A CONJECTURE ON MARTINGALES AND ROTATIONS

AIMO HINKKANEN

Abstract. We conjecture that continuous complex-valued piecewise affine maps of compact support in the complex plane have a probabilistic structure: associated with certain combinations of the first partial derivatives of such functions, there are two fields of rotations, and two martingales that are martingale transforms of each other, starting from constants of equal modulus, and ending at what one obtains after rotating these combinations of the derivatives. We prove this result in certain cases of continuous piecewise affine functions in the plane depending on 13 complex parameters.

The motivation for this is that such a result would be sufficient to prove the conjectured value for the sharp $p-$norm of the Beurling–Ahlfors transformation in the plane. Indeed the result for the norms of these transformations would then follow from Burkholder's estimates for the norms of two martingales that are martingale transforms of each other. On the other hand, it is shown that if we look for a way of obtaining the desired estimate for the norm of the Beurling–Ahlfors transformation, then we are naturally lead to considering martingales that are obtained after rotations from a function and its Beurling–Ahlfors transformation.

1. Introduction: conjectured probabilistic structure of functions

In this paper, we present a conjecture; state and prove a theorem on a very special case of the conjecture; give motivation for the conjecture; and discuss a possible approach to proving the conjecture.

1.1. **The Conjecture.** Let D be a domain in the complex plane $\mathbb{C}$ whose closure $\overline{D}$ can be expressed as the union of finitely many closed triangles with pairwise disjoint interiors. We say that a function f :

Date: February 3, 2010.

1991 *Mathematics Subject Classification.* Primary 42B20; Secondary 30E20.

Key words and phrases. Martingales, martingale transforms, Beurling–Ahlfors transform.

This material is based upon work supported by the National Science Foundation under Grants No. 0457291 and 0758226.

$\mathbb{C} \to \mathbb{C}$ is a continuous piecewise affine function of compact support if, for some such domain D, f vanishes identically outside D, if f is continuous on $\mathbb{C}$, and if in each triangle T we used to represent D, we have $f(z) = az + b\overline{z} + c$ for some complex numbers a, b, c that depend on T.

We can partition $\overline{D}$ into finitely many pairwise disjoint sets such that each set contains the interior of exactly one of the triangles T we used, together with a suitably chosen part of its boundary ∂T. These sets generate a $\sigma-$algebra $\mathcal{F}$. If some of the sets generating $\mathcal{F}$ are replaced by their union, resulting in a smaller number of pairwise disjoint sets, then these sets generate a $\sigma-$algebra contained in $\mathcal{F}$. The smallest such $\sigma-$algebra is $\mathcal{F}_0 = \{\emptyset, \overline{D}\}$. There are many ways of constructing finite sequences $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n = \mathcal{F}$ consisting of $\sigma-$algebras $\mathcal{F}_j$ defined in this way, such that $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for $0 \le j \le n-1$. One can then use such sequences of $\sigma-$algebras to define martingales with respect to the Lebesgue area measure on $\overline{D}$ multiplied by a positive constant to make the total measure of $\overline{D}$ equal to 1. (We would set $\mathcal{F}_k = \mathcal{F}_n$ for all $k \ge n$.)

We propose the following conjecture. We will provide the definitions of the concepts occurring in this conjecture in Section 3.

Conjecture 1. *For each continuous piecewise affine complex-valued function f of compact support in the plane, there exist piecewise constant functions $c_1(z)$ and $c_2(z)$ (constant in the same triangles as where $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ are constant) with*

$$|c_1(z)| \equiv 1 \equiv |c_2(z)|$$

such that from the two functions

$$c_1(z)\frac{\partial f}{\partial \overline{z}} \qquad \text{and} \qquad c_2(z)\frac{\partial f}{\partial z}$$

one can construct martingales

$$X_n \qquad \text{and} \qquad Y_n$$

that are martingale transforms of each other, going from constants X_1, Y_1 of the same modulus and depending on f to

$$c_1(z)\frac{\partial f}{\partial \overline{z}} \qquad \text{and} \qquad c_2(z)\frac{\partial f}{\partial z}.$$

This means that for some large integer N, we have

$$X_n(z) = c_1(z)(\partial f/\partial \overline{z})$$

and

$$Y_n(z) = c_2(z)(\partial f/\partial z)$$

for all $n \geq N$. We have thus "rotated" $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ and then related the rotated derivatives by martingales. Hence the title of this paper.

Starting with constants X_1 and Y_1 of equal modulus is, of course, from a martingale point of view, equivalent to starting with $X_0 \equiv 0$, $Y_0 \equiv 0$; essentially we duplicate the situation under consideration (c.f. [11], p 11). We have mentioned these constants only since it is easier not to deal with such duplication.

The sigma-algebras $\mathcal{F}_n$ with respect to which X_n and Y_n are measurable, as well as the functions $c_1(z)$ and $c_2(z)$, would have to depend on f. The derivatives $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ are well defined in the interiors of the triangles, hence almost everywhere in D. It is convenient to define each such derivative to be constant on the entire set in the $\sigma-$algebra $\mathcal{F} = \mathcal{F}_N$ containing the interior of a particular triangle. It is irrelevant for purposes of integration how we define the functions c_1 and c_2 on the boundaries of the triangles, but for the sake of measurability with respect to the appropriate $\sigma-$algebras, we take each of c_1 and c_2 to be constant on each set in $\mathcal{F}$.

1.2. **The theorem.** In each case where Conjecture 1 is valid, it also implies a number of integral inequalities involving the derivatives of f. To discuss these, we follow Burkholder and define for a real number p with $p > 1$,

$$p^* = \max\{p, p/(p-1)\}.$$

So $p^* = p$ if $p \geq 2$. We further set

$$\alpha_p = p(1 - 1/p^*)^{p-1} \quad \text{for} \quad 1 < p < \infty, \tag{1}$$

and consider

$$u(z,w) = (|w| - (p^* - 1)|z|)(|z| + |w|)^{p-1} \tag{2}$$

for $z, w \in \mathbb{C}$. Burkholder proved that for all $z, w \in \mathbb{C}$,

$$|w|^p - (p^* - 1)^p |z|^p \leq \alpha_p \, u(z,w). \tag{3}$$

In fact, Burkholder denoted the right hand side of (3) by $u(z,w)$. Since the constant α_p is unimportant for most of our considerations, we find it more convenient to omit it from the definition of u and to include it separately in any one of the few formulas where it is really needed.

Burkholder also considered the function u_0 defined by

$$u_0(z,w) = |w|^2 - |z|^2 \qquad \text{for} \qquad |z| + |w| \leq 1 \tag{4}$$

and by

$$u_0(z,w) = 1 - 2|z| \qquad \text{for} \qquad |z| + |w| > 1. \tag{5}$$

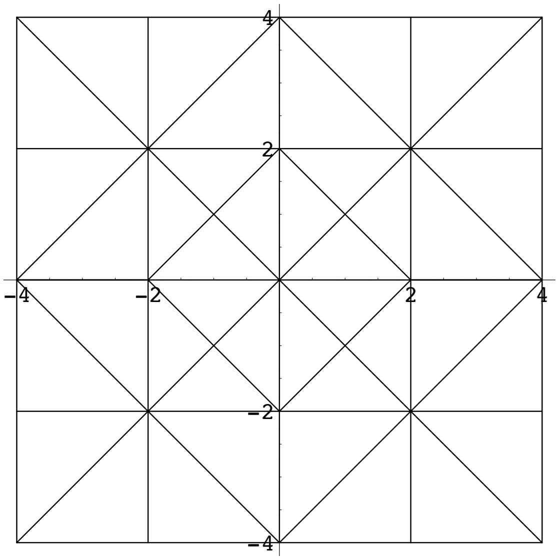

FIGURE 1. The case with 13 complex parameters. The big square with vertices ±4 ± 4i is the square Q.

A. Baernstein and S. Montgomery–Smith ([6], 1997) proved that with

$$u_1(z,w) = u_0(z,w) - (|w|^2 - |z|^2),$$

$$\beta_p = 2/(p(2-p)), \qquad \gamma_p = 2/(p(p-1)(p-2)),$$

we have

$$\int_0^\infty t^{p-1} u_0\left(\frac{z}{t}, \frac{w}{t}\right)\,dt = \beta_p u(z,w) \quad \text{if} \quad 1 < p < 2$$

and

$$\int_0^\infty t^{p-1} u_1\left(\frac{z}{t}, \frac{w}{t}\right)\,dt = \gamma_p u(w,z) \quad \text{if} \quad 2 < p < \infty,$$

so that to prove suitable inequalities for the functions u, it would suffice to consider u_0. However the definition of u_0 in two parts may make u_0 complicated to deal with.

The specific theorem that we prove in this paper is the following.

