{31}$$

since

$$\tilde{u}(b,-a) = u\left(\frac{1}{2}(b-ia), \frac{1}{2}(b+ia)\right) = u\left(\frac{-i}{2}(a+ib), \frac{i}{2}(a-ib)\right)$$
$$= u\left(\frac{1}{2}(a+ib), \frac{1}{2}(a-ib)\right) = \tilde{u}(a,b),$$

which means that the roles of H and V can be interchanged provided that one of them is also multiplied by -1. This is understandable since it only means that if, earlier, H corresponded to the direction c, then we take, instead, H to correspond to the direction ic or to $-ic$.

In view of (22), using the notation immediately preceding (22), we have

$$\tilde{u}(H,V) = u\left(\frac{1}{2}(H+iV), \frac{1}{2}(H-iV)\right) = u(\bar{c}b, ca) = u(b,a). \tag{32}$$

The counterpart of the fact that $u(z,w)$ depends only on the moduli $|z|$ and $|w|$ is

$$\tilde{u}(H,\lambda H) \le 0 \quad \text{whenever} \quad \lambda \in \mathbb{R} \ \text{ and } \ H \in \mathbb{C}. \tag{33}$$

Namely,

$$\tilde{u}(H,\lambda H) = u\left(\frac{1}{2}\left(1+i\lambda\right)H, \frac{1}{2}\left(1-i\lambda\right)H\right) \le 0$$

by (11) since

$$|(1+i\lambda)\,H| = |(1-i\lambda)\,H|$$

when λ is real.

Suppose that $H_1 = H_2 = H$, say, and note that then

$$V_1 = i(a_1c_1 - b_1\overline{c_1}), \quad V_2 = i(a_2c_2 - b_2\overline{c_2}).$$

Then using the notation (28), and taking into account (22) and (30), we may write (26) as

$$\begin{aligned} & \tilde{u}(H,V_1) + \tilde{u}(H,V_2) \\ = \ & u\left(\frac{1}{2}(H+iV_1), \frac{1}{2}(H-iV_1)\right) + u\left(\frac{1}{2}(H+iV_2), \frac{1}{2}(H-iV_2)\right) \\ = \ & 2u\left(\frac{1}{2}\left(\overline{c_1}b_1 + \overline{c_2}b_2\right), \frac{1}{2}\left(c_1a_1 + c_2a_2\right)\right) \end{aligned}$$

while

$$\tilde{u}\left(H, \frac{V_1+V_2}{2}\right) = u\left(\frac{1}{2}\left(H + i\frac{V_1+V_2}{2}\right), \frac{1}{2}\left(H - i\frac{V_1+V_2}{2}\right)\right)$$
$$= u\left(\frac{1}{2}(b_1\overline{c_1} + b_2\overline{c_2}), \frac{1}{2}(a_1c_1 + a_2c_2)\right)$$

which imply that

$$\tilde{u}(H,V_1)+\tilde{u}(H,V_2)\le 2\tilde{u}\left(H,\frac{V_1+V_2}{2}\right). \tag{34}$$

Suppose that $V_1=V_2=V$, say, and note that then

$$H_1=a_1c_1+b_1\overline{c_1},\quad H_2=a_2c_2+b_2\overline{c_2}.$$

Then, using the notation (28), we may similarly write (27) as

$$\tilde{u}(H_1,V)+\tilde{u}(H_2,V)\le 2\tilde{u}\left(\frac{H_1+H_2}{2},V\right). \tag{35}$$

If we have a situation where $V_1=V_2$, we could always make use of (31) first and thereby reduce the situation to the case where $H_1=H_2$. Thus it is somewhat redundant to discuss the case $V_1=V_2$. However, there may be practical cases where it is more convenient to proceed instead of using (31) first, so it may be worth having also (35) available.

Of course, both (34), and (35) can be verified directly by appealing to (8), (9), (10), and (28). Above, we wanted to emphasize the view of considering mappings rather than abstract numbers.

More generally, it follows by a calculation from (8), (9), (10), and (28) that

$$\sum_{j=1}^{\ell}\tilde{u}(H,V_j)\le \ell\tilde{u}\left(H,\frac{1}{\ell}\sum_{j=1}^{\ell}V_j\right) \tag{36}$$

and

$$\sum_{j=1}^{\ell}\tilde{u}(H_j,V)\le \ell\tilde{u}\left(\frac{1}{\ell}\sum_{j=1}^{\ell}H_j,V\right) \tag{37}$$

and this can be generalized to some extent by using unequal weights.

8. A conjecture for multipliers of modulus 1

8.1. **Propagation of multipliers.** Suppose that we manage to find multipliers of the type α and β of modulus 1 and combine two or more $u-$terms in (18) into a single term. We may be able to do this for several pairs or larger groups of initial $u-$terms. After that, we may use further multipliers to combine terms, some of which may still be among the initial terms and some of which may be new terms obtained by combinations. If we are able to continue with this process until we find an upper bound for the left hand side of (18) that is a positive multiple of a term of the form $u(z,w)$, where $|z|=|w|$ (possibly even with $z=w$), which guarantees that this last single term $u(z,w)\le 0$, then we will have proved (18) and hence (14) in that particular case.

We now see that in retrospect, there is a propagation of multipliers. We could have initially multiplied all the a_j and b_j by suitable numbers of modulus 1, so that after that, no further multipliers would have been required, and we would have used only the martingale properties of the function u. In view of this possibility of expanding our chances of finding matching lengths if we multipy a and b by constants of modulus 1, we are lead to the following question, which we formulate as a conjecture.

Conjecture. *For each continuous piecewise affine complex-valued function f of compact support in the plane, there exist piecewise constant functions $c_1(z)$ and $c_2(z)$ (constant in the same triangles as where $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ are constant) with*

$$|c_1(z)| \equiv 1 \equiv |c_2(z)|$$

such that from the two functions

$$c_1(z)\frac{\partial f}{\partial \overline{z}} \qquad \text{and} \qquad c_2(z)\frac{\partial f}{\partial z}$$

one can construct martingales

$$X_n \qquad \text{and} \qquad Y_n$$

that are martingale transforms of each other, going from constants X_1, Y_1 of the same modulus and depending on f to

$$c_1(z)\frac{\partial f}{\partial \overline{z}} \qquad \text{and} \qquad c_2(z)\frac{\partial f}{\partial z}.$$

This means that for some large integer N, we have

$$X_n(z) = c_1(z)(\partial f/\partial \overline{z})$$

and

$$Y_n(z) = c_2(z)(\partial f/\partial z)$$

for all $n \geq N$. The constants X_1 and Y_1 of equal modulus correspond to the last z and w in the paragraph above the Conjecture. We have thus "rotated" $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ and then related them by martingales.

Starting with constants X_1 and Y_1 of equal modulus is, of course, from a martingale point of view, equivalent to starting with $X_0 \equiv 0$, $Y_0 \equiv 0$; essentially we duplicate the situation under consideration. We have mentioned these constants only since it is easier not to deal with such duplication.

The sigma-algebras $\mathcal{F}_n$ with respect to which X_n and Y_n are measurable, as well as the functions $c_1(z)$ and $c_2(z)$, would have to depend on f. The $\mathcal{F}_n$ indicate the pattern according to which $u-$terms are combined. For functions f in suitable larger classes of functions, there would then have to be the corresponding result involving continuous-time martingales.

We mention in passing that a special case of the results of Baernstein and Montgomery–Smith [6] relevant to us states that (16) holds for u_0 and hence for u if $f(re^{i\theta}) = g(r)e^{i\theta}$, where $g : (0,\infty) \to (0,\infty)$ is Lipschitz with $g(0) \equiv \lim_{r\to 0+} g(r) = 0$, and $f_z, f_{\overline{z}} \in L^p(\mathbb{C})$. Their proof has nothing to do with martingales and hence does not address the question of whether an analogue of the above Conjecture holds for these f.

One should note that the affine maps that one obtains in when combining $u-$terms are purely auxiliary maps that usually cannot be put together to define a single **continuous** piecewise affine map.

Some combinations are simple. For example, it is easily seen that if a square is divided into four triangles by means of the two diagonals, then the resulting four $u-$terms can always be combined into a single $u-$term, with H and V corresponding to the differences in function values along the two diagonals (rendering the value at the vertex at the center of the square irrelevant). However it need not be possible to combine such results from two adjacent squares, making it impossible to apply a trivial strategy of combining adjacent squares into larger squares that one might initially be tempted to try.

9. Outline of the proof of Theorem 1

Martingale methods and rotations indeed yield

$$||f_z||_p \leq (p^* - 1)||f_{\overline{z}}||_p$$

in a case with 13 arbitrary independent complex parameters, as claimed in Theorem 1. This situation is illustrated in Figure 2 (which is the same as Figure 1 but is reproduced here for easier reference).

9.1. **The set-up and outline of the proof.** Consider a big square Q with vertices at $\pm 4 \pm 4i$, with 13 vertices inside, at $j + ik$ for $j, k \in \{0, 2, -2\}$ and at $\pm 1 \pm i$. The square is divided into 16 triangles of area 1, such as the triangle with vertices at $2i$, $2 + 2i$, and $1 + i$, and 24 triangles of area 2, such as the triangle with vertices at $2 + 4i$, $4 + 4i$, and $2 + 2i$. We assume that f is a continuous complex-valued piecewise affine map in $\mathbb{C}$, with $f \equiv 0$ on ∂Q and outside Q, and taking any preassigned complex values at each of the 13 inner vertices. This determines f uniquely, and hence f depends on 13 arbitrary independent complex parameters. This is because given any non-degenerate triangle with vertices A, B, C and any three not necessarily distinct

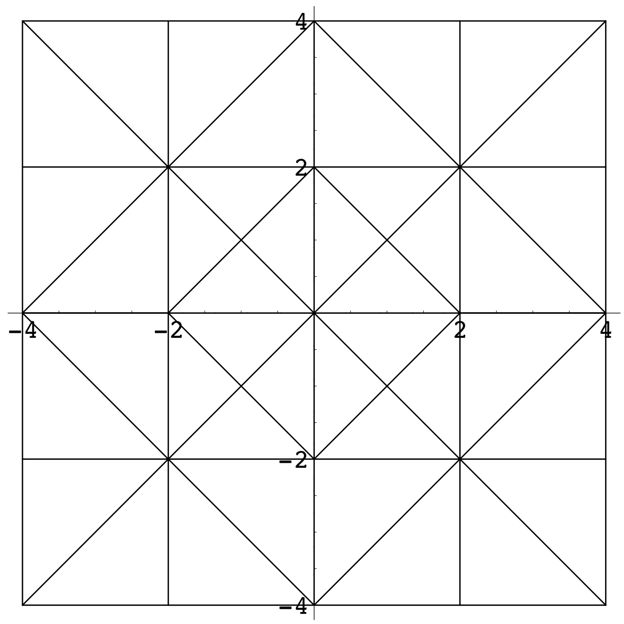

FIGURE 2. The case with 13 complex parameters. The big square with vertices $\pm 4 \pm 4i$ is the square Q.

complex numbers α, β, γ, there is a unique affine mapping of the form $g(z) = az + b\overline{z} + c$ with $g(A) = \alpha$, $g(B) = \beta$, and $g(C) = \gamma$.