Theorem 1. *Consider the square Q in $\mathbb{C}$ triangulated as in Figure 1. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous piecewise affine function that vanishes outside Q and is affine in each triangle in Figure 1. Thus f is uniquely determined by its values at the vertices of the triangulation strictly inside Q, so that f depends on 13 independent complex parameters.*

Then Conjecture 1 is valid for any such function f. Furthermore, we have

$$\int_{\mathbb{C}} u\left(\frac{\partial f}{\partial \bar{z}}, \frac{\partial f}{\partial z}\right)\, dx\, dy \leq 0 \tag{6}$$

where u is given by (2), for any real $p \in (1, \infty)$, and also if we set $u = u_0$ where u_0 is given by (4) and (5). Consequently, by (1) and (3), we have for $1 < p < \infty$ that

$$||f_z||_p \leq (p^* - 1)||f_{\bar{z}}||_p\,. \tag{7}$$

Since the problem in invariant under translations, dilations, and rotations, the conclusion of Theorem 1 obviously applies to any square that has been triangulated in the same way as the square Q in Figure 1. In particular, it is irrelevant that the vertices of Q in Figure 1 are given as $\pm 4 \pm 4i$, since any square divided into triangles in the same way would do.

Clearly Theorem 1 proves a very special case of Conjecture 1.

Motivation for Conjecture 1 is obtained by considering the Beurling–Ahlfors transformation. This will be discussed in Section 2.

We will prove Theorem 1 in Section 7 after devoting several sections to discussing the concepts necessary for the proof.

In Section 8 we will suggest an algorithm for approaching the problem of proving Conjecture 1 in the general case.

2. The Beurling–Ahlfors transformation

The Beurling–Ahlfors transformation was introduced by Arne Beurling in a talk that he gave in Uppsala in November 1949. If $f : \mathbb{C} \to \mathbb{C}$ is in $C_0^\infty(\mathbb{C})$, the space of infinitely differentiable functions of compact support, we define its *Beurling–Ahlfors transform* Sf at $z \in \mathbb{C}$ to be the Cauchy principal value integral

$$(Sf)(z) = \lim_{\varepsilon \to 0} \frac{-1}{\pi} \int_{|\zeta - z| > \varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta \tag{8}$$

where $\zeta = \xi + i\eta$. Beurling [7] provided the definition and proved that S extends as an isometry to $L^2(\mathbb{C} \to \mathbb{C})$. The terms "Beurling transformation" ([3], [19]) and "two-dimensional Hilbert transformation" ([1], [23]) have also been used of S.

Calderon and Zygmund ([13], 1952) proved that many singular integral operators in $\mathbb{R}^n$, where $n \geq 2$, including S in $\mathbb{R}^2$, extend to bounded linear operators in L^p for $1 < p < \infty$. Further, their analysis shows that for $f \in L^p$, (8) still holds for almost every $z \in \mathbb{C}$.

The Beurling–Ahlfors transformation soon found use in the theory of quasiconformal mappings, first independently by Ahlfors ([1], 1955) and by Vekua ([25], 1955), then particularly in the seminal paper by Bojarski ([8], 1957).

2.1. **Known facts concerning the L^p−norm of S.**

We write

$$||f||_p^p = \int_{\mathbb{C}} |f(x+iy)|^p \, dx \, dy$$

and

$$||S||_p = \sup\{ \, ||Sf||_p \, : \, ||f||_p \le 1 \, \}.$$

Beurling proved that $||S||_2 = 1$ and indeed $||Sf||_2 = ||f||_2$ for all $f \in L^2(\mathbb{C})$.

It is easy to see that if $1 < p < \infty$ and $1/p + 1/q = 1$ then $||S||_p = ||S||_q$.

Lehto [22] proved in 1965 that if $p > 2$ then $||S||_p \ge p - 1$. So if $1 < p < 2$ then $||S||_p \ge 1/(p-1)$.

The first upper bound for $||S||_p$ for $p \ne 2$ arises from the work of Calderon and Zygmund ([13], 1952) who proved that $||S||_p = O(p)$ as $p \to \infty$.

Conjecture (T. Iwaniec [18], 1982) We have $||S||_p = p^* - 1$.

This amounts to suggesting that the lower bound obtained by Lehto is the sharp bound.

Motivated by this and by Burkholder's sharp inequalities for martingale transforms ([9], [10]) also involving the quantity $p^* - 1$, several authors have obtained upper bounds of the form $C(p^* - 1)$ for $||S||_p$, for absolute constants $C > 1$.

Bañuelos and Wang ([5], 1995) showed that $||S||_p \le 4(p^* - 1)$. Nazarov and Volberg ([26], 2003) obtained $||S||_p \le 2(p^* - 1)$. Dragičević and Volberg ([15], 2005) proved that

$$||S||_p \le \sqrt{2}(p-1) \left(\int_0^{2\pi} |\cos\theta|^p \, d\theta \right)^{-1/p} \qquad \text{for} \qquad 2 \le p < \infty.$$

Further, they were able to get the better asymptotic bound

$$\limsup_{p \to \infty} ||S||_p / p \le \sqrt{2}. \tag{9}$$

The best bound known at this time is due to Bañuelos and Janakiraman ([4], 2008). It says that $||S||_p \le \sqrt{2p(p-1)}$ for $2 \le p < \infty$, and in particular,

$$||S||_p \le 1.575 \, (p-1).$$

They obtained from this the same asymptotic bound (9) as Dragičević and Volberg.

There are generalizations of S to weighted L^p-spaces (e.g., [24]) and to $\mathbb{R}^n$ ([20], [21]). There is a lot of recent literature on the Beurling–Ahlfors transformation on weighted L^p-spaces, but we will not attempt to provide more references since that subject is outside the scope of this paper.

At one point, an important application of $||S||_p = p^* - 1$ would have been a proof of Gehring's [16] conjecture on the integrability of the derivatives of a quasiconformal map, but this was settled in another way by Astala [2]. Questions on the norm of S and its generalizations are nonetheless still of interest in quasiconformal theory, c.f. [24].

We do not discretize the integral transformation S, but consider the application of S to functions that can be described by finitely many parameters. In terms of the usual Wirtinger derivatives

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$

and

$$f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

the transformation S has the basic property that (c.f. [23], p. 160)

$$S\left(\frac{\partial f}{\partial \overline{z}}\right) = \frac{\partial f}{\partial z}. \tag{10}$$

Thus the question associated with the Iwaniec conjecture becomes the following. Find the smallest C_p such that for all $f : \mathbb{C} \to \mathbb{C}$ with L^p-derivatives, we have

$$\left|\left|\frac{\partial f}{\partial z}\right|\right|_p \leq C_p \left|\left|\frac{\partial f}{\partial \overline{z}}\right|\right|_p. \tag{11}$$

Indeed, the conjecture is that

$$\left|\left|\frac{\partial f}{\partial z}\right|\right|_p \leq (p^* - 1) \left|\left|\frac{\partial f}{\partial \overline{z}}\right|\right|_p. \tag{12}$$

2.2. **Approximation by continuous piecewise affine mappings of compact support.** It is routine to see that it would suffice to consider questions like (11) for functions in special classes instead of L^p, where $1 < p < \infty$. We could certainly assume that $f_{\overline{z}}$ in (11) is in the class $C_0^\infty(\mathbb{C})$, consisting of infinitely differentiable complex-valued functions of compact support defined in $\mathbb{C}$, as this class is dense in L^p. But we may go further and assume that $f : \mathbb{C} \to \mathbb{C}$ is continuous, piecewise affine, affine on certain triangles, and of compact support in the plane. A rigorous (and fairly routine) justification for such approximation can be obtained, for example, as follows.

We first note that if we are given a function h in $C_0^\infty(\mathbb{C})$, to which we wish to apply the operator S, then Sh need not be of compact support. To deal with this, note that the usual Cauchy transform integral operator $f(z) = Th(z) = \frac{-1}{\pi}\int_{\mathbb{C}} h(\zeta)/(\zeta - z)\, d\xi\, d\eta$, where $\zeta = \xi + i\eta$, gives a function f with $f_{\overline{z}} = h$ and such that this f is analytic outside the closed support of h. Of course, f need not be of compact support.

Now modify f far away (so that nothing is changed in a large compact set containing the support of h) to a function f_0 that has compact support. Then the $z-$ and $z-$bar-derivatives of f_0 have compact support and S takes one to the other, that is, $S((f_0)_{\overline{z}}) = (f_0)_z$. All that is needed now is to show that the $z-$bar derivatives of f (that is, h) and of f_0 are close to each other in the L^p-norm. Note that these $z-$bar derivatives will differ from each other only in that domain far away where we modify f.

If we choose a large R and take $f_0(z) = f(z)\eta(z)$ for $R < |z| \le R+1$ and $f_0(z) \equiv 0$ for $|z| > R+1$ (and $f_0 = f$ when $|z| \le R$), then we get f_0 with continuous derivatives if, e.g., $\eta(z) = g(|z|^2)$ where $g \ge 0$, $g(x) \to 1$ as $x \to R^2$, $g(x) \to 0$ as $x \to (R+1)^2$, and $g'(x) \to 0$ fast enough when x tends to R^2 and to $(R+1)^2$. Note that mostly $|g'(x)|$ can be of the order of magnitude $1/R$ for $R^2 < x < (R+1)^2$.

We can also choose g so that in this construction

$$\int_{R<|z|<R+1} |(f_0)_{\overline{z}}|^p$$

tends to zero as $R \to \infty$, which of course depends on having $p > 1$. This then shows that we can make $||h - (f_0)_{\overline{z}}||_p$ arbitrarily small by doing this for a sufficiently large R.

This allows us to limit our consideration to functions f_0 in $C_0^\infty(\mathbb{C})$ such that also Sf_0 has compact support.

Before proceeding, we make a side-remark. This argument did not discuss the integral $\int_{\mathbb{C}} h$ of h at all. We need not have $\int_{\mathbb{C}} h = 0$ but we will have $\int_{\mathbb{C}} (f_0)_{\overline{z}} = 0$. This shows another aspect of a limitation that we impose when we consider what the operator S does to $(f_0)_{\overline{z}}$ instead of $f_{\overline{z}}$. This phenomenon can be explained in general terms as follows.

If $1 < p < \infty$, let h be an L^p-function of compact support in the plane such that $\int_{\mathbb{C}} h = b \in \mathbb{C}$. For a large $A > 0$, define h_0 to be the constant $-b/A$ on a set (say, a disk, in any location, possibly disjoint from the support of h) of area A and zero elsewhere, and define

$h_1 = h + h_0$. Then $\int_{\mathbb{C}} h_1 = 0$. We have

$$||h_1 - h||_p^p = ||h_0||_p^p = A(| - b/A|^p) = |b|^p A^{1-p}$$

which tends to zero as A tends to infinity since $p > 1$. Obviously we could make h_0 smooth, if desired, and still get the same effect.

So by approximation in the L^p-norm one can assume that $\int_{\mathbb{C}} h = 0$.

Let us consider the function $h \in C_0^\infty(\mathbb{C})$ that we started with in this light. If we assume that $\int_{\mathbb{C}} h \neq 0$ and consider what $Th = f$ looks like far away, we see that the function $(f_0 - f)_{\overline{z}}$ does here exactly the kind of job that h_0 did above, and the set of area A (where now A is comparable to R) is the annulus with radii R and $R+1$. The difference is that $(f_0 - f)_{\overline{z}}$ is not constant in this annulus due to the other desired smoothness conditions, but it is close enough. The author is grateful to P. Janakiraman for the remark that the property of $\int_{\mathbb{C}} h$ becoming zero after applying the approximation may be worth pointing out explicitly.

Now we are in a position to consider (10) for $f \in C_0^\infty(\mathbb{C})$. We proceed by quoting the results of Ciarlet and Raviart [14]. Suppose that we use triangles T such that there is a constant $C_0 > 1$ for which

$$\operatorname{diam} T \leq C_0 R(T) \tag{13}$$

where $R(T)$ is the radius of the largest disk contained in T. Pick $f \in C_0^\infty(\mathbb{C})$ and cover the support of f by finitely many triangles T, each with diameter not exceeding $\delta > 0$, for which the above holds for a fixed C_0. Suppose that for all $z \in \mathbb{C}$, the modulus of each derivative of f of order 2 does not exceed C_1. Then, by Theorem 2 on page 184 of [14], there is a continuous piecewise affine mapping $f_0 : \mathbb{C} \to \mathbb{C}$ that vanishes outside this cover of the support of f such that for all $z \in \mathbb{C}$, we have

$$|f_0(z) - f(z)| \leq C_1 C_2 \delta^2$$

and

$$|Df_0(z) - Df(z)| \leq C_1 C_2 \delta$$

where $|Df|$ denotes the matrix norm of the derivative matrix Df of f. Here C_2 depends only on C_0. It follows that $||f_z - (f_0)_z||_p \leq A^{1/p} C_1 C_2 \delta$ and $||f_{\overline{z}} - (f_0)_{\overline{z}}||_p \leq A^{1/p} C_1 C_2 \delta$ where A is the area of the union of those triangles T needed to cover the support of f. Thus, even if we were to vary the triangulations that we use and decrease δ, we may choose an upper bound for A to be a fixed constant only depending on f but not on f_0. Since we may take δ to be arbitrarily small, we can approximate f_z and $f_{\overline{z}}$ as closely as we like in L^p-norm by the derivatives of continuous piecewise affine mappings of compact support. Note that we may take C_0 to be fixed if all triangles used have all their angles bounded away from zero by a fixed constant. For example, some regular standard triangulation may be used, such as one by equilateral triangles or by right-angled isosceles triangles.

2.3. **A reformulation of the problem of estimating norms.** Thus our situation is now as follows. We assume that $f : \mathbb{C} \to \mathbb{C}$ is a continuous piecewise affine mappings of compact support. Each of

$$\frac{\partial f}{\partial z} \qquad \text{and} \qquad \frac{\partial f}{\partial \overline{z}}$$

is piecewise constant, constant on each triangle used, and of compact support. It does not matter exactly what kinds of triangulations are used as long as we allow triangles of arbitrarily small diameter and all angles are bounded away from zero, this latter condition being what we need to satisfy (13) with a fixed constant C_0.

In order to prove (11) with $C_p = p^* - 1$ at least for certain functions f of this type, we follow the ideas that Burkholder used to prove a similar inequality in the setting of martingale transforms. Thus our goal becomes to prove for such a function f the stronger condition

$$\begin{aligned} \frac{1}{\alpha_p} \int_{\mathbb{C}} \left(\left| \frac{\partial f}{\partial z} \right|^p - (p^* - 1)^p \left| \frac{\partial f}{\partial \overline{z}} \right|^p \right) \, dx\, dy \\ \leq \int_{\mathbb{C}} u \left(\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial z} \right) \, dx\, dy \leq 0, \end{aligned} \tag{14}$$

where u is as in (2) (or $u = u_0$, where u_0 is given by (4) and (5)).

Suppose that the bounded region in the plane where f need not be identically 0 is the closure of the union of disjoint open triangles T_j, and suppose that we write

$$f(z) = a_j z + b_j \overline{z} + c_j \tag{15}$$

for all $z \in T_j$. For simplicity, suppose that we choose the triangles so that all the T_j have equal areas. Then we need to prove that

$$\sum_j u(b_j, a_j) \leq 0 \tag{16}$$

since the last integral in (14) is equal to a positive constant (the common area of the triangles T_j) times the left hand side of (16). If, instead, the triangle T_j has area A_j, then (14) becomes

$$\sum_j A_j u(b_j, a_j) \leq 0. \tag{17}$$

The question is now how to get upper bounds for the left-hand side of (16). The strategy of Burkholder, to prove similar inequalities for pairs of martingales that are martingale transforms of each other, was to find a few terms at a time and show that their sum does not exceed a positive constant times a single $u-$term, e.g., $\sum_{j=1}^{N} u(b_j, a_j) \leq Nu(b, a)$ for some suitable a and b. For such an inequality to be true, there needs to be a relationship between the a_j and b_j, such as something corresponding to (19) and (23) below.

The above is intended to explain how a certain formal similarity between what Burkholder has proved for martingales and what has been conjectured for the Beurling–Ahlfors transformation, can lead one to ask whether some modification of Burkholder's ideas might yield results for the operator S. In particular, the question arises as to whether one can find pairs of martingales that are martingale transforms of each other, that are related to the $z-$ and $z-$bar-derivatives of a function. So far we have mentioned Burkholder's work but not defined more carefully what it entails. In the next section we will pursue a more careful analysis of how one might proceed and how, in inequalities of the form (17), one might combine terms in the way described.

3. Definitions

3.1. **Burkholder's auxiliary functions.** Starting in the early 1980's, D.L. Burkholder obtained the sharp solutions to several extremal problems involving martingale transforms, and developed various methods of proof to study such situations, in a series of papers, the principal ones for our purposes being [9], [10]. For a survey, see [12].

We recall the definition of a martingale. Usually martingales are defined on probability spaces. For our purposes, it is convenient to consider finite measure spaces. There could be two ways of proceeding. We could multiply any finite measure by the reciprocal of the total measure, getting a probability measure. Or we could observe that practically any results proved for martingales, and certainly all those that we will use, are valid even if we were to define all concepts (suitably modified when necessary) for finite measure spaces instead of probability spaces. Since either approach would work for us, we do not need to take sides on which to choose. It should be clear in each instance that either alternative would be acceptable. With these caveats, we will now recall some definitions in their customary setting.

3.2. **Definitions.** Let Ω be a probability space with a $\sigma-$algebra $\mathcal{F}$ of measurable sets for the measure P.

A discrete-time complex–valued **martingale** on Ω is a sequence of complex-valued functions g_n in $L^1(\Omega)$ such that g_n is measurable with respect to a $\sigma-$algebra $\mathcal{F}_n$, where $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$, such that for each

$A \in \mathcal{F}_n$, we have

$$\int_A (g_{n+1} - g_n)\, dP = 0.$$

In a **finite martingale**, each $\mathcal{F}_n$ contains only finitely many sets and each g_n takes only finitely many values, and $g_n, \mathcal{F}_n$ are the same for all sufficiently large n. In particular, if, for example, some minimal non-empty set $A \in \mathcal{F}_n$ is the disjoint union of only two minimal non-empty sets $A_1, A_2 \in \mathcal{F}_{n+1}$ of equal measure, we have, for some $\alpha \in \mathbb{C}$, $g_{n+1} = g_n + \alpha$ on A_1, and $g_{n+1} = g_n - \alpha$ on A_2, while g_n is constant on A.