Each of the functions $\frac{\partial f}{\partial \overline{z}}$ and $\frac{\partial f}{\partial z}$ is piecewise constant (constant on each of the 40 triangles). In each of the 16 triangles with 2 vertices on the boundary, we have two vertices mapped to the same point (the origin) by f so that f maps the whole triangle into a line. This implies that in this triangle $\left|\frac{\partial f}{\partial \overline{z}}\right| = \left|\frac{\partial f}{\partial z}\right|$, and hence $u\left(\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial z}\right) \leq 0$. While these terms might be helpful in proving (18), it turns out that their help is not needed, and that indeed the sum of the remaining 24 $u-$terms is also ≤ 0.

The combining of the remaining 24 $u-$terms starts, e.g., with the two triangles with vertices at $-4i, -2i, -2-2i$ and at $-4, -2, -2-2i$. The way to get started with finding matching lengths is the observation that since $f = 0$ on the boundary, we have $f(-2-2i) - f(-4) = f(-2-2i) - f(-4i)(= f(-2-2i))$. Thus we designate the directions from -4 to $-2-2i$ in one triangle, and from $-4i$ to $-2-2i$ in the other triangle, as the horizontal directions (these two line segments have the same length, so we need not worry about the fact that it is not unit length right now as that can be adjusted) and we obtain what we called matching lengths. The affine map resulting from this combination turns out to be comparable to a certain other affine map obtained by further combinations (indeed, by first using the triangles with vertices at $-2, 0, -1-i$ and at $0, -1-i, -2i$, then those with vertices at $-2, -2-2i, -1-i$ and at $-1-i, -2i, -2-2i$, and then combining those two results together). One can continue all the way to prove (18) in this case. This has only been an outline, and we now proceed to discuss the details step by step. Even though it will take some space to write down all the details below, the principles are quite simple, and after looking at some cases like this it becomes much easier to look at a picture geometrically to see what is going on instead of performing computations.

10. A detailed proof of Theorem 1

As mentioned in the outline, it suffices to consider only the terms arising from those 24 triangles that have at most one vertex on ∂Q since each individual term $u\left(\frac{\partial f}{\partial \bar z}, \frac{\partial f}{\partial z}\right) \le 0$ for each triangle that has two vertices on ∂Q, hence mapped to the origin by f. (Of course, one cannot exclude the possibility that in some larger case, the help obtained from these trivial terms might yet be needed.)

Consider the two triangles with vertices at $-4i, -2i, -2-2i$ (triangle T_1, say) and at $-4, -2, -2-2i$ (triangle T_2). Since $f = 0$ on the boundary of Q, we have $f(-4) = f(-4i) = 0$, so that $f(-2-2i) - f(-4) = f(-2-2i) - f(-4i)$. In T_1, we designate the direction from -4 to $-2-2i$ as the horizontal direction, corresponding to the choice $c = (1-i)/\sqrt{2}$ for the complex number c of modulus 1 as discussed when we defined horizontal and vertical directions. Thus the vertical direction is $ic = (1+i)/\sqrt{2}$. For T_1, we thus have

$$H = H_1 = f(-2-2i)/(2\sqrt{2}) \text{ and } V = V_1 = (f(-2) - f(-3-i))/\sqrt{2}.$$

In T_2, we designate the direction from $-4i$ to $-2-2i$ as the horizontal direction. Here $c = (-1+i)\sqrt{2}$ so that $ic = -(1+i)\sqrt{2}$. For T_2, we thus have

$$H = H_2 = f(-2-2i)/(2\sqrt{2}) \text{ and } V = V_2 = (f(-1-3i) - f(-2i))/\sqrt{2}.$$

Note that since we are dealing with affine mappings and since $f(-4) = f(-4i)(= 0)$, we have

$$f(-3-i) = f(-1-3i).$$

In other words, if two affine mappings map the end points of two line segments to the same two points, respectively, then they also map the midpoints of these line segments to the same point. When performing combinations, it is often necessary to make use of this fact.

Since $H_1 = H_2$, we obtain from (34) that

$$\tilde{u}(H_1, V_1) + \tilde{u}(H_2, V_2) \leq 2\tilde{u}\left(H_1, \frac{V_1 + V_2}{2}\right) = 2\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2) - f(-2i)}{2\sqrt{2}}\right). \tag{38}$$

The $u-$term of each triangle must be taken into account with a weight proportional to the area of the triangle. Each triangle inside Q that we start with has area 1 or 2, so it seems convenient to use the area itself as the weight (indeed a proportionality factor different from 1, whose choice is always a matter of taste anyway, makes sense only if the areas themselves are complicated enough to justify it). Since each of T_1 and T_2 has area equal to 2, the term that we need to use in the sequel is twice what we just obtained, that is,

$$4\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2) - f(-2i)}{2\sqrt{2}}\right). \tag{39}$$

10.1. **Combination of 4 triangles that form a square.** Let us then discuss the square with vertices at $0, -2, -2-2i, -2i$. The computation that we now perform illustrates how one can always combine the $u-$terms of four triangles that form a square in this way, with one vertex at the center of the square. Let us denote by T_3, T_4, T_5, T_6 the triangles with vertices at $(-2, -2-2i, -1-i)$, $(0, -2, -1-i)$, $(0, -2i, -1-i)$, and $(-2i, -1-i, -2-2i)$, respectively.

We may now take any two adjacent triangles (adjacent meaning that they share an edge) out of these four triangles and combine their $u-$terms together, then do the same for the remaining two triangles (which are necessarily also adjacent), and then combine the results. The final result is the same, no matter how we choose the pairs of triangles.

Let us start, for example, by combining the $u-$terms of the triangles T_3 and T_4. We choose horizontal and vertical directions so that

$$H_3 = (f(-2) - f(-1-i))/\sqrt{2}, \quad V_3 = (f(-2-2i) - f(-1-i))/\sqrt{2},$$

and

$$H_4 = H_3, \quad V_4 = (f(-1-i) - f(0))/\sqrt{2}.$$

Since $H_3 = H_4$, (34) yields

$$\tilde{u}(H_3, V_3) + \tilde{u}(H_4, V_4) \leq 2\tilde{u}\left(H_3, \frac{V_3 + V_4}{2}\right) = 2\tilde{u}\left(\frac{f(-2) - f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i) - f(0)}{2\sqrt{2}}\right). \tag{40}$$

Let us then combine the $u-$terms of the triangles T_5 and T_6. We choose horizontal and vertical directions so that

$$H_5 = (f(-2i) - f(-1-i))/\sqrt{2}, \quad V_5 = (f(0) - f(-1-i))/\sqrt{2},$$

and

$$H_6 = H_5, \quad V_6 = (f(-1-i) - f(-2-2i))/\sqrt{2}.$$

Since $H_5 = H_6$, (34) and (30) yield

$$\begin{aligned}
\tilde{u}(H_5, V_5) + \tilde{u}(H_6, V_6) &\leq 2\tilde{u}\left(H_5, \frac{V_5 + V_6}{2}\right) \\
&= 2\tilde{u}\left(\frac{f(-2i) - f(-1-i)}{\sqrt{2}}, -\frac{f(-2-2i) - f(0)}{2\sqrt{2}}\right) \\
&= 2\tilde{u}\left(-\frac{f(-2i) - f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i) - f(0)}{2\sqrt{2}}\right).
\end{aligned} \tag{41}$$

The last terms in (40) and (41) have identical $V-$terms (that is, second variables for the function $\tilde{u}$). Hence by (35) we obtain

$$\begin{aligned}
& 2\tilde{u}\left(\frac{f(-2) - f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i) - f(0)}{2\sqrt{2}}\right) \\
+ \; & 2\tilde{u}\left(-\frac{f(-2i) - f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i) - f(0)}{2\sqrt{2}}\right) \\
\leq \; & 4\tilde{u}\left(\frac{f(-2) - f(-2i)}{2\sqrt{2}}, \frac{f(-2-2i) - f(0)}{2\sqrt{2}}\right)
\end{aligned} \tag{42}$$

Note that the quantities $f(-2)-f(-2i)$ and $f(-2-2i)-f(0)$ appearing here correspond to the difference in the values of the function f across the two diagonals of the square with vertices $0, -2, -2-2i, -2i$, while the value of f at the center $-1-i$ of the square has disappeared. This also completes our discussion of what can be done in any square like this.

10.2. **Further combinations in the "southwest" quadrant of Q.** Let us compare the term in (39) and the last term in (42). By (31) and (35), we have

$$
\begin{aligned}
&4\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) \\
&+4\tilde{u}\left(\frac{f(-2)-f(-2i)}{2\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right) \\
&= 4\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) \\
&+4\tilde{u}\left(-\frac{f(-2-2i)-f(0)}{2\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) \\
&\leq 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right).
\end{aligned} \tag{43}
$$

Now we have combined the $u-$terms from all the 6 triangles that we are going to use at all, from the "southwest" quadrant of Q: the square with vertices at $0, -4, -4-4i, -4i$. Of course, the triangles that we have used form the larger triangle with vertices at $0, -4, -4i$, but this is a mere coincidence. If we had a larger example and were to leave out all small triangles with at least two vertices on the boundary of the big square, the remaining triangles would form a more complicated figure. The term involving $f(0)$ should really be considered as a difference of the values of f at the origin and at a boundary point (where $f \equiv 0$ so that the choice of the boundary point does not matter).

10.3. **The terms from the other three quadrants of Q.** Proceeding in the same way, we find that the following quantities form an upper bound for the 6 original $u-$terms in each of the other three quadrants of the big square Q:

$$8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2)}{2\sqrt{2}}\right),$$

$$8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2)-f(2i)}{2\sqrt{2}}\right),$$

$$8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2i)-f(2)}{2\sqrt{2}}\right).$$

10.4. **Combining the terms from the four quadrants.** There are now two possibilities. Using (34) repeatedly, we may note that by (29),

$$\begin{aligned}
&8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2)}{2\sqrt{2}}\right) \\
&\leq 16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2i)}{4\sqrt{2}}\right),
\end{aligned}$$

$$\begin{aligned}
&8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2)-f(2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2i)-f(2)}{2\sqrt{2}}\right) \\
&\leq 16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, -\frac{f(2i)-f(-2i)}{4\sqrt{2}}\right),
\end{aligned}$$

$$\begin{aligned}
&16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2i)}{4\sqrt{2}}\right) + 16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, -\frac{f(2i)-f(-2i)}{4\sqrt{2}}\right) \\
&\leq 32\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, 0\right) \leq 0.
\end{aligned}$$

This then proves (19) in this case, and hence completes the proof of Theorem 1.

Alternatively, we may apply (10) to four terms in one go and deduce by (36) that

$$\begin{aligned}
&8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2)}{2\sqrt{2}}\right) \\
&+8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2)-f(2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2i)-f(2)}{2\sqrt{2}}\right) \\
&\leq 32\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, 0\right) \leq 0.
\end{aligned}$$

10.5. **Discussion.** The case of 13 parameters is particularly simple since it is possible to find matching lengths in terms of parametric expressions (using function values such as $f(-2-2i)$ as parameters). Also the pattern according to which the matching happens, that is, the sequence of sigma-algebras created for the corresponding martingales, and also the functions c_1 and c_2 of modulus 1, depend only on the general setting we have but not on the particular values of f. This cannot be true in more complicated cases involving a large number of parameters, obtained, for example, when we subdivide the 40 triangles in this special case into smaller triangles.

11. Area identities

11.1. **A potential obstruction for a proof using formal methods: identities for signed area.** If one is to squeeze several terms $u(b_j, a_j)$ (or $A_j u(b_j, a_j)$) into a single term $u(b, a)$ and get a sharp bound, equality must hold at every stage when $p = 2$. For if $p = 2$, then $u(b, a) = |a|^2 - |b|^2$, that is,

$$u\left(\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial z}\right) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2 = J = J(f)$$

where J is the Jacobian determinant, so that $u(b, a)$ (with an implied multiplication by the area of a triangle or the union of triangles) corresponds to the signed area of the image under f of a triangle or the union of triangles.

It may be easier to view this using a change of variables. With $a = c + d$, $b = c - d$, we have $|a|^2 - |b|^2 = 4\mathrm{Re}\,\overline{c}d$. Thus one needs to be able to express a sum of the form $\sum_j \mathrm{Re}\,\overline{c_j}d_j$ as a single term $A\mathrm{Re}\,\overline{c}d$ for some positive constant A (such as the number of terms put together).

For given c_j, d_j, this is naturally possible if c, d can be anything. But if we are to be able to solve this problem in a straightforward formal fashion, as we were able to do above in the case of 13 parameters, then c and d must be linear combinations of the parameters in c_j, d_j with complex coefficients independent of the parameters c_j, d_j, and it is easy to verify that this is usually not possible (even though it works in certain small cases of which we have seen an example above).

11.2. **A case where there is no obvious match involving parametric expressions.** When trying to obtain $||f_z||_p \leq (p^* - 1)||f_{\overline{z}}||_p$ in a case of 25 arbitrary independent complex parameters (see Figure 3, where there are 25 vertices strictly inside the same big square Q as in Theorem 1, more finely triangulated), after a few obvious combinations we run into the problem of having to prove (using a suitable parametrization) the case $n = 4$ of the inequality

$$\sum_{j=1}^{n} \left(\tilde{u}(v_{j-1} - v_j, w_j) + 2\tilde{u}\left(v_j, \frac{w_j - w_{j+1}}{2}\right)\right) \leq 0 \tag{44}$$

where $v_0 = v_n$, $w_{n+1} = w_1$. Considering a more general example, one sees that one would expect (44) to hold for any $n \geq 2$.

We may take

$$2\sqrt{2}v_1 = f(-2),$$

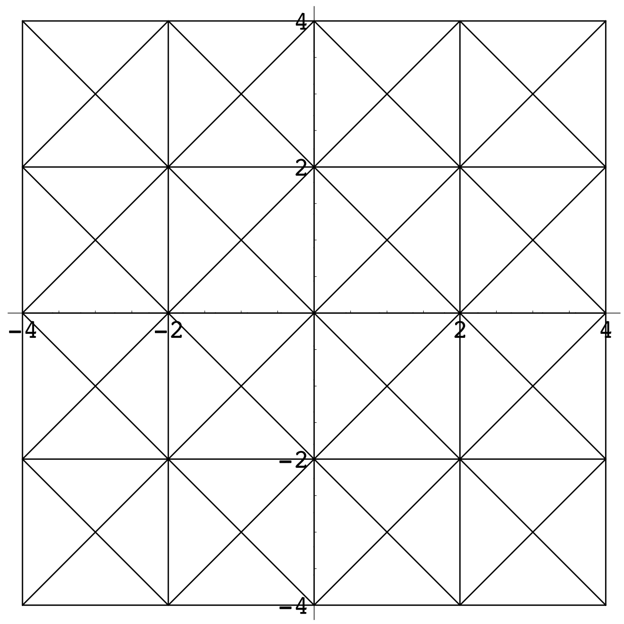

FIGURE 3. The case with 25 complex parameters.

$$
\begin{aligned}
2\sqrt{2}v_2 &= f(-2i),\\
2\sqrt{2}v_3 &= f(2),\\
2\sqrt{2}v_4 &= f(2i),\\
2\sqrt{2}w_1 &= f(-2+2i) - f(0),\\
2\sqrt{2}w_2 &= f(-2-2i) - f(0),\\
2\sqrt{2}w_3 &= f(2-2i) - f(0),\\
2\sqrt{2}w_4 &= f(2+2i) - f(0).
\end{aligned}
$$

11.3. **How to get (44) with $n = 4$ from Figure 3.** Consider any one of the four corner squares in Figure 3, such as the one with vertices at $(-4-4i, -2-4i, -2-2i, -4-2i)$. Reducing the four $u-$terms arising from the four original triangles inside this square into a single term, we get a term that is trivially ≤ 0 since the term $f(-4-2i) - f(-2-4i)$ from one diagonal of the square is equal to zero (since $f(-4-2i) = f(-2-4i) = 0$). We ignore the terms arising from these four corner squares.

There are further triangles with two vertices on ∂Q, such as the triangle with vertices at $(-4, -3-i, -4-2i)$ and one could consider ignoring the $u-$terms arising from these triangles also. However, we will combine terms in a different way. Thus there are other approaches that could be tried.

We combine the four original triangles in the square with vertices $(-4, -2, -2-2i, -4-2i)$ and get, with a proper choice of horizontal and vertical directions,

$$H = H_1 = (f(-2) - f(-4-2i))/(2\sqrt{2}) = f(-2)/(2\sqrt{2}) = v_1,$$

$$V = V_1 = (f(-4) - f(-2-2i))/(2\sqrt{2}) = -f(-2-2i)/(2\sqrt{2}),$$

say. Then we combine the four triangles in the square with vertices $(-4+2i, -2+2i, -2, -4)$ and get

$$H = H_2 = (f(-2) - f(-4+2i))/(2\sqrt{2}) = f(-2)/(2\sqrt{2}),$$

$$V = V_2 = (f(-2+2i) - f(-4))/(2\sqrt{2}) = f(-2+2i)/(2\sqrt{2}).$$

Since $H_1 = H_2$, we may combine the resulting two terms and get $H = H_1 = v_1$,

$$V = \frac{V_1 + V_2}{2} = \frac{f(-2+2i) - f(-2-2i)}{4\sqrt{2}} = \frac{w_1 - w_2}{2}.$$

This last term gives rise to one term in (44).

We have now reduced the rectangle with vertices $(-4+2i, -2+2i, -2-2i, -4-2i)$ to a single $u-$term. We do the same in the other three rectangles in a similar position.

Finally we reduce each of the central squares to a single $u-$term. For example, from the one with vertices at $(-2, -2-2i, -2i, 0)$, we get, with an appropriate choice of horizontal and vertical directions,

$$H = \frac{f(-2) - f(-2i)}{2\sqrt{2}} = v_1 - v_2, \quad V = \frac{f(-2-2i) - f(0)}{2\sqrt{2}} = w_2.$$

A computation now shows that this gives rise to (44) with $n = 4$ and with the v_j and w_j as listed above. The difference in the coefficients (1 and 2) in the terms on the left hand side of (44) is explained by the fact that the area of each rectangle is twice the area of each square.

11.4. **Lack of area identities when $n \geq 3$.** Now, for $n \geq 3$, there is no pair of terms that would be comparable for obvious algebraic reasons (looking at the parameters but ignoring what their relative values might be), and a lack of identities for signed area shows that no selection of at least two terms (if we do not take them all, or all but one) can be combined to a single $u-$term on a formal basis.

Two or more terms could still be comparable, with matching lengths, if the **values** of the parameters allow. This is a key point. It is worth emphasizing what happens here by giving the following examples.

11.5. **Proof of (44) when $n = 2$.** If $n = 2$, the proof of (44) is given by the following reductions that make use of the various properties of the function $\tilde{u}$:

$$\begin{aligned} & 2\tilde{u}\,(v_1, (w_1 - w_2)/2) + 2\tilde{u}\,(v_2, (w_2 - w_1)/2) \\ = \;& 2\tilde{u}\,(v_1, (w_1 - w_2)/2) + 2\tilde{u}\,(-v_2, (w_1 - w_2)/2) \\ \le \;& 4\tilde{u}\,((v_1 - v_2)/2, (w_1 - w_2)/2)\,; \end{aligned}$$

$$\begin{aligned} & \tilde{u}\,(v_1 - v_2, w_2) + \tilde{u}\,(v_2 - v_1, w_1) \\ = \;& \tilde{u}\,(v_1 - v_2, w_2) + \tilde{u}\,(v_1 - v_2, -w_1) \\ \le \;& 2\tilde{u}\,(v_1 - v_2, (w_2 - w_1)/2) = 2\tilde{u}\,(v_2 - v_1, (w_1 - w_2)/2)\,; \end{aligned}$$

$$\begin{aligned} & 4\tilde{u}\,((v_1 - v_2)/2, (w_1 - w_2)/2) + 2\tilde{u}\,(v_2 - v_1, (w_1 - w_2)/2) \\ \le \;& 6\tilde{u}\,(0, (w_1 - w_2)/2) \le 0. \end{aligned}$$

The presence of both $w_1 - w_2$ and $w_2 - w_1$ allowed us to find matching lengths for formal reasons. For $n \ge 3$, the cyclic nature of (44) prevents such coincidences from happening.