Let X_n and Y_n be two martingales on Ω with respect to the same sequence $\mathcal{F}_n$ of $\sigma-$algebras. We say that $Y = \{Y_n\}$ is a **martingale transform** of X if

$$|Y_n - Y_{n-1}| \leq |X_n - X_{n-1}|$$

$P-$almost everywhere, for each $n \geq 1$. Thus X and Y are martingale transforms of each other if

$$|Y_n - Y_{n-1}| = |X_n - X_{n-1}|$$

$P-$almost everywhere, for each $n \geq 1$. In particular, if $X_0 \equiv 0$, $Y_0 \equiv 0$, then $|X_1| = |Y_1|$ $P-$almost everywhere. This latter property will be sufficient for us later on, instead of $X_0 \equiv 0$, $Y_0 \equiv 0$.

Burkholder [9] proved that if $X_0 \equiv 0$, $Y_0 \equiv 0$, and if Y is a martingale transform of X, and if

$$||X||_p = \sup_n ||X_n||_p < \infty,$$

for some $p \in (1, \infty)$, then

$$||Y||_p \leq (p^* - 1)||X||_p, \tag{18}$$

with equality if $p = 2$ and if X is also a martingale transform of Y. He also obtained more complicated inequalities when X_0 and Y_0 are constants other than zero.

It is essential for us to discuss the method of proof introduced by Burkholder. It is based on considering the auxiliary functions u given by (2).

3.3. Properties of Burkholder's functions.

Burkholder showed that u (and u_0) has the following **concavity property**. Suppose that $z, w, h, k \in \mathbb{C}$ with $z \neq 0$, $w \neq 0$ and $|k| \leq |h|$. Then

$$u(z + h, w + k) \leq u(z, w) + \mathrm{Re}\,\{u_z(z, w)\overline{h} + u_w(z, w)\overline{k}\}.$$

Here we have used the complex partial derivatives of u given by

$$u_z = (z/|z|)[(p-p^*)|w| - p(p^*-1)|z|](|z|+|w|)^{p-2},$$

$$u_w = (w/|w|)[p|w| + (p+p^*-pp^*)|z|](|z|+|w|)^{p-2}.$$

Note that since u takes only real values, we have $u_{\overline{z}} = \overline{u_z}$ and $u_{\overline{w}} = \overline{u_w}$.

This concavity property has the following important consequence. Suppose that $A_i > 0$ where the index i runs over finitely many values, and that the following **martingale conditions** are satisfied:

$$\sum_i A_i h_i = \sum_i A_i k_i = 0. \tag{19}$$

Suppose further that

$$|k_i| \le |h_i| \qquad \text{for all } i. \tag{20}$$

Then

$$\sum_i A_i u(z+h_i, w+k_i) \le \left(\sum_i A_i\right) u(z,w). \tag{21}$$

This is the property that Burkholder used to prove that

$$||Y||_p \le (p^*-1)||X||_p$$

for martingales X and Y (with values, more generally, in a Hilbert space) and for $1 < p < \infty$ if $X_0 \equiv 0$, $Y_0 \equiv 0$, and Y is a martingale transform of X.

We note that

$$u(z,w) = u(|z|,|w|) = u(|z|,|z|) \le 0 \quad \text{if} \quad |z| = |w|. \tag{22}$$

In spite of its simplicity and triviality, this is an essential property. We expect to have a symmetry between the two functions we will consider, so we will only use the assumption that

$$|k| = |h|. \tag{23}$$

We mention in passing that we can replace the martingale conditions (19) by

$$\sum_i A_i \mathrm{Re}\,(\overline{z} h_i) = \sum_i A_i \mathrm{Re}\,(\overline{w} k_i) = 0 \tag{24}$$

which together with $|k_i| \le |h_i|$ for all i yield (21). We call the conditions (24) weakened martingale conditions. Already Burkholder ([9], (13.1) and Theorems 13.1, 13.2, p. 695) noticed that such weaker conditions are sufficient for his results, and he called processes that satisfy conditions that in our discrete setting substantially correspond to the conditions (24), very weak martingales.

4. Search for martingales

Suppose that on two adjacent triangles, our continuous piecewise mapping is defined by the expressions $f(z) = az+b\overline{z}$ and $g(z) = cz+d\overline{z}$. Here we ignore translations as we may, by assuming that the origin is a common vertex of the two triangles and that the origin is mapped to itself. These assumptions do not affect the derivatives of the function. Suppose that f and g agree on a line segment joining the common vertex at 0 to another common vertex $\zeta \neq 0$. (In fact, it suffices for 0 and ζ to be any distinct points on the common part of the boundaries of the triangles.) Then $f(\zeta) = g(\zeta)$, so that

$$a\zeta + b\overline{\zeta} = c\zeta + d\overline{\zeta}$$

and hence

$$(a-c)\zeta = -(b-d)\overline{\zeta}.$$

Thus

$$a-c = \alpha(b-d) \qquad \text{where} \qquad \alpha = -\overline{\zeta}/\zeta$$

so that $|\alpha| = 1$, that is,

$$|a-c| = |b-d|. \tag{25}$$

This relationship that refers to moduli of the changes we have in the $z-$ and $\overline{z}-$derivatives of the function, is of the same type as (23). Thus this is the kind of relation that we are looking for. Having an equality is needed so as to get two martingales that are transforms of each other, rather than only one being a transform of the other.

However, once one starts combining terms arising from different triangles, even if one were to begin with neighboring triangles, one will already at the next stage get to a situation that is no longer equally simple. Then it is typically not possible to identify pairs of affine mappings for which the moduli of the differences of the $z-$ and $\overline{z}-$derivatives are equal. Thus something more needs to be done. This fact is reminiscent of the following. In various earlier treatments of the problem of the norm $||S||_p$, the given function f for which Sf is to be estimated, was used to define pairs of martingales, after extending f from the complex plane to the upper half space using a fixed kernel such as the Poisson kernel or the heat kernel. Also, this association was not symmetric: f was extended to a half-space, a martingale was defined, another martingale was defined that is a martingale transform of the first one but not vice versa, and the second martingale was used to get a connection to the Beurling–Ahlfors transform Sf. The processes used were not reversible. This shows that it is not easy to come up with martingales associated directly with the $z-$ and $\overline{z}-$derivatives of a function. Those derivatives themselves certainly do not define martingales as such, even if we were to look at a process by which a function (say a continuous piecewise affine mapping) is built up from similar functions, making the triangulation finer at each step.

4.1. **Effect of rotations.** Thus, since there turn out to be rather few situations where equations such as (25) are satisfied, we need to do something further to get closer to **repeatedly** having circumstances under which (19) and (21) are satisfied. Since $u(z, w)$ depends only on $|z|$ and $|w|$, we may independently replace z and w by αz and βw, where $|\alpha| = |\beta| = 1$. This is a *key observation* that in spite of its simplicity was not used in earlier approaches to this problem.

4.2. **Matching lengths.** If it is not initially true that $|a-c| = |b-d|$, we may consider the effect of replacing

$$(a, b) \qquad \text{by} \qquad (\alpha a, \beta b), \qquad \text{where } |\alpha| = |\beta| = 1.$$

Now $|\alpha a - c| = |\beta b - d|$ means that there is a complex number γ with $|\gamma| = 1$ such that

$$\alpha a - c = \gamma(\beta b - d). \tag{26}$$

This can be written in the more useful form

$$aA + b\overline{A} = E(cB + d\overline{B}) \tag{27}$$

where

$$A^2 = -\alpha\overline{\beta\gamma}, \; B^2 = -\overline{\gamma}, \; E = -\overline{\beta\gamma AB},$$

so that

$$|A| = |B| = |E| = 1.$$

With the notation $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$, (27) states that

$$|f(A)| = |g(B)|, \qquad \text{where} \qquad |A| = |B| = 1.$$

The conclusion is that for us to be able to compare the derivatives of f and g after using appropriate rotations, then $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$ must map *line segments of equal length* in **some** directions to *line segments of equal length*. When this occurs, let us say that f and g have **matching lengths**, or that f and g are **comparable affine maps**.

This raises the question of how we may determine whether given affine f and g actually have matching lengths. Of course, one could use the definition and perform some computations, but that might be too clumsy in many cases, so we look for methods for seeing that this is the case.

4.3. **Intervals of lengths.** To be able to detect whether there are matching lengths, we need to characterize the possible lengths that can occur for any particular affine mapping.