11.6. **On (44) when $n = 3$.** In the case $n = 3$ in (44) we argue as follows. We only consider the special case when $v_3 = v_1$. Then

$$\begin{aligned} & 2\tilde{u}\,(v_1, (w_1 - w_2)/2) + 2\tilde{u}\,(v_3, (w_3 - w_1)/2) \\ \le \;& 4\tilde{u}\,(v_1, (w_3 - w_2)/4)\,; \end{aligned}$$

$$\begin{aligned} & \tilde{u}\,(v_1 - v_2, w_2) + \tilde{u}\,(v_2 - v_3, w_3) + \tilde{u}\,(v_3 - v_1, w_1) \\ \le \;& \tilde{u}\,(v_1 - v_2, w_2) + \tilde{u}\,(v_1 - v_2, -w_3) \\ \le \;& 2\tilde{u}\,(v_1 - v_2, (w_2 - w_3)/2)\,; \end{aligned}$$

since $\tilde{u}\,(v_3 - v_1, w_1) = \tilde{u}\,(0, w_1) \le 0$, while

$$\begin{aligned} & 2\tilde{u}\,(v_1 - v_2, (w_2 - w_3)/2) + 2\tilde{u}\,(v_2, (w_2 - w_3)/2) \\ \le \;& 4\tilde{u}\,(v_1/2, (w_2 - w_3)/2)\,, \end{aligned}$$

and the terms

$$4\tilde{u}\,(v_1, (w_3 - w_2)/4) \quad \text{and} \quad 4\tilde{u}\,(v_1/2, (w_2 - w_3)/2) \tag{45}$$

can be compared, and one can show that (44) follows. To see geometrically that the comparison of the terms in (45) is possible, note that the two ellipses centered at the origin, one containing the vectors v_1 and $(w_3 - w_2)/4$, and the other one containing the vectors $v_1/2$ and $(w_2 - w_3)/2$ (and hence also $(w_3 - w_2)/2$), must intersect, and the point of intersection gives rise to matching lengths.

11.7. **Computations for (45).** Let us now verify algebraically that the comparison for (45) is possible and that indeed the resulting $u-$term is ≤ 0.

First note that if $f(0)=0$, $f(z)=az+b\overline{z}$, $|c|=1$, and

$$f(c)=H \qquad \text{and} \qquad f(ic)=V,$$

then, by (22), for real θ,

$$f(e^{i\theta}c)=\frac{1}{2c}\,(H-iV)\,e^{i\theta}c+\frac{1}{2\overline{c}}\,(H+iV)\,\overline{e^{i\theta}c}=H\cos\theta+V\sin\theta.$$

Replacing θ by $\theta+\pi/2$ we obtain

$$f(ie^{i\theta}c)=-H\sin\theta+V\cos\theta.$$

Thus if we replace the direction c by $e^{i\theta}c$, we obtain new values for H and V, namely,

$$H'=H\cos\theta+V\sin\theta \qquad \text{and} \qquad V'=-H\sin\theta+V\cos\theta.$$

We can choose θ so that

$$\cos\theta=\frac{1}{\sqrt{5}} \qquad \text{and} \qquad \sin\theta=\frac{2}{\sqrt{5}}.$$

Using this choice of θ in the first case when we initially have

$$(H,V)=(v_1,(w_3-w_2)/4),$$

we get the new values, say,

$$(H_1,V_1)=\left(\frac{1}{\sqrt{5}}\left(v_1+2\frac{w_3-w_2}{4}\right),\frac{1}{\sqrt{5}}\left(-2v_1+\frac{w_3-w_2}{4}\right)\right).$$

In the second case when we initially have

$$(H,V)=(v_1/2,(w_2-w_3)/2),$$

we choose θ so that

$$\cos\theta=\frac{2}{\sqrt{5}} \qquad \text{and} \qquad \sin\theta=-\frac{1}{\sqrt{5}}.$$

This gives us the new values, say,

$$(H_2,V_2)=\left(\frac{1}{\sqrt{5}}\left(2\frac{v_1}{2}-\frac{w_2-w_3}{2}\right),\frac{1}{\sqrt{5}}\left(\frac{v_1}{2}+2\frac{w_2-w_3}{2}\right)\right).$$

It is possible to perform these changes since we are merely using different directions for the same affine mappings. This is equivalent to using the property that $u(z,w)$ depends only on the moduli $|z|$ and $|w|$ so that rotations can be used. Thus $H_1=H_2$ and

$$\frac{V_1+V_2}{2}=\frac{1}{2\sqrt{5}}\left(-\frac{3}{2}v_1+\frac{3}{4}(w_2-w_3)\right)=\frac{-3}{8\sqrt{5}}(2v_1+w_3-w_2)=-\frac{3}{4}H_1,$$

so that by (34) and (33),

$$4\tilde{u}\left(v_1, \frac{w_3 - w_2}{4}\right) + 4\tilde{u}\left(\frac{v_1}{2}, \frac{w_2 - w_3}{2}\right) = 4(\tilde{u}(H_1, V_1) + \tilde{u}(H_2, V_2))$$
$$\leq 8\tilde{u}\left(H_1, -\frac{3}{4}H_1\right) \leq 0.$$

This also completes the proof that (44) holds when $n = 3$ and $v_1 = v_3$.

11.8. **Different choices for parameters in (44) when $n = 3$.** Considering now other special cases such as $v_2 = v_1$, we see that the pattern of combination (hence the sigma-algebras $\mathcal{F}_n$) as well as the functions c_1 and c_2 must depend on the actual **values of the parameters** v_j and w_j, even in such a small case as (44) with $n = 3$. This leads us to the following problem.

Problem. *Find a pattern according to which combinations can be performed, presumably depending on the geometric mapping properties of f.*

Of course, the solution of this problem for any particular triangulation amounts to proving the inequality (1) for continuous piecewise affine mappings f of compact support defined using that triangulation.

12. A topological picture of a continuous piecewise affine mapping

We next provide a purely heuristic discussion concerning hypothetical ways of combining $u-$terms of triangles. For this reason, we will not prove anything rigorously in this section.

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous piecewise affine mapping of compact support. Suppose that $\mathbb{C}$ has been triangulated in some way. For example, we may use equilateral triangles, or isosceles right-angled triangles as in Figures 2 and 3. Suppose that there is a domain consisting of finitely many such triangles, without loss of generality a square Q, such that $f \equiv 0$ outside Q (and hence also on ∂Q). If desired, we could always approximate f by a function of the same type such that f has a non-zero Jacobian determinant J in each triangle contained in Q that does not have more than one vertex on ∂Q. Thus in each such triangle f is either homeomorphic and sense-preserving (if $J > 0$) or homeomorphic and sense-reversing (if $J < 0$).

Let us split the image $f(\mathbb{C})$ of the plane under f into maximal triangles T not containing images of any edges in their interior, so that f is affine in each component U of any set $f^{-1}(T)$. These components U are triangles and are subsets of the triangles in the original triangulation considered. Thus the triangles U need not have any specific regularity properties and they can be of variable shape and size.

By the theory of topological degree (or by a simple direct argument for this type of functions f), the degree of f is zero, and for each triangle T covered at all, the following holds: T is covered the same number of times in triangles U where f is sense-preserving, and in triangles U where f is sense-reversing. For some T, some such U are adjacent, while for some T, all such U may be far apart.

The question arises as to whether one can use this decomposition as a guide when combining ($u-$terms arising from certain) triangles. We now discuss this question.

Trying to simply combine any two U mapping onto the same T, one in a sense-preserving way and the other in a sense-reversing way, fails without matching lengths as one can verify by a simple computation, and hence may, in general, fail in two such sets U without some common boundary segment.

There are many difficult situations that can arise, probably there being no limit to how complicated they may be. To illustrate the principles of what can happen, we therefore consider a particular simple situation that nonetheless manifests the most important characteristics of the problem.

Thus we note that a simple prototype of the problematic situations that can arise is the following: f expands a small triangle U greatly to a large triangle T in a sense-preserving way, and all other U map into pieces of T in a sense-reversing way. It is not difficult to find concrete examples of this.

An example. For example, in Figure 3, let the triangle with vertices $0, 2, 1+i$ be mapped into a large triangle T with vertices w_1, w_2, w_3 in a sense-preserving way, pick three distinct points z_1, z_2, z_3 strictly inside T, and use them to subdivide T into 7 smaller triangles with vertices (assuming the points are so placed that this gives rise to triangles with disjoint interiors) (w_1, w_2, z_3), (w_1, w_3, z_2), (w_2, w_3, z_1), (z_1, z_2, z_3), (z_1, z_2, w_3), (z_1, z_2, w_2), (z_2, z_3, w_1). Suppose that f maps the points $0, 2, 1+i, 2+2i, 2i$ onto the points w_1, w_2, w_3, z_1, z_2, respectively, and that f maps all other vertices to z_3 (we may assume by translation that $z_3 = 0$). Then we have the kind of situation just described (some of the other triangles are mapped to line segments, so this is not a completely non-degenerate example).

Remark. In connection with the proof of Theorem 1, we ignored the $u-$terms that were trivially non-positive, arising from boundary triangles that had two vertices mapped to the origin. Interior triangles with a similar degeneracy property can certainly *not* be ignored; their help will most certainly be needed. For example, we could modify the example above so that at least one of the inner vertices z_1, z_2, z_3, say z_3, lies on the line segment joining w_1 to w_2. In this case the $u-$term for the triangle with vertices $(0, 2, 1-i)$, mapped by f into this line, would be trivially non-positive, but it could play an essential role in cancelling out positive terms as a computation shows.

Strategy ? In such a case as the example above, it seems that a good place to start is the edges between neighboring triangles U, such that f is sense-preserving in one of them and sense-reversing in the other. Note that in the example above, some triangles U would be subsets of the triangle with vertices $(0, 2, 1+i)$. Then the two triangles U are mapped by f onto the same triangle T, and upon combining the corresponding $u-$values one gets a term that involves all lengths on an interval $[0, M]$; this can be verified by a computation. Note that there is no need for the two triangles U to have equal areas. Thus, as a result of this particular type of combination, one gets not only an interval of lengths, but an interval of all lengths from 0 to a positive length.

This is an optimal situation when one hopes to find further matching lengths. One length in $[0, M]$ will do, and if this can be found, then the next length range can be arranged to be again of the form $[0, M]$ by including in the new combination not only the result of the original combination but also a pair of other triangles U mapping to the same (new) T by opposite orientations.

The principal problem of continuing is to show that further matching lengths can be found. The consideration of particular examples may convince one that it is likely that this should work, in other words, that this may well be the truth about how functions actually behave. But the problem with this approach is its complication since the number of triangles to be considered is arbitrarily large and the way that the sense-preserving and sense-reversing triangles occur can greatly vary.

13. On numerical work

One may ask whether numerical calculations might give an indication of the potential truth of (1) in more general situations than the one covered in Theorem 1. The author has performed numerical experiments with the software package Mathematica, using random numbers to test the various inequalities such as (18), and the further inequalities resulting from various ways of combining terms, for situations involving up to 49 complex parameters, in millions of cases. This is of importance since, whenever we combine terms, we increase the sum of the $u-$terms, so that even if the original sum in (18) were non-positive, it is conceivable that some combinations could lead to positive values for the resulting new sum.

The results of these experiments suggest that (18) holds, and that it is always legitimate to divide the domain of f, where f need not be zero (say a square to start with as in Figure 3) into smaller squares with four triangles inside each, and combine those four triangles together as we have mentioned one can always do. Even though this goes against the spirit of saying that whatever we do should depend on f, it seems that such basic combinations independent of f still result in valid inequalities, hence even stronger than (18); what one does after that, should depend on f, however. Indeed, the possibility of further combining the resulting terms then will, in general, depend on the numerical values of the parameters in question, as one can show by demonstrating a lack of identities for signed area.

In particular, we have found examples where no two squares inside the outermost boundary layer of squares are comparable. This means that one cannot get started with combinations deeper inside the big square, but one must start with the boundary layer.

For any given assortment of smaller squares, there are examples where any particular boundary squares are not comparable to other squares (inside or next to the boundary) next to them. This means that if combinations start at the boundary, they cannot start just anywhere next to the boundary, but the particular spot(s) where one can get started will in general depend on the numerical values of the parameters as well.

Therefore, either reductions must begin using the outermost boundary layer of squares (at a spot depending on f), or, perhaps, they may sometimes begin at a spot possibly inside and depending on the function f. This could be a spot where f changes from being sense-preserving to being sense-reversing, in accordance with what was said in Section 12.

14. A generalization of the Beurling–Ahlfors transformation to space

Iwaniec and Martin ([19], [20]) defined a counterpart $\mathcal{S}$ of the Beurling–Ahlfors operator on the L^p-space of differential forms in $\mathbb{R}^n$, where $n \geq 2$. For $I = (i_1, i_2, \ldots, i_k)$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, write $d_I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, and for each I, let f_I be a complex-valued function in $L^p(\mathbb{R}^n)$, where $1 < p < \infty$. Let $|I|$ denote the cardinality of I, so that $|I| = k$ when $I = (i_1, i_2, \ldots, i_k)$. For each I, one defines

$\varepsilon_I \in \{1, -1\}$ so that if

$$I' = \{1, 2, \ldots, n\} \setminus I,$$

we have

$$\varepsilon_I d_I \wedge d_{I'} = d_J \qquad \text{where } J = \{1, 2, \ldots, n\}$$

and in particular,

$$\varepsilon_I \varepsilon_{I'} = (-1)^{|I|(n-|I|)}.$$

For example, if $n = 3$, then $\varepsilon_I = -1$ if $I = \{2\}$ or $I = \{1, 3\}$, while $\varepsilon_I = 1$ for all the other six subsets I of $\{1, 2, 3\}$, including $I = \emptyset$.

We have

$$d(f_I d_I) = \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j}\, dx_j \wedge d_I$$

and

$$\delta(f_I d_I) = (-1)^{|I|(n-|I|)} \varepsilon_I \sum_{j \in I} \varepsilon_{\{j\} \cup I'} \frac{\partial f_I}{\partial x_j}\, d_{I \setminus \{j\}}.$$

We do not recall the definition of $\mathcal{S}$ since it suffices for our purposes to note that in terms of the exterior derivative d and the Hodge codifferential δ, we have

$$\mathcal{S}(d + \delta)\alpha = (d - \delta)\alpha$$

for all

$$\alpha = \sum_I f_I d_I$$

where all $f_I \in L^p(\mathbb{R}^n)$ and the sum is over all I, including $I = \emptyset$. We define

$$||\alpha||_p^p = \int_{\mathbb{R}^n} \left(\sum_I |f_I(x)|^2 \right)^{p/2}$$

where the integral is with respect to the Lebesgue measure. Iwaniec and Martin conjectured that for all $n \geq 2$ and all $p \in (1, \infty)$, we have $||\mathcal{S}||_p = p^* - 1$, the non-trivial part of which is

$$||(d - \delta)\alpha||_p \leq (p^* - 1)||(d + \delta)\alpha||_p.$$

They noted that equality always holds when $p = 2$.

We think that this problem can be approached in the same way as in the plane. We consider again continuous piecewise affice complex-valued functions of compact support defined in $\mathbb{R}^n$. Such a function f is assumed to be affine in each of finitely many simplices. If we want to reduce an integral to a finite sum with equal weights, then the only property needed is that the simplices should have equal volume, while they or their faces need not be congruent in any way. This is because we compare only affine maps, and not affine maps on faces, and for the comparison of affine maps, we only need to know how they behave on $(n-1)-$dimensional hyperplanes containing common faces of various simplices. Otherwise, we get a finite sum with unequal weights.

14.1. **Approximation.** By approximation, we may assume that each $f_I \in \mathcal{F}(\mathbb{R}^n, \mathbb{C})$, where for a triangulation we may use any simplices as long as the maximum diameter of the simplices may be arbitrarily small and the angles are bounded away from zero (more precisely, the counterpart of (15) holds for the largest sphere inscribed in each simplex). We would like to indicate more concretely what this means, so we assume that $n = 3$ to avoid too much notation. We use notation such as

$$f_{12}^j = \partial f_I / \partial x_j \quad \text{where} \quad I = \{1, 2\},$$

and write f_0 for f_I when $I = \emptyset$. Then

$$(d+\delta)\alpha = \sum_I A_I d_I$$

and

$$(d-\delta)\alpha = \sum_I B_I d_I$$

where

$$\begin{aligned}
A_0 = -B_0 &= f_1^1 + f_2^2 + f_3^3, \\
A_{123} = B_{123} &= f_{12}^3 - f_{13}^2 + f_{23}^1, \\
A_1 &= f_0^1 + f_{12}^2 + f_{13}^3, \\
A_2 &= f_2^0 - f_{12}^1 + f_{23}^3, \\
A_3 &= f_0^3 - f_{13}^1 - f_{23}^2, \\
A_{12} &= f_{123}^3 - f_1^2 + f_2^1, \\
A_{13} &= -f_{123}^2 - f_1^3 + f_3^1, \\
A_{23} &= f_{123}^1 - f_2^3 + f_3^2, \\
B_1 &= f_0^1 - f_{12}^2 - f_{13}^3, \\
B_2 &= f_2^0 + f_{12}^1 - f_{23}^3, \\
B_3 &= f_0^3 + f_{13}^1 + f_{23}^2, \\
B_{12} &= -f_{123}^3 - f_1^2 + f_2^1, \\
B_{13} &= f_{123}^2 - f_1^3 + f_3^1, \\
B_{23} &= -f_{123}^1 - f_2^3 + f_3^2.
\end{aligned}$$

To prove that

$$\text{(46)} \qquad \int_{\mathbb{R}^n} \left(\sum_I |B_I(x)|^2 \right)^{p/2} \leq (p^* - 1)^p \int_{\mathbb{R}^n} \left(\sum_I |A_I(x)|^2 \right)^{p/2}$$

we may therefore ignore the terms $A_0, B_0, A_{123}, B_{123}$, noting that this does not affect the equality for $p = 2$. The groups of terms

$$(A_1, A_2, A_3) \quad \text{and} \quad (B_1, B_2, B_3),$$

are structurally similar to the groups

$$(A_{12}, A_{23}, A_{13}) \quad \text{and} \quad (B_{12}, B_{23}, B_{13}),$$

so that it suffices to consider only the former.

We also see that the transition from the $A-$terms to the $B-$terms merely amounts to changing certain signs, so that this affects the real and imaginary parts of the functions f_I in the same way. Thus we see that we may assume that the f_I are real-valued, and if the problem is solved in that case, the result immediately generalizes not only to complex-valued but also, e.g., $\mathbb{R}^n-$valued functions f_I.

Initially, we have 8 complex-valued functions f_I, but after these simplifications, we are down to looking at only 3 essentially different real-valued functions due to the relations that exist between the functions still remaining. Thus we have made a reduction from $\mathcal{F}(\mathbb{R}^3, \mathbb{C}^8)$ to $\mathcal{F}(\mathbb{R}^3, \mathbb{R}^3)$.

We write $-\varphi$ for f_0 and $\mathbf{A}$ for $(f_{23}, -f_{13}, f_{12})$. Then the conjectured inequality (46) assumes the simpler form

$$||\nabla\varphi + \nabla \times \mathbf{A}||_p \leq (p^* - 1)||\nabla\varphi - \nabla \times \mathbf{A}||_p. \tag{47}$$

Note that even though $\mathbf{A}$ depends on three real functions, for all practical purposes $\mathbf{A}$ depends on only two real functions since $\nabla \times \mathbf{A}$ is not affected if $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\psi_0$ for some real-valued function ψ_0. We should really identify $\mathbf{A}$ with an equivalence class of functions with the same $\nabla \times \mathbf{A}$. Thus our information $(\varphi, \mathbf{A})$ really depends on only three real-valued functions.

We get a physically interesting situation by interpreting φ as the electric potential and $\mathbf{A}$ as the magnetic vector potential, noting that then $\mathbf{E} = -\nabla\varphi$ is the electric field and $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field. Thus (47) reads

$$||\mathbf{E} - \mathbf{B}||_p \leq (p^* - 1)||\mathbf{E} + \mathbf{B}||_p. \tag{48}$$

Of course, we could replace $\mathbf{A}$ by $-\mathbf{A}$ and hence $\mathbf{B}$ by $-\mathbf{B}$ in (48). Thus there is nothing special about having the difference $\mathbf{E} - \mathbf{B}$ on the left rather than the right hand side of (48).