For every affine map F given by $F(z) = az+b\overline{z}+c$, there are numbers m, M with

$$0 \le m \le M < \infty$$

such that the interval $[m, M]$ is precisely the set of lengths of the images of segments of unit length under the map normalized to $F(0) = 0$. More precisely, for any affine mapping F, the image of the unit circle under $F - F(0)$ is an ellipse (possibly a degenerate ellipse) with semi-axes m and M, say. Here we have $m = 0 < M$ when $F(\mathbb{C})$ is a line, and $m = M = 0$ when $F(\mathbb{C})$ is a point. Indeed, $M = |a| + |b|$ and $m = ||a| - |b||$. Thus $m = 0$ if, and only if, $|a| = |b|$, that is, $|\partial F/\partial z| = |\partial F/\partial \overline{z}|$.

Two affine mappings with intervals $[m_1, M_1]$ and $[m_2, M_2]$ are obviously comparable if, and only if,

$$[m_1, M_1] \cap [m_2, M_2] \neq \emptyset,$$

that is,

$$\max\{m_1, m_2\} \le \min\{M_1, M_2\},$$

and in that case the common length that we would make use of would be one of the lengths in $[m_1, M_1] \cap [m_2, M_2] \neq \emptyset$. In the same way we may settle the question of whether a single affine mapping is comparable to each of a larger number of affine mappings. This question arises when trying to apply (21) with more than two terms on the left hand side.

Given finitely many affine maps F_j with intervals $[m_j, M_j]$ for $1 \le j \le n$, it is easy to find affine maps F with interval $[m, M]$ that are comparable to every given map F_j. For example, choose m and M so that

$$0 \le m \le \min\{m_j : 1 \le j \le n\}, \qquad M \ge \max\{M_j : 1 \le j \le n\}.$$

However, usually the martingale conditions (19) are not satisfied for these comparisons. Indeed, some thought shows that much greater care is required to obtain the necessary cancellations.

5. DIRECTIONS LABELLED AS “HORIZONTAL” AND “VERTICAL”

A calculation shows that since we may use rotations, we may designate, for each triangle independently, two orthogonal directions given by c and ic, where $|c| = 1$, as “horizontal” and “vertical” directions, and denote their images under the affine map F in the triangle by H and V, so $H = F(c) - F(0)$, $V = F(ic) - F(0)$.

The partial derivatives of F can be expressed in terms of H and V. For simplicity, suppose that $F(0)=0$. Write $F(z)=az+b\overline{z}$. Then

$$H = ac + b\overline{c}, \qquad V = i(ac - b\overline{c}).$$

Thus

$$a = \frac{1}{2c}\left(H - iV\right), \qquad b = \frac{1}{2\overline{c}}\left(H + iV\right), \tag{28}$$

so that

$$\begin{aligned} u\left(\frac{\partial F}{\partial \overline{z}}, \frac{\partial F}{\partial z}\right) &= u\left(\frac{1}{2\overline{c}}\left(H+iV\right), \frac{1}{2c}\left(H-iV\right)\right) \\ &= u\left(\frac{1}{2}\left(H+iV\right), \frac{1}{2}\left(H-iV\right)\right) \end{aligned} \tag{29}$$

If we have two such triangles, then it turns out that we may combine their $u-$terms in (16) into a single term if $H_1 = H_2$ or if $V_1 = V_2$. (These are special cases of the most general sufficient condition.)

To see this, let us first verify that we have matching lengths in those cases. Suppose that the horizontal directions used are c_1 and c_2. Thus $|c_1| = |c_2| = 1$, and if the two mappings are written as $a_1 z + b_1\overline{z}$ and $a_2 z + b_2 \overline{z}$, then $H_1 = H_2$ means that

$$a_1 c_1 + b_1\overline{c_1} = a_2 c_2 + b_2 \overline{c_2}, \tag{30}$$

and in particular $|a_1c_1 + b_1\overline{c_1}| = |a_2c_2 + b_2\overline{c_2}|$, while $V_1 = V_2$ means that

$$a_1(ic_1) + b_1\overline{ic_1} = a_2(ic_2) + b_2\overline{ic_2}, \tag{31}$$

and hence $|a_1(ic_1) + b_1\overline{ic_1}| = |a_2(ic_2) + b_2\overline{ic_2}|$. In both cases it is obvious that we have matching lengths in a particularly simple way.

5.1. **Rewriting (21) when $H_1 = H_2$ or $V_1 = V_2$.**

Let us see what (21) implies when $H_1 = H_2$ and notation is as above. By (21) and (30) we have

$$\begin{aligned} u(a_1,b_1) + u(a_2,b_2) &= u(c_1a_1, \overline{c_1}b_1) + u(c_2a_2, \overline{c_2}b_2) \\ &\leq 2u\left(\frac{1}{2}\left(c_1a_1 + c_2a_2\right), \frac{1}{2}\left(\overline{c_1}b_1 + \overline{c_2}b_2\right)\right) \end{aligned} \tag{32}$$

since

$$\left(\frac{1}{2}\left(c_1a_1+c_2a_2\right) - c_1a_1\right) + \left(\frac{1}{2}\left(c_1a_1+c_2a_2\right) - c_2a_2\right) = 0,$$

$$\left(\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right) - \overline{c_1}b_1\right) + \left(\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right) - \overline{c_2}b_2\right) = 0,$$

$$\left|\frac{1}{2}\left(c_1a_1+c_2a_2\right) - c_1a_1\right| = \left|\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right) - \overline{c_1}b_1\right|,$$

and

$$\left|\frac{1}{2}\left(c_1a_1+c_2a_2\right)-c_2a_2\right|=\left|\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right)-\overline{c_2}b_2\right|.$$

Similarly, if $V_1=V_2$, we obtain by (21) and (31) that

$$\begin{aligned}(33)\qquad & u(a_1,b_1)+u(a_2,b_2)=u(c_1a_1,-\overline{c_1}b_1)+u(c_2a_2,-\overline{c_2}b_2)\\ \leq\ & 2u\left(\frac{1}{2}\left(c_1a_1+c_2a_2\right),-\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right)\right)\\ =\ & 2u\left(\frac{1}{2}\left(c_1a_1+c_2a_2\right),\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right)\right).\end{aligned}$$

5.2. **The function $\tilde{u}$.** When discussing H and V, it turns out to be convenient to introduce new notation and write

$$(34)\qquad \tilde{u}(a,b)=u\left(\frac{1}{2}(a+ib),\frac{1}{2}(a-ib)\right).$$

Some useful properties of $\tilde{u}$ for any $a,b\in\mathbb{C}$ are

$$(35)\qquad \tilde{u}(a,0)=u(a/2,a/2)\leq 0,\quad \tilde{u}(0,b)=u(ib/2,-ib/2)\leq 0,$$

and

$$(36)\qquad \tilde{u}(a,b)=\tilde{u}(-a,-b)=\tilde{u}(ca,cb)\quad\text{whenever}\quad |c|=1.$$

Furthermore, we have

$$(37)\qquad \tilde{u}(a,b)=\tilde{u}(b,-a)=\tilde{u}(-b,a)$$

since

$$\begin{aligned}\tilde{u}(b,-a)&=u\left(\frac{1}{2}(b-ia),\frac{1}{2}(b+ia)\right)=u\left(\frac{-i}{2}(a+ib),\frac{i}{2}(a-ib)\right)\\ &=u\left(\frac{1}{2}(a+ib),\frac{1}{2}(a-ib)\right)=\tilde{u}(a,b),\end{aligned}$$

which means that the roles of H and V can be interchanged provided that one of them is also multiplied by -1. This is understandable since it only means that if, earlier, H corresponded to the direction c, then we take, instead, H to correspond to the direction ic or to $-ic$.

In view of (28), using the notation immediately preceding (28), we have

$$(38)\qquad \tilde{u}(H,V)=u\left(\frac{1}{2}(H+iV),\frac{1}{2}(H-iV)\right)=u(\overline{c}b,ca)=u(b,a).$$

The counterpart of the fact that $u(z,w)$ depends only on the moduli $|z|$ and $|w|$ is

$$(39)\qquad \tilde{u}(H,\lambda H)\leq 0\quad\text{whenever}\quad \lambda\in\mathbb{R}\ \text{ and }\ H\in\mathbb{C}.$$

Namely,

$$\tilde{u}(H,\lambda H)=u\left(\frac{1}{2}\left(1+i\lambda\right)H,\frac{1}{2}\left(1-i\lambda\right)H\right)\leq 0$$

by (22) since

$$|(1+i\lambda)\,H|=|(1-i\lambda)\,H|$$

when λ is real.

Suppose that $H_1=H_2=H$, say, and note that then

$$V_1=i(a_1c_1-b_1\overline{c_1}),\quad V_2=i(a_2c_2-b_2\overline{c_2}).$$

Then using the notation (34), and taking into account (28) and (36), we may write (32) as

$$\begin{aligned}&\tilde{u}(H,V_1)+\tilde{u}(H,V_2)\\ =\;& u\left(\frac{1}{2}(H+iV_1),\frac{1}{2}(H-iV_1)\right)+u\left(\frac{1}{2}(H+iV_2),\frac{1}{2}(H-iV_2)\right)\\ =\;& 2u\left(\frac{1}{2}\left(\overline{c_1}b_1+\overline{c_2}b_2\right),\frac{1}{2}\left(c_1a_1+c_2a_2\right)\right)\end{aligned}$$

while

$$\begin{aligned}&\tilde{u}\left(H,\frac{V_1+V_2}{2}\right)=u\left(\frac{1}{2}\left(H+i\frac{V_1+V_2}{2}\right),\frac{1}{2}\left(H-i\frac{V_1+V_2}{2}\right)\right)\\ =\;& u\left(\frac{1}{2}(b_1\overline{c_1}+b_2\overline{c_2}),\frac{1}{2}(a_1c_1+a_2c_2)\right)\end{aligned}$$

which imply that

$$\tilde{u}(H,V_1)+\tilde{u}(H,V_2)\leq 2\tilde{u}\left(H,\frac{V_1+V_2}{2}\right). \tag{40}$$

Suppose that $V_1=V_2=V$, say, and note that then

$$H_1=a_1c_1+b_1\overline{c_1},\quad H_2=a_2c_2+b_2\overline{c_2}.$$

Then, using the notation (34), we may similarly write (33) as

$$\tilde{u}(H_1,V)+\tilde{u}(H_2,V)\leq 2\tilde{u}\left(\frac{H_1+H_2}{2},V\right). \tag{41}$$

If we have a situation where $V_1=V_2$, we could always make use of (37) first and thereby reduce the situation to the case where $H_1=H_2$. Thus it is somewhat redundant to discuss the case $V_1=V_2$. However, there may be practical cases where it is more convenient to proceed instead of using (37) first, so it may be worth having also (41) available.

Of course, both (40), and (41) can be verified directly by appealing to (19), (20), (21), and (34). Above, we wanted to emphasize the view of considering mappings rather than abstract numbers.

More generally, it follows by a calculation from (19), (20), (21), and (34) that

$$\sum_{j=1}^{\ell} \tilde{u}(H, V_j) \le \ell \tilde{u}\left(H, \frac{1}{\ell}\sum_{j=1}^{\ell} V_j\right) \tag{42}$$

and

$$\sum_{j=1}^{\ell} \tilde{u}(H_j, V) \le \ell \tilde{u}\left(\frac{1}{\ell}\sum_{j=1}^{\ell} H_j, V\right) \tag{43}$$

and this can be generalized to some extent by using unequal weights.

5.3. **Propagation of multipliers.** Suppose that we manage to find multipliers of the type α and β of modulus 1 and combine two or more $u-$terms in (16) into a single term. We may be able to do this for several pairs or larger groups of initial $u-$terms. After that, we may use further multipliers to combine terms, some of which may still be among the initial terms and some of which may be new terms obtained by combinations. If we are able to continue with this process until we find an upper bound for the left hand side of (16) that is a positive multiple of a term of the form $u(z, w)$, where $|z| = |w|$ (possibly even with $z = w$), which guarantees by (22) that this last single term $u(z, w) \le 0$, then we will have proved (16) and hence (12) in that particular case.

We now see that in retrospect, there is a propagation of multipliers. We could have initially multiplied all the a_j and b_j by suitable numbers of modulus 1, so that after that, no further multipliers would have been required, and we would have used only the martingale properties of the function u. In view of this possibility of expanding our chances of finding matching lengths if we multipy a and b by constants of modulus 1, we are lead to Conjecture 1. Thus we have now explained how Conjecture 1 arises from an attempt to prove (12) using martingale methods and observing that rotations can be used since a norm depends only on the modulus.

6. Outline of the proof of Theorem 1

6.1. **The set-up and outline of the proof.** Consider a big square Q as in Figure 1, with vertices at $\pm 4 \pm 4i$, with 13 vertices inside, at $j + ik$ for $j, k \in \{0, 2, -2\}$ and at $\pm 1 \pm i$. The square is divided into 16 triangles of area 1, such as the triangle with vertices at $2i$, $2 + 2i$, and $1 + i$, and 24 triangles of area 2, such as the triangle with vertices at $2 + 4i$, $4 + 4i$, and $2 + 2i$. We assume that f is a continuous complex-valued piecewise affine map in $\mathbb{C}$, with $f \equiv 0$ on ∂Q and outside Q, and taking any preassigned complex values at each of the 13 inner vertices. This determines f uniquely, and hence f depends on 13 arbitrary independent complex parameters. This is because given any non-degenerate triangle with vertices A, B, C and any three not necessarily distinct complex numbers α, β, γ, there is a unique affine mapping of the form $g(z) = az + b\bar{z} + c$ with $g(A) = \alpha$, $g(B) = \beta$, and $g(C) = \gamma$.

Each of the functions $\frac{\partial f}{\partial \bar{z}}$ and $\frac{\partial f}{\partial z}$ is piecewise constant (constant on each of the 40 triangles). In each of the 16 triangles with 2 vertices on the boundary, we have two vertices mapped to the same point (the origin) by f so that f maps the whole triangle into a line. This implies that in this triangle $\left|\frac{\partial f}{\partial \bar{z}}\right| = \left|\frac{\partial f}{\partial z}\right|$, and hence $u\left(\frac{\partial f}{\partial \bar{z}}, \frac{\partial f}{\partial z}\right) \leq 0$. While these terms might be helpful in proving (16), it turns out that their help is not needed, and that indeed the sum of the remaining 24 $u-$terms is also ≤ 0.

The combining of the remaining 24 $u-$terms starts, e.g., with the two triangles with vertices at $-4i, -2i, -2-2i$ and at $-4, -2, -2-2i$. The way to get started with finding matching lengths is the observation that since $f = 0$ on the boundary, we have $f(-2-2i) - f(-4) = f(-2-2i) - f(-4i) (= f(-2-2i))$. Thus we designate the directions from -4 to $-2-2i$ in one triangle, and from $-4i$ to $-2-2i$ in the other triangle, as the horizontal directions (these two line segments have the same length, so we need not worry about the fact that it is not unit length right now as that can be adjusted) and we obtain what we called matching lengths. The affine map resulting from this combination turns out to be comparable to a certain other affine map obtained by further combinations (indeed, by first using the triangles with vertices at $-2, 0, -1-i$ and at $0, -1-i, -2i$, then those with vertices at $-2, -2-2i, -1-i$ and at $-1-i, -2i, -2-2i$, and then combining those two results together). One can continue all the way to prove (16) in this case. This has only been an outline, and we now proceed to discuss the details step by step. Even though it will take some space to write down all the details below, the principles are quite simple, and after looking at some cases like this it becomes much easier to look at a picture geometrically to see what is going on instead of performing computations.

7. A detailed proof of Theorem 1

As mentioned in the outline, it suffices to consider only the terms arising from those 24 triangles that have at most one vertex on ∂Q since each individual term $u\left(\frac{\partial f}{\partial \bar{z}}, \frac{\partial f}{\partial z}\right) \leq 0$ for each triangle that has two vertices on ∂Q, hence mapped to the origin by f. (Of course, one cannot exclude the possibility that in some larger case, the help obtained from these trivial terms might yet be needed.)

Consider the two triangles with vertices at $-4i, -2i, -2-2i$ (triangle T_1, say) and at $-4, -2, -2-2i$ (triangle T_2). Since $f = 0$ on the boundary of Q, we have $f(-4) = f(-4i) = 0$, so that $f(-2-2i) - f(-4) = f(-2-2i) - f(-4i)$. In T_1, we designate the direction from -4 to $-2-2i$ as the horizontal direction, corresponding to the choice $c = (1-i)/\sqrt{2}$ for the complex number c of modulus 1 as discussed when we defined horizontal and vertical directions. Thus the vertical direction is $ic = (1+i)/\sqrt{2}$. For T_1, we thus have

$$H = H_1 = f(-2-2i)/(2\sqrt{2}) \text{ and } V = V_1 = (f(-2) - f(-3-i))/\sqrt{2}.$$

In T_2, we designate the direction from $-4i$ to $-2-2i$ as the horizontal direction. Here $c = (-1+i)\sqrt{2}$ so that $ic = -(1+i)\sqrt{2}$. For T_2, we thus have

$$H = H_2 = f(-2-2i)/(2\sqrt{2}) \text{ and } V = V_2 = (f(-1-3i) - f(-2i))/\sqrt{2}.$$

Note that since we are dealing with affine mappings and since $f(-4) = f(-4i)(= 0)$, we have

$$f(-3-i) = f(-1-3i).$$

In other words, if two affine mappings map the end points of two line segments to the same two points, respectively, then they also map the midpoints of these line segments to the same point. When performing combinations, it is often necessary to make use of this fact.