Even though neither (47) or (48) appears in [19] or [20], and the above reduction of the problem in $\mathbb{R}^3$ seems new, it follows from the results of Iwaniec and Martin [20] that these inequalities hold for some constants depending on p, and the conjecture is that the constant $p^* - 1$ works.

That we have an isometry in L^2, that is,

$$\int_{\mathbb{R}^3} |\nabla\varphi + \nabla\times\mathbf{A}|^2 = \int_{\mathbb{R}^3} |\nabla\varphi - \nabla\times\mathbf{A}|^2,$$

is equivalent to saying that

$$\int_{\mathbb{R}^3} (\nabla\varphi)\cdot(\nabla\times\mathbf{A}) = 0,$$

which follows by a suitable substitution from Gauss's theorem since φ is of compact support.

Since each $f \in \mathcal{F}(\mathbb{R}^3, \mathbb{R}^3)$ can be written in the form $\nabla\varphi + \nabla\times\mathbf{A}$ by Helmholtz's theorem, we obtain, in fact, a bounded linear operator from $L^p(\mathbb{R}^3 \to \mathbb{R}^3)$ into itself, taking

$$\nabla\varphi + \nabla\times\mathbf{A} \quad \text{to} \quad \nabla\varphi - \nabla\times\mathbf{A},$$

whose $p-$norm is then conjectured to be $p^* - 1$. This operator is perhaps the proper counterpart of the Beurling–Ahlfors transformation in $3-$space, and the rest is extra formalism.

It should now be obvious that if we are to approach the problem of proving (47) using martingale transforms and rotations in the same way as we did in the plane, we end up with the following problem, which could also be stated as a conjecture.

Problem. *Do there exist (piecewise constant) functions $P_1(x)$ and $P_2(x)$ for $x \in \mathbb{R}^3$ whose values are 3×3 orthogonal matrices (with real entries), and do there exist $\mathbb{R}^3-$valued martingales X_n and Y_n that are martingale transformations of each other, start at constants X_1, Y_1 of the same length, and end after finitely many steps at $P_1(\mathbf{E}+\mathbf{B})$ and $P_2(\mathbf{E}-\mathbf{B})$? This would imply (47)–(48).*

14.2. **Rotations in space.** Here the application of P_1 and P_2, which does not change the lengths of the vectors $\mathbf{E}+\mathbf{B}$ and $\mathbf{E}-\mathbf{B}$, respectively, is the counterpart of multiplying, in the plane, complex-valued functions $\partial f/\partial\bar{z}$ and $\partial f/\partial z$ by complex-valued functions $c_1(z)$ and $c_2(z)$ of modulus 1. This is the way of making use, in space, of the fact that $u(z,w)$ only depends on $|z|$ and $|w|$ when $z, w \in \mathbb{R}^n$.

Also it is not P_1 and P_2 as such that are important, but the way that they vary. If there were P_1 and P_2 as above, we could obviously choose constant orthogonal matrices P_3 and P_4 and replace P_1 and P_2 by P_3P_1 and P_4P_2, respectively. Thus we could choose any simplex and normalize, for example, each of P_1 and P_2 to be the identity mapping there. So it is the derivatives of P_1 and P_2 that are important, even though, in the present normalization, those derivatives are concentrated on the faces of the simplices and hence depend on delta-distributions.

14.3. **Non-local quantities in physics ?** If the answer to the above Problem is affirmative, then the Burkholder formalism implies immediately that (48) holds. On the other hand, the proof of (48) in any particular case by means of combining various $u-$terms together after suitable rotations, amounts to proving the existence, in that case, of P_1, P_2, $\{X_n\}$, and $\{Y_n\}$. Also one can then ask whether the "**fields of rotations**" P_1 and P_2 have any physical interpretation. They cannot be determined by local properties of the **static** fields $\mathbf{E}$ and $\mathbf{B}$ alone, but would have to depend, in general, of what $\mathbf{E}$ and $\mathbf{B}$ are as a whole. This would be a rather interesting "non-local" quantity in a classical physical theory.

It may be noted that even classically, the energy and momentum of the electromagnetic field are not considered by all to be localized, either, in spite of their densities being given by the apparently local quantities $|\mathbf{E}|^2 + |\mathbf{B}|^2$ and $\mathbf{E} \times \mathbf{B}$ (apart from multiplicative physical constants), but that seems somewhat different from what happens with P_1 and P_2 as the dependence of these densities on $\mathbf{E}$ and $\mathbf{B}$ is straightforward ([15], Ch. 27, particularly p. 27–6).

The discussion becomes now highly speculative, but if all this were to work out, one could ask what happens with dynamic electromagnetic fields. Such a question would probably make sense only if one were to first find a physical interpretation for P_1 and P_2. Since P_1 and P_2 depend on $\mathbf{E}$ and $\mathbf{B}$ as a whole, it would take some time for any change in $\mathbf{E}$ or $\mathbf{B}$ in one location to propagate its effect to other locations. Thus it would take some time for P_1 and P_2 (or more precisely for the derivatives of P_1 and P_2) to change.

14.4. **Counterpart of matching lengths in space.** In this formulation it is also perhaps easiest to see that the ideas presented above, in the plane case, to deal with Burkholder's $u-$function also work in this setting in $3-$space. We may assume that two neighboring simplices have their common face in the plane $T = \{(x_1, x_2, x_3) : x_3 = 0\}$, one vertex at the origin, and that $\varphi(0) = 0$, $\mathbf{A}(0) = \mathbf{0}$. We use notation such as $(\nabla\varphi)^+$ and $(\nabla\varphi)^-$ for the values of $\nabla\varphi$ in the upper and lower simplex, respectively. With $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and

$$\varphi(x) = a_1x_1 + a_2x_2 + a_3x_3$$

in the upper simplex and

$$\varphi(x) = b_1x_1 + b_2x_2 + b_3x_3$$

in the lower simplex, the fact that φ is continuous across T implies that $a_1 = b_1$ and $a_2 = b_2$. Since $(\nabla\varphi)^+ = (a_1, a_2, a_3)$, this means that

$$(\nabla\varphi)^+ - (\nabla\varphi)^- = (0, 0, a_3 - b_3).$$

Since the counterpart of the conditions $(a_1 = b_1, a_2 = b_2)$ must also hold for each component of $\mathbf{A}$, it follows that the third component of $(\nabla \times \mathbf{A})^+ - (\nabla \times \mathbf{A})^-$ is zero. Hence

$$((\nabla\varphi)^+ - (\nabla\varphi)^-) \cdot ((\nabla \times \mathbf{A})^+ - (\nabla \times \mathbf{A})^-) = 0.$$

This is equivalent to saying that

$$|(\nabla\varphi + \nabla \times \mathbf{A})^+ - (\nabla\varphi + \nabla \times \mathbf{A})^-| = |(\nabla\varphi - \nabla \times \mathbf{A})^+ - (\nabla\varphi - \nabla \times \mathbf{A})^-|,$$

which is the counterpart in this $\mathbb{R}^3$–situation of the formula (20) in the plane. This seems to us to be the most transparent setting that displays the essentials of the problem. The quantities $\nabla\varphi$ and $\nabla \times \mathbf{A}$ behave well under rotations and changes of variables by rotations. In this way one can see that (48) holds in certain small cases.

15. Banach spaces and the UMD-property

We briefly mention a possible generalization and therefore refer to the survey [11] for a more complete discussion and references. One says by definition that a Banach space $\mathcal{B}$ has the UMD-property (*unconditional for martingale differences*) if

$$||Y||_p \leq \beta ||X||_p$$

for some finite $\beta = \beta(p, \mathcal{B})$, for $\mathcal{B}-$valued martingales X and Y starting at 0 and for $1 < p < \infty$ if

$$Y_n - Y_{n-1} = \pm(X_n - X_{n-1}),$$

so Y is a special martingale transform of X. For example, the sequence space ℓ^p with values in a Hilbert space $\mathcal{H}$ and the space L^p of functions on $[0, 1]$ with values in $\mathcal{H}$ are UMD-spaces for $1 < p < \infty$.

The above questions for the various Beurling–Ahlfors transformations and martingales generalize from real-valued functions to $\mathcal{H}-$valued functions. In a Banach space the problems may be harder. Thus raising these same questions for $\mathcal{B}-$valued functions may suggest new interesting problems for Banach spaces with the UMD-property.

16. The role of discreteness

We have presented various problems discussing only simple classes of functions such as $\mathcal{F}(\mathbb{R}^3, \mathbb{R}^3)$ since it is important to bring out (and discover) the fundamental new ideas, while after the problems have been solved for these simple classes, the transition to larger classes of functions should be only a matter of technicalities. In some sense, for example, the fundamental theorem of calculus and its higher-dimensional counterparts such as Stokes's theorem are merely appropriate continuous counterparts of the discrete identity

$$a_n - a_0 = \sum_{k=1}^{n} [(a_k - a_{k-1})/\delta_k]\delta_k,$$

and its multidimensional counterparts, where the $\delta_k > 0$ and the a_k belong to a space such as $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^d$. The form of this identity is independent of the a_k. What we are proposing is that hidden in functions such as $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$ or $f \in \mathcal{F}(\mathbb{R}^3, \mathbb{R}^3)$, there are complicated identities whose **form** also varies and depends on f (through the dependence on f of the rotations and of the sigma-algebras for the associated martingales).

17. Other extremal problems

Burkholder [9] solved the $p-$norm problem for martingale transforms by solving a more general extremal problem of maximizing $||Y||_p$ for a martingale transform Y of the martingale X when $||X||_p$ is given and also the starting points X_0 and Y_0 are given and need not be zero. Later [10] he simplified the proof so that one can now obtain (4) directly without solving a more general problem.

Nonetheless there may be some merit in trying to solve more general problems than the one at hand, both to get better results and to obtain a better understanding of the underlying situation. In that spirit, we now briefly discuss possible problems to consider for the Beurling–Ahlfors transformation. The treatment is heuristic to expose ideas, so we prove nothing rigorously and omit questions such as exactly what function classes functions can be allowed to belong to.

Statement of the problem. *Let D be a bounded domain in $\mathbb{C}$ with ∂D consisting of finitely many disjoint rectifiable Jordan curves. Let $f : \overline{D} \to \mathbb{C}$ be a function continuous in $\overline{D}$ and, furthermore, in $C^2(\overline{D})$. Denote the boundary value function of f by $F = f|\partial D$. Choose $t > 0$ large enough so that there is $h : \overline{D} \to \mathbb{C}$ in $C^2(\overline{D})$ with $h|\partial D = F$ and* *with*

$$||h_{\overline{z}}||_p \equiv \left(\int_D |h_{\overline{z}}|^p \right)^{1/p} \leq t,$$

where $p \in (1, \infty)$ *is a fixed real number. The problem is to maximize* $||h_z||_p$ *under these constraints.*

It is easily seen that the supremum of $||h_z||_p$ under these constraints is finite.

Remark. It would be of some interest, from the point of view of approximation by piecewise affine mappings, to solve this problem in the special case when D is a triangle (possibly even a special kind of triangle such as an equilateral triangle, or a right-angled isosceles triangle) and F is given by an affine mapping, with t large enough so that F itself is in the competing class of mappings.

Properties of maximizers. Suppose that the choice $h = f$ maximizes $||h_z||_p$ (we do not address here the question of the existence of a maximizer). Let $\eta : \overline{D} \to \mathbb{C}$ be a function in $C^2(\overline{D})$ with $\eta|\partial D \equiv 0$. Then if

$$||(f + \eta)_{\overline{z}}||_p \leq t,$$

we have

$$||(f + \eta)_z||_p \leq ||f_z||_p.$$

If $|\eta_z(z)|$ and $|\eta_{\overline{z}}(z)|$ are uniformly small, we may approximate (ignoring points where $f_{\overline{z}} = 0$ or $f_z = 0$)

$$|(f + \eta)_{\overline{z}}| = |f_{\overline{z}}| \left| 1 + \frac{\eta_{\overline{z}}}{f_{\overline{z}}} \right| \approx |f_{\overline{z}}| \left(1 + \mathrm{Re}\, \frac{\eta_{\overline{z}}}{f_{\overline{z}}} \right) = |f_{\overline{z}}| + |f_{\overline{z}}| \mathrm{Re}\, \frac{\eta_{\overline{z}}}{f_{\overline{z}}}$$

and

$$|(f + \eta)_{\overline{z}}|^p \approx |f_{\overline{z}}|^p + p|f_{\overline{z}}|^p \mathrm{Re}\, \frac{\eta_{\overline{z}}}{f_{\overline{z}}} = |f_{\overline{z}}|^p + p|f_{\overline{z}}|^{p-2} \mathrm{Re}\, (\eta_{\overline{z}} \overline{f_{\overline{z}}})$$

and similarly

$$|(f + \eta)_z|^p \approx |f_z|^p + p|f_z|^{p-2} \mathrm{Re}\, (\eta_z \overline{f_z}).$$

Thus

$$\int_D |f_z|^{p-2} \mathrm{Re}\, (\eta_z \overline{f_z}) \leq 0 \tag{49}$$

and, if $||f_{\overline{z}}||_p = t$, we are required to choose η so that

$$\int_D |f_{\overline{z}}|^{p-2} \mathrm{Re}\, (\eta_{\overline{z}} \overline{f_{\overline{z}}}) \leq 0. \tag{50}$$

By Green's theorem and the fact that $\eta \equiv 0$ on ∂D, we have

$$\begin{aligned} 0 &= \frac{1}{2i}\int_{\partial D} |f_z|^{p-2}\left(\overline{\eta} f_z\, dz - \eta \overline{f_z}\, d\overline{z}\right) \\ &= \int_D \left(\left(|f_z|^{p-2}\overline{\eta} f_z\right)_{\overline{z}} + \left(|f_z|^{p-2}\eta\overline{f_z}\right)_z\right) \\ &= \int_D \left(2|f_z|^{p-2}\mathrm{Re}\,(\eta_z\overline{f_z}) + \eta\left(|f_z|^{p-2}\overline{f_z}\right)_z + \overline{\eta}\left(|f_z|^{p-2} f_z\right)_{\overline{z}}\right) \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{1}{2i}\int_{\partial D} |f_{\overline{z}}|^{p-2}\left(\eta \overline{f_{\overline{z}}}\, dz - \overline{\eta} f_{\overline{z}}\, d\overline{z}\right) \\ &= \int_D \left(\left(|f_{\overline{z}}|^{p-2}\eta \overline{f_{\overline{z}}}\right)_{\overline{z}} + \left(|f_{\overline{z}}|^{p-2}\overline{\eta} f_{\overline{z}}\right)_z\right) \\ &= \int_D \left(2|f_{\overline{z}}|^{p-2}\mathrm{Re}\,(\eta_{\overline{z}}\overline{f_{\overline{z}}}) + \eta\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}} + \overline{\eta}\left(|f_{\overline{z}}|^{p-2} f_{\overline{z}}\right)_z\right) \end{aligned}$$

so that by (49) and (50),

$$\int_D \left(\eta\left(|f_z|^{p-2}\overline{f_z}\right)_z + \overline{\eta}\left(|f_z|^{p-2} f_z\right)_{\overline{z}}\right) \geq 0 \tag{51}$$

and

$$\int_D \left(\eta\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}} + \overline{\eta}\left(|f_{\overline{z}}|^{p-2} f_{\overline{z}}\right)_z\right) \geq 0. \tag{52}$$

Pick $z_1 \in D$. By approximation, we may get as closely as we like to the situation where $\eta(z_1) = \alpha$ is any non-zero complex number while $\eta(z) = 0$ for all $z \in D \setminus \{z_1\}$. This gives by (51)

$$\alpha\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1) + \overline{\alpha}\left(|f_z|^{p-2} f_z\right)_{\overline{z}}(z_1) \geq 0,$$

which is the same as

$$\mathrm{Re}\,\left(\alpha\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1)\right) \geq 0 \tag{53}$$

since

$$\overline{\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1)} = \left(|f_z|^{p-2} f_z\right)_{\overline{z}}(z_1),$$

and by (52)

$$\alpha\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1) + \overline{\alpha}\left(|f_{\overline{z}}|^{p-2} f_{\overline{z}}\right)_z(z_1) \geq 0.$$

which is the same as

$$\mathrm{Re}\,\left(\alpha\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1)\right) \geq 0. \tag{54}$$

The logic here is that η must be so chosen that (50) holds, and if this is so, then (49) must hold. Hence α must be chosen so that (54) holds, and then (53) must hold.

If $\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1) \neq 0$, and if every α that satisfies (54) is to also satisfy (53), then it must be the case that

$$\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1) = \lambda\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1) \tag{55}$$

for some $\lambda \geq 0$. This number λ could conceivably depend on z_1. If $\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1) = 0$, then any α can be used, so that (53) implies that $\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1) = 0$. Thus again (55) holds, now for any λ.

Let us next choose two distinct points $z_1, z_2 \in D$. By approximation, we may get as closely as we like to the situation where $\eta(z_1) = \alpha$ and $\eta(z_1) = \beta$ are any non-zero complex numbers while $\eta(z) = 0$ for all $z \in D \setminus \{z_1, z_2\}$. Now we are required to choose α and β so that

$$\mathrm{Re}\,\left(\alpha\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1)\right) + \mathrm{Re}\,\left(\beta\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_2)\right) \geq 0, \tag{56}$$

and then we must have

$$\mathrm{Re}\,\left(\alpha\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1)\right) + \mathrm{Re}\,\left(\beta\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_2)\right) \geq 0. \tag{57}$$

We know that there are $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that

$$\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_1) = \lambda_1\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1)$$

and

$$\left(|f_z|^{p-2}\overline{f_z}\right)_z(z_2) = \lambda_2\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_2).$$

Thus, whenever (56) holds, we must have

$$\lambda_1\mathrm{Re}\,\left(\alpha\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1)\right) + \lambda_2\mathrm{Re}\,\left(\beta\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_2)\right) \geq 0. \tag{58}$$

If $\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_1) \neq 0$, $\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z_2) \neq 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$, it is easily seen that this is possible only if $\lambda_1 = \lambda_2$.

Thus, if a hypothetical maximizer f is sufficiently regular, then the following should hold:

If $p \in (1, \infty)$, $p \neq 2$, and f maximizes $||f_z||_p$ for given boundary values $f|D$ and a given upper bound for $||f_{\overline{z}}||_p$, then there is a constant $\lambda > 0$ such that

$$\left(|f_z|^{p-2}\overline{f_z}\right)_z(z) = \lambda\left(|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right)_{\overline{z}}(z) \tag{59}$$

for all $z \in D$.

Incidentally, we should take $p \neq 2$ in these considerations.

Hence f should also be a maximizer for its boundary values in each subdomain G of D, for the same p, provided that the value of t used is $\left(\int_G |f_{\overline{z}}|^p\right)^{1/p}$ so as to admit f as a competing function.

An auxiliary function. If, furthermore, D is simply connected, we find by (59) a function g in D such that

$$g_z = \lambda|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}, \qquad g_{\overline{z}} = |f_z|^{p-2}\overline{f_z}. \tag{60}$$

A special null-Lagrangian. If G is a subdomain of D whose boundary is a rectifiable Jordan curve then by Green's theorem,

$$\begin{aligned}
&\int_G \left(\lambda|f_{\overline{z}}|^p - |f_z|^p\right) \\
(61)\qquad = \; &\int_G \left(f_{\overline{z}}\left(\lambda|f_{\overline{z}}|^{p-2}\overline{f_{\overline{z}}}\right) - f_z\left(|f_z|^{p-2}\overline{f_z}\right)\right) \\
= \; &\int_G \left(f_{\overline{z}}g_z - f_z g_{\overline{z}}\right) \\
= \; &\int_G \left((fg_z)_{\overline{z}} - (fg_{\overline{z}})_z\right) \\
= \; &\frac{1}{2i}\int_{\partial G} f(g_z\,dz + g_{\overline{z}}\,d\overline{z}) \\
= \; &\frac{1}{2i}\int_{\partial G} f\,dg
\end{aligned}$$

so that the integral over G depends only on values of f and g on ∂G. This shows that $\lambda|f_{\overline{z}}|^p - |f_z|^p$ has the properties of a null-Lagrangian in D when f is a maximizer.