Since $H_1 = H_2$, we obtain from (40) that

$$\begin{aligned} \tilde{u}(H_1, V_1) + \tilde{u}(H_2, V_2) &\le 2\tilde{u}\left(H_1, \frac{V_1 + V_2}{2}\right) \\ &= 2\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2) - f(-2i)}{2\sqrt{2}}\right). \end{aligned} \tag{44}$$

The $u-$term of each triangle must be taken into account with a weight proportional to the area of the triangle. Each triangle inside Q that we start with has area 1 or 2, so it seems convenient to use the area itself as the weight (indeed a proportionality factor different from 1, whose choice is always a matter of taste anyway, makes sense only if the areas themselves are complicated enough to justify it). Since each of T_1 and T_2 has area equal to 2, the term that we need to use in the sequel is twice what we just obtained, that is,

$$4\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2) - f(-2i)}{2\sqrt{2}}\right). \tag{45}$$

7.1. **Combination of 4 triangles that form a square.** Let us then discuss the square with vertices at $0, -2, -2-2i, -2i$. The computation that we now perform illustrates how one can always combine the $u-$terms of four triangles that form a square in this way, with one vertex at the center of the square. Let us denote by T_3, T_4, T_5, T_6 the triangles with vertices at $(-2, -2-2i, -1-i)$, $(0, -2, -1-i)$, $(0, -2i, -1-i)$, and $(-2i, -1-i, -2-2i)$, respectively.

We may now take any two adjacent triangles (adjacent meaning that they share an edge) out of these four triangles and combine their $u-$terms together, then do the same for the remaining two triangles (which are necessarily also adjacent), and then combine the results. The final result is the same, no matter how we choose the pairs of triangles.

Let us start, for example, by combining the $u-$terms of the triangles T_3 and T_4. We choose horizontal and vertical directions so that

$$H_3 = (f(-2) - f(-1-i))/\sqrt{2}, \quad V_3 = (f(-2-2i) - f(-1-i))/\sqrt{2},$$

and

$$H_4 = H_3, \quad V_4 = (f(-1-i) - f(0))/\sqrt{2}.$$

Since $H_3 = H_4$, (40) yields

$$\begin{aligned} \tilde{u}(H_3, V_3) + \tilde{u}(H_4, V_4) &\leq 2\tilde{u}\left(H_3, \frac{V_3+V_4}{2}\right) \\ &= 2\tilde{u}\left(\frac{f(-2)-f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right). \end{aligned} \tag{46}$$

Let us then combine the $u-$terms of the triangles T_5 and T_6. We choose horizontal and vertical directions so that

$$H_5 = (f(-2i) - f(-1-i))/\sqrt{2}, \quad V_5 = (f(0) - f(-1-i))/\sqrt{2},$$

and

$$H_6 = H_5, \quad V_6 = (f(-1-i) - f(-2-2i))/\sqrt{2}.$$

Since $H_5 = H_6$, (40) and (36) yield

$$\begin{aligned} \tilde{u}(H_5, V_5) + \tilde{u}(H_6, V_6) &\leq 2\tilde{u}\left(H_5, \frac{V_5+V_6}{2}\right) \\ &= 2\tilde{u}\left(\frac{f(-2i)-f(-1-i)}{\sqrt{2}}, -\frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right) \\ &= 2\tilde{u}\left(-\frac{f(-2i)-f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right). \end{aligned} \tag{47}$$

The last terms in (46) and (47) have identical $V-$terms (that is, second variables for the function $\tilde{u}$). Hence by (41) we obtain

$$\begin{aligned} &2\tilde{u}\left(\frac{f(-2)-f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right) \\ +\ &2\tilde{u}\left(-\frac{f(-2i)-f(-1-i)}{\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right) \\ \leq\ &4\tilde{u}\left(\frac{f(-2)-f(-2i)}{2\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right) \end{aligned} \tag{48}$$

Note that the quantities $f(-2)-f(-2i)$ and $f(-2-2i)-f(0)$ appearing here correspond to the difference in the values of the function f across the two diagonals of the square with vertices $0, -2, -2-2i, -2i$, while the value of f at the center $-1-i$ of the square has disappeared. This also completes our discussion of what can be done in any square like this.

7.2. **Further combinations in the "southwest" quadrant of Q.** Let us compare the term in (45) and the last term in (48). By (37) and (41), we have

$$\begin{aligned} &4\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) \\ &+4\tilde{u}\left(\frac{f(-2)-f(-2i)}{2\sqrt{2}}, \frac{f(-2-2i)-f(0)}{2\sqrt{2}}\right) \\ &= 4\tilde{u}\left(\frac{f(-2-2i)}{2\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) \\ &+4\tilde{u}\left(-\frac{f(-2-2i)-f(0)}{2\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) \\ &\leq 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right). \end{aligned} \tag{49}$$

Now we have combined the $u-$terms from all the 6 triangles that we are going to use at all, from the "southwest" quadrant of Q: the square with vertices at $0, -4, -4-4i, -4i$. Of course, the triangles that we have used form the larger triangle with vertices at $0, -4, -4i$, but this is a mere coincidence. If we had a larger example and were to leave out all small triangles with at least two vertices on the boundary of the big square, the remaining triangles would form a more complicated figure. The term involving $f(0)$ should really be considered as a difference of the values of f at the origin and at a boundary point (where $f \equiv 0$ so that the choice of the boundary point does not matter).

7.3. **The terms from the other three quadrants of Q.** Proceeding in the same way, we find that the following quantities form an upper bound for the 6 original $u-$terms in each of the other three quadrants of the big square Q:

$$8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2)}{2\sqrt{2}}\right),$$

$$8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2)-f(2i)}{2\sqrt{2}}\right),$$

$$8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2i)-f(2)}{2\sqrt{2}}\right).$$

7.4. **Combining the terms from the four quadrants.** There are now two possibilities. Using (40) repeatedly, we may note that by (35),

$$\begin{aligned}
&8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2)}{2\sqrt{2}}\right)\\
&\leq 16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2i)}{4\sqrt{2}}\right),
\end{aligned}$$

$$\begin{aligned}
&8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2)-f(2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2i)-f(2)}{2\sqrt{2}}\right)\\
&\leq 16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, -\frac{f(2i)-f(-2i)}{4\sqrt{2}}\right),
\end{aligned}$$

$$\begin{aligned}
&16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2i)}{4\sqrt{2}}\right) + 16\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, -\frac{f(2i)-f(-2i)}{4\sqrt{2}}\right)\\
&\leq 32\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, 0\right) \leq 0.
\end{aligned}$$

This then proves (17) in this case, and hence completes the proof of Theorem 1.

Alternatively, we may apply (21) to four terms in one go and deduce by (42) that

$$\begin{aligned}
&8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2)-f(-2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2i)-f(-2)}{2\sqrt{2}}\right)\\
&+8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(2)-f(2i)}{2\sqrt{2}}\right) + 8\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, \frac{f(-2i)-f(2)}{2\sqrt{2}}\right)\\
&\leq 32\tilde{u}\left(\frac{f(0)}{4\sqrt{2}}, 0\right) \leq 0.
\end{aligned}$$

7.5. **Discussion.** The case of 13 parameters is particularly simple since it is possible to find matching lengths in terms of parametric expressions (using function values such as $f(-2-2i)$ as parameters). Also the pattern according to which the matching happens, that is, the sequence of $\sigma-$algebras created for the corresponding martingales, and also the functions c_1 and c_2 of modulus 1, depend only on the general setting we have but not on the particular values of f. This cannot be true in more complicated cases involving a large number of parameters, obtained, for example, when we subdivide the 40 triangles in this special case into smaller triangles. This is discussed in greater detail in [17].

8. An algorithm

It remains an unsolved problem whether Theorem 1 can be extended to an arbitrary continuous piecewise affine function of compact support defined in any domain that has been divided into the union of finitely many triangles with disjoint interiors. We propose now an algorithm for solving this problem, even though it remains open whether the algorithm always works.

We have described how one can start with two affine maps f_1 and f_2 associated with sets of area A_1 and A_2, respectively. If f_j has minimum stretch m_j and maximum stretch M_j, and if $[m_1, M_1] \cap [m_2, M_2] \neq \emptyset$, then we can combine the maps to a new affine map F, associated with area $A_1 + A_2$. The combination can, in general, be perfomed in many ways since we could choose any length $\mu \in [m_1, M_1] \cap [m_2, M_2]$ and find horizontal directions H_1 and H_2 with, initially, $|H_1| = |H_2| = \mu$. By means of suitable rotations, we can arrange to have $H_1 = H_2$. Then for the corresponding vertical directions V_1 and V_2, we obtain a combination by $H = H_1$ and $V = (A_1V_1 + A_2V_2)/(A_1 + A_2)$. This defines the map F. Since further rotations could be applied, only a certain class of maps F is uniquely determined, but it is only this class that matters. Any representative of it could be used in the sequel.

The question arises as to how to combine f_1 and f_2 if there are many possibilities, and whether we are always left with pairs of maps for which $[m_1, M_1] \cap [m_2, M_2] \neq \emptyset$, so that there would be at least one way of performing one more combination.

In general, the maximum stretch of F is smaller than $\max\{M_1, M_2\}$. Also, since we could choose H_1 with $|H_1| = \min([m_1, M_1] \cap [m_2, M_2])$, it is always possible to achieve a minimum stretch for F that does not exceed $\min([m_1, M_1] \cap [m_2, M_2])$.

At the beginning, we have a triangulation of a domain D and the corresponding affine maps. If a particular map f_1 with $m_1 > 0$ is ignored for a long time when performing combinations, it is conceivable that at some point, the largest maximum stretch for all the other remaining maps is $< m_1$, which means that f_1 cannot be included in any further combination. To avoid this, it seems prudent to include in combinations first those maps whose minimum stretch is large.