Solution to the problem for the conjugate exponent. Write $q = p/(p-1)$ for the conjugate exponent of p, so that $(q-1)(p-1) = 1$. Now with $h = \overline{g}$, we have

$$(62)\qquad |h_{\overline{z}}|^{q-2}\overline{h_{\overline{z}}} = |g_z|^{q-2}g_z = \lambda^{q-1}\overline{f_{\overline{z}}}, \quad |h_z|^{q-2}\overline{h_z} = |g_{\overline{z}}|^{q-2}g_{\overline{z}} = \overline{f_z},$$

which imply that

$$(63)\qquad \left(|h_z|^{q-2}\overline{h_z}\right)_z = \left(\overline{f_z}\right)_z = \left(\overline{f_{\overline{z}}}\right)_{\overline{z}} = \lambda^{p-1}\left(|h_{\overline{z}}|^{q-2}\overline{h_{\overline{z}}}\right)_{\overline{z}},$$

so that (59) holds with f, p, λ replaced by h, q, λ^{p-1}, respectively. Hence h is a maximizer *for its boundary values*, for the exponent q, provided that we take $t \geq ||h||_q$. Of course, we may not be able to tell much, if anything, about the boundary values of h; see the examples below.

Examples of solutions. The following functions are solutions to (59), hence probably maximizers for their boundary values in any reasonable domain, for suitable values of t. They are very special, but it seems hard to find explicit solutions to (59) in general. Of course, if f is a solution of (59), then so is $\alpha f + \beta$ for any $\alpha, \beta \in \mathbb{C}$.

(i) We may take in a suitable domain

$$f(z) = \alpha \log z + \beta \log \overline{z}$$

where α and β are complex numbers such that α/β is a positive real number. Then

$$\lambda = \frac{|\alpha|^{p-2}\overline{\alpha}}{|\beta|^{p-2}\overline{\beta}},$$

$$g(z) = \frac{-2}{p-2}\,\frac{|\alpha|^{p-2}\overline{\alpha}}{|z|^{p-2}}, \quad h(z) = \frac{-2}{p-2}\,\frac{|\alpha|^{p-2}\alpha}{|z|^{p-2}}.$$

(ii) We may take, for positive α,

$$f(z) = z|z|^{\alpha} = z^{(\alpha+2)/2}\overline{z}^{\alpha/2},$$

so that

$$g(z) = \overline{z}|z|^{\alpha(p-1)}\left(\frac{\alpha+2}{2}\right)^{p-1}\left(1+\frac{\alpha(p-1)}{2}\right)^{-1},$$

$$\lambda = \left(\frac{\alpha+2}{\alpha}\right)^{p-1}\frac{\alpha(p-1)}{2}\left(1+\frac{\alpha(p-1)}{2}\right)^{-1},$$

and $h(z) = \overline{g(z)} = Cz|z|^{\alpha(p-1)}$ where the positive constant C is clear from the formula for g.

If we consider constant multiples of this f in any disk $\{z \in \mathbb{C} : |z| \le R\}$, and require that $f(z) = Az$ whenever $|z| = R$, for a given $A > 0$, then we get $f(z) = Bz|z|^{\alpha}$ for some $B > 0$. The two conditions $A = BR^{\alpha}$ and $||f_{\overline{z}}||_p = t$, where p and t are given, determine B and α, and hence also determine g (which is now B times what was given above) and λ (which is still the same, regardless of the value of B). Namely, α is the unique positive solution to (recall that $1 < p < \infty$)

$$\frac{\alpha^p}{\alpha p + 2} = \frac{2^p t^p}{2\pi R^2 A^p},$$

and once α is known, we take $B = AR^{-\alpha}$.

If we were to take $\alpha = 0$, we would get $f(z) = z$. More generally, any affine f satisfies (59) for any λ, but this does not necessarily mean that such an f is a maximizer. Also, if $p = 2$, then any f satisfies (59) for $\lambda = 1$.

(iii) If a and b are non-zero complex numbers such that ab is real, then we may take

$$f(z) = \exp\{az + b\overline{z}\},$$

$$g(z) = \frac{2\overline{a}|a|^{p-2}}{p\overline{a} + (p-2)b}\exp\left\{(p\overline{b} + (p-2)a)\frac{z}{2} + (p\overline{a} + (p-2)b)\frac{\overline{z}}{2}\right\},$$

$$\lambda = \left|\frac{a}{b}\right|^{p-2}\frac{\frac{p}{2}\overline{ab} + \frac{p-2}{2}|a|^2}{\frac{p}{2}\overline{ab} + \frac{p-2}{2}|b|^2}.$$

Here we have the relation

$$\frac{p-2}{2}\{|a|^p - \lambda|b|^p\} = \frac{p}{2}\overline{(ab)}\{\lambda|b|^{p-2} - |a|^{p-2}\},$$

where, of course, $\overline{ab} = ab$, but it may be easier to verify using $\overline{ab}$ that this is the relation that arises.

(iv) It might be of interest to look for solutions to (59) of the form

$$f = \sum_{j=1}^{k} f_j \overline{g_j}$$

where the f_j and g_j are analytic in D. If we do this for $k = 1$ and write $f = F\overline{G}$ where F and G are analytic and non-constant, then (59) becomes

$$\lambda = \left| \frac{F'/F}{G'/G} \right|^{p-2} \frac{\frac{p}{2} + \frac{p-2}{2} \frac{F''/F'}{G'/G}}{\overline{\frac{p}{2} + \frac{p-2}{2} \left(\frac{G''/G'}{F'/F} \right)}}.$$

Special solutions can probably be obtained by taking F and G to be powers of $az + b$, and ae^{bz}. However, that would be very limited, but it may be difficult to find other explicit solutions.

Third order equations. It is possible to eliminate λ and derive lengthy third order partial differential equations from (59), with p as the only remaining parameter. We omit the details.

Connection to the $p-$harmonic equation. Each side of (59) might be viewed as a complex-valued counterpart of the $p-$harmonic equation, which reads $\nabla \cdot (|\nabla v|^{p-2} \nabla v) = 0$ for a real-valued function v in D.

Another extremal problem. Suppose that we try to maximize, subject to prescribed boundary values,

$$\int_D u_0 \left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \overline{z}} \right),$$

where u_0 is as defined by (6) and (7). An analysis similar to the one above suggests that a hypothetical maximizer should satisfy a partial differential equation with one type of particular solution given by

$$f_z = -\lambda \psi \left(1 + \frac{1}{|\psi|} \right)$$

or

$$f_z = -\lambda \psi \left(1 - \frac{1}{|\psi|} \right)$$

where $\lambda > 0$ and ψ is analytic in that part of D (depending on f, of course, and hence depending on the boundary values) where

$$\left| \frac{\partial f}{\partial z} \right| + \left| \frac{\partial f}{\partial \overline{z}} \right| > 1. \tag{64}$$

On the boundary of any component G of the subset of D where (64) holds, we have

$$\frac{f_z}{|f_z|}\,dz = \frac{f_{\overline{z}}}{|f_{\overline{z}}|}\,d\overline{z}.$$

References

[1] L.V. Ahlfors, Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn. A I 206 (1955).

[2] K. Astala, Area distortion of quasiconformal mappings, Acta Math. 173 (1994), 37–60.

[3] K. Astala, T. Iwaniec, and E. Saksman, Beltrami operators in the plane, Duke Math. J. 107 (2001), 27–56.

[4] R. Bañuelos and P. Janakiraman, L^p-bounds for the Beurling–Ahlfors transform, preprint, 2006.

[5] R. Bañuelos and Gang Wang, Sharp inequalities for martingales with applications to the Beurling–Ahlfors and Riesz transforms, Duke Math. J. 80 (1995), 575–600.

[6] A. Baernstein II and S.J. Montgomery-Smith, Some conjectures about integral means of ∂f and $\overline{\partial} f$, pp. 92–109 in Complex analysis and differential equations, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 64, Uppsala Univ., Uppsala, 1999.

[7] A. Beurling, The Collected Works of Arne Beurling, Volume 1, Birkhäuser, Boston, 1989, a seminar talk at Uppsala, "On the Hilbert Transform in Two Variables", November 8, 1949, pp. 460–461.

[8] B. Boyarskiĭ, Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients, Mat. Sb. N.S. 43(85) 1957, 451–503.

[9] D.L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, Ann. Probability 12 (1984), 647–702.

[10] D.L. Burkholder, Sharp inequalities for martingales and stochastic integrals, Asterisque 157–158 (1988), 75–94.

[11] Martingales and singular integrals in Banach spaces, Handbook of the geometry of Banach spaces, Vol. I, 233–269, North-Holland, Amsterdam, 2001.

[12] A.P. Calderón and A. Zygmund, On the existence of certain singular integrals, Acta Math. 88 (1952), 85–139.

[13] P.G. Ciarlet and P.-A. Raviart, General Lagrange and Hermite interpolation in $\mathbf{R}^n$ with applications to finite element methods, Arch. Rational Mech. Anal. 46 (1972), 177–199.

[14] O. Dragičević and A. Volberg, Bellman function, Littlewood-Paley estimates and asymptotics for the Ahlfors–Beurling operator in $L^p(\mathbb{C})$, Indiana Univ. Math. J. 54 (2005), 971–995.

[15] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics, vol. II, Addison–Wesley, Reading, MA, 1964.

[16] F.W. Gehring, The L^p-integrability of the partial derivatives of a quasiconformal mapping, Acta Math. 130 (1973), 265–277.

[17] T. Iwaniec, Extremal inequalities in Sobolev spaces and quasiconformal mappings, Z. Anal. Anwendungen 1 (1982), 1–16.

[18] T. Iwaniec and G.J. Martin, Quasiconformal mappings and capacity, Indiana Univ. Math. J. 40 (1991), 101–122.
[19] T. Iwaniec and G.J. Martin, Quasiregular mappings in even dimensions, Acta Math. 170 (1993), no. 1, 29–81.
[20] T. Iwaniec and G.J. Martin, Riesz transforms and related singular integrals, J. Reine Angew. Math. 473 (1996), 25–57.
[21] O. Lehto, Remarks on the integrability of the derivatives of quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I No. 371 (1965), 8 pp.
[22] O. Lehto and K.I. Virtanen, Quasiconformal mappings in the plane, Springer, Berlin, 1973.
[23] S. Petermichl and A. Volberg, Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular, Duke Math. J. 112 (2002), 281–305.
[24] I.N. Vekua, The problem of reduction to canonical form of differential forms of elliptic type and the generalized Cauchy-Riemann system, Dokl. Akad. Nauk SSSR (N.S.) 100, (1955), 197–200.
[25] A. Volberg and F. Nazarov, Heat extension of the Beurling operator and estimates for its norm (Russian), Algebra i Analiz 15 (2003), 142–158.

University of Illinois at Urbana–Champaign, Department of Mathematics, 1409 West Green Street, Urbana, IL 61801 USA
E-mail address: aimo@uiuc.edu