Thus we propose the following algorithm. At the beginning, we have the affine maps arising from the original triangulation and a given continuous piecewise affine function of compact support. Each original affine map is associated with the area of the triangle in which it is defined. At each stage we take that affine map f_1 still considered with the largest minimum stretch m_1. We have a problem with this algorithm only if this map cannot be combined with any other remaining map, that is, if m_1 is greater than the largest maximum stretch for all the other remaining maps. Assuming that this is not the case, we try the combinations of f_1 with all those other remaining maps with which a combination is possible. Since there may be several possible combinations for each pair of maps, we choose one that minimizes the resulting minimum stretch for the new affine map. Out of all these possibilities, we then use the one that minimizes the new minimum stretch; it is associated with an area that is the sum of the areas of the two maps used in the combination, even though we do not visualize the maps as being defined on a union of triangles (which may be a disconnected set, if we were to keep track of the original triangles that have been taken into account when coming up with this combination). This results in a new set of affine maps, one less than before. Some of the maps may be ones arising from the original triangulation, namely those maps that have not been used in any combination yet. Other maps arise from combinations. When combining two maps, it may be that they both arise from the original triangulation, or one does while the other one was earlier obtained in a combination, or they both were earlier obtained in some combinations.

An affine map $az+b\overline{z}+c$ with associated area A has the **signed area** $A(|a|^2 - |b|^2)$, which is the signed area of the image set of a set of area A under the map (positive for orientation preserving maps, negative for orientation reversing maps, zero for non-homeomorphic maps). The combinations we use preserve the total signed area. For a continuous piecewise affine function of compact support, the total signed area is zero. Thus the total signed area is zero for our collection of affine maps at each stage. The minimum stretch of a map is zero if, and only if, the signed area of the map is zero. Hence, when applying the above algorithm, we cannot arrive at a situation of exactly one map with positive minimum stretch.

Thus, if we are able to apply the algorithm up to a point where we have only mappings left with minimum stretch zero, we will have proved the claim of Theorem 1 for the piecewise affine continuous function that we started with. Namely, the original sum (17) for the mapping does not exceed such a sum for the final mappings, and for each of the last mappings f we have $|\partial f/\partial z| = |\partial f/\partial \overline{z}|$ since the minimum stretch is zero. Hence each term in the sum for the final mappings is ≤ 0, by (22), making the final sum of the form (17) also ≤ 0.

8.1. **A proposed algorithm for performing combinations, to construct pairs of martingales.**

To summarize, this is the algorithm we propose for a continuous piecewise affine function of compact support:

(1) We start with finitely many affine maps, one for each triangle, and the corresponding "weight", the area of the triangle.

(2) At each stage, we combine two of the remaining maps that are comparable to get another one whose weight is the sum of the two weights.

(3) Of the two maps to be composed, one has the largest minimum stretch at the time. The other one is chosen among those that are comparable to the first map, so that the combination affine map has the smallest possible minimum stretch.

(4) If we can continue in this way until there is at most one map with a positive minimum stretch (and hence none) then the desired pair of martingales can be constructed and hence the inequality $\sum_j A_j u(b_j, a_j) \leq 0$ holds.

(5) At each stage, we can continue provided that there is at least one other map comparable to the map with the largest minimum stretch.

We have proved that the above description of the process implies that one can continue expect possibly in the following situation: the largest minimum stretch occurs for a composite map and every other remaining map corresponds to one of the original triangles that has not been used yet. We omit the details here but note that this proof is based only on properties described above and not on the formulas for the minimal minimum stretch to be given in Section 9. The author believes that to get further, a more quantitative argument, possibly making use of such formulas, would be required.

9. How to minimize the minimum stretch of a combination

Since rotations (and translations) do not matter, we may assume that we are dealing with comparable affine maps $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$, with weights $A_1, A_2 > 0$, where $a, b, c, d \geq 0$ and

$$|c - d| \leq a - b \leq c + d.$$

Namely, both maps could be replaced by their complex conjugates (in which case the same holds for the combination), so that we may assume that the map with the larger minimum stretch is orientation preserving.

A lengthy calculation, which we omit here, shows that the smallest minimum stretch for any composition of the two maps is

$$\frac{\left|A_1 a + A_2 c - \sqrt{X_1}\right|}{A_1 + A_2},$$

$$X_1 = (A_1 + A_2)(A_1 b^2 + A_2 d^2) - A_1 A_2 (a - c)^2,$$

if $c + d \leq a + b$, and

$$\frac{\left|A_1 b + A_2 d - \sqrt{X_2}\right|}{A_1 + A_2},$$

$$X_2 = (A_1 + A_2)(A_1 a^2 + A_2 c^2) - A_1 A_2 (b - d)^2,$$

if $a + b < c + d$.

Numerical experiments suggest that these minima decrease, in the process of performing combinations, fast enough so that at least one of the remaining maximum stretches will be larger than the minimum so obtained.

References

[1] L.V. Ahlfors, Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn. A I 206 (1955).

[2] K. Astala, Area distortion of quasiconformal mappings, Acta Math. 173 (1994), 37–60.

[3] K. Astala, T. Iwaniec, and E. Saksman, Beltrami operators in the plane, Duke Math. J. 107 (2001), 27–56.

[4] R. Bañuelos and P. Janakiraman, L^p-bounds for the Beurling–Ahlfors transform, Trans. Amer. Math. Soc. 360 (2008), no. 7, 3603–3612.

[5] R. Bañuelos and Gang Wang, Sharp inequalities for martingales with applications to the Beurling–Ahlfors and Riesz transforms, Duke Math. J. 80 (1995), 575–600.

[6] A. Baernstein II and S.J. Montgomery-Smith, Some conjectures about integral means of ∂f and $\overline{\partial} f$, pp. 92–109 in Complex analysis and differential equations, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 64, Uppsala Univ., Uppsala, 1999.

[7] A. Beurling, The Collected Works of Arne Beurling, Volume 1, Birkhäuser, Boston, 1989, a seminar talk at Uppsala, "On the Hilbert Transform in Two Variables", November 8, 1949, pp. 460–461.

[8] B. Boyarskiĭ, Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients, Mat. Sb. N.S. 43(85) 1957, 451–503.

[9] D.L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, Ann. Probability 12 (1984), 647–702.

[10] D.L. Burkholder, Sharp inequalities for martingales and stochastic integrals, Asterisque 157–158 (1988), 75–94.

[11] D.L. Burkholder, Explorations in martingale theory and its applications, École d'Ét de Probabilités de Saint-Flour XIX—1989, 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991.

[12] Martingales and singular integrals in Banach spaces, Handbook of the geometry of Banach spaces, Vol. I, 233–269, North-Holland, Amsterdam, 2001.

[13] A.P. Calderón and A. Zygmund, On the existence of certain singular integrals, Acta Math. 88 (1952), 85–139.

[14] P.G. Ciarlet and P.-A. Raviart, General Lagrange and Hermite interpolation in $\mathbf{R}^n$ with applications to finite element methods, Arch. Rational Mech. Anal. 46 (1972), 177–199.

[15] O. Dragičević and A. Volberg, Bellman function, Littlewood-Paley estimates and asymptotics for the Ahlfors–Beurling operator in $L^p(\mathbb{C})$, Indiana Univ. Math. J. 54 (2005), 971–995.

[16] F.W. Gehring, The L^p-integrability of the partial derivatives of a quasiconformal mapping, Acta Math. 130 (1973), 265–277.

[17] A. Hinkkanen, On the norm of the Beurling–Ahlfors transformation, a preprint, 2008, available at http://www.math.uiuc.edu/ aimo/preprints.html

[18] T. Iwaniec, Extremal inequalities in Sobolev spaces and quasiconformal mappings, Z. Anal. Anwendungen 1 (1982), 1–16.

[19] T. Iwaniec and G.J. Martin, Quasiconformal mappings and capacity, Indiana Univ. Math. J. 40 (1991), 101–122.

[20] T. Iwaniec and G.J. Martin, Quasiregular mappings in even dimensions, Acta Math. 170 (1993), no. 1, 29–81.

[21] T. Iwaniec and G.J. Martin, Riesz transforms and related singular integrals, J. Reine Angew. Math. 473 (1996), 25–57.

[22] O. Lehto, Remarks on the integrability of the derivatives of quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I No. 371 (1965), 8 pp.

[23] O. Lehto and K.I. Virtanen, Quasiconformal mappings in the plane, Springer, Berlin, 1973.

[24] S. Petermichl and A. Volberg, Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular, Duke Math. J. 112 (2002), 281–305.

[25] I.N. Vekua, The problem of reduction to canonical form of differential forms of elliptic type and the generalized Cauchy-Riemann system, Dokl. Akad. Nauk SSSR (N.S.) 100, (1955), 197–200.

[26] A. Volberg and F. Nazarov, Heat extension of the Beurling operator and estimates for its norm (Russian), Algebra i Analiz 15 (2003), 142–158.

Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801 U.S.A.
E-mail address: `aimo@math.uiuc.edu`