ON CRITICAL VALUES OF POLYNOMIALS WITH REAL CRITICAL POINTS

AIMO HINKKANEN AND ILGIZ KAYUMOV

Abstract. Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the zeros of f' are real. We show that there is a zero ζ of f' such that $|f(\zeta)/\zeta| \leq 2/3$, and that this inequality can be taken to be strict unless f is of the form $f(z) = z + cz^3$.

1. Introduction and results

Smale [12], in connection with his investigations on effective algorithms that solve polynomial systems, formulated the following conjecture. Let f be a polynomial degree of $n \geq 2$ normalized by $f(0) = 0$ and $f'(0) = 1$. Then there exists a critical point ζ of f (that is, a zero of f') such that

$$\left| \frac{f(\zeta)}{\zeta} \right| < 1. \tag{1}$$

Smale proved this result with the constant 4 on the right hand side. At this time no constant < 4 has been obtained that would be valid for the class of all polynomials. It is only known that for $n \leq 4$, the inequality

$$\left| \frac{f(\zeta)}{\zeta} \right| \leq 1 - \frac{1}{n} \tag{2}$$

holds for some critical point ζ of f, with strict inequality unless $f(z) = z + cz^n$ (this result was proven by J.-C. Sikorav, see [13]). The same conclusion is supported by numerical experiments for $n \leq 10$, obtained by Sendov and Marinov, see [8].

There have been several serious attempts to solve Smale's problem, but no considerable improvements of the constant 4 have been obtained.

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Beardon, Minda and Ng [1] reduced the constant 4 to $4^{1-1/n}$, which was slightly improved in the papers [2], [7]. At this time the best upper estimate belongs to Crane (see [4], [5]). In the case when the critical points of f have equal modulus or the values of f at the critical points have equal modulus, Sheil-Small ([10], pp. 361–362) used the method of Córdova and Rusheweyh [3] to establish (1). Dubinin [6] obtained (2) when the critical points have equal modulus. Tischler [13] proved (2) when the non-zero zeros of f have equal modulus.

In the case when the polynomial f has only real zeros, Palais proved (1) (see [11], p. 159) while Tischler [13] obtained (2). If f has only real zeros, it follows from Rolle's theorem that f' has only real zeros. Since the converse does not hold, the case when f has only real critical points is more general than that when f has only real zeros.

Suppose that the polynomial f has only real critical points. Sheil-Small ([10], p. 368) obtained (1) in this case with the constant $e-2$ on the right hand side. Rahman and Schmeisser ([9], p. 217) obtained the slightly better result (here $n \geq 3$)

$$\left|\frac{f(\zeta)}{\zeta}\right| \leq \frac{n-2}{n}\left(\left(\frac{n-1}{n-2}\right)^{n-1}-2\right) < e-2.$$

We will obtain the sharp result when f has only real critical points (Corollary 1.3 below), as part of the appropriate special case of the following more general conjecture.

Conjecture. We conjecture that if f is a polynomial of degree at least 2 with $f(0)=0$ and $f'(0)=1$ such that the critical points of f are contained in the union of $k \geq 1$ rays from zero to infinity, then there is a critical point ζ of f such that $|f(\zeta)/\zeta| \leq 1-1/(k+1)$. Furthermore, we suggest that ζ can be chosen so that $|f(\zeta)/\zeta| < 1-1/(k+1)$ unless f is of the form $f(z)=z+cz^{k+1}$ for some non-zero complex constant c, in which case we have $|f(\zeta)/\zeta| = 1-1/(k+1)$ for all critical points ζ of f.

In this paper we prove this conjecture when $k=1$ and $k=2$.

Theorem 1.1. *Let f be a polynomial of degree at least 2 with $f(0)=0$ and $f'(0)=1$. Suppose that all the critical points of f lie on a single ray from zero to infinity. Then there is a critical point ζ of f such that $|f(\zeta)/\zeta| < 1/2$, unless f is of the form $f(z)=z+cz^2$ for some non-zero complex constant c, in which case we have $|f(\zeta)/\zeta| = 1/2$ for the only critical point ζ of f.*

Our main result is the following.

Theorem 1.2. *Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the critical points of f lie in the union of two rays from zero to infinity. Then there is a critical point ζ of f such that $|f(\zeta)/\zeta| < 2/3$, unless f is of the form $f(z) = z+cz^3$ for some non-zero complex constant c, in which case we have $|f(\zeta)/\zeta| = 2/3$ for both critical points ζ of f.*

Taking the two rays so that they form a straight line, we immediately obtain the following consequence of Theorem 1.2.

Corollary 1.3. *Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the zeros of f' are real. If f is not of the form $f(z) = z + cz^3$ for a negative real constant c, then there is a zero ζ of f' such that $|f(\zeta)/\zeta| < 2/3$. If f is of the form $f(z) = z + cz^3$ for a negative real constant c, then for both zeros ζ of f' we have $|f(\zeta)/\zeta| = 2/3$.*

Thus Corollary 1.3 applies, in particular, when f has only real zeros.

This result is surprising because the extremal polynomial has the fixed degree 3. As a general principle, one would think that one could get a larger value for some $|f(\zeta)/\zeta|$ by increasing the degree of f.

2. Proof of Theorem 1.1

Suppose that f is of degree $n \geq 2$, and let the critical points of f be denoted by z_j for $1 \leq j \leq n-1$, with due count of multiplicity. Without loss of generality, we may replace f by the function $f(az)/a$ for any non-zero complex number a. Therefore we may assume that $z_{n-1} = 1$ and that $|z_j| \geq 1$ for $1 \leq j \leq n-2$. We will use this same observation throughout the paper.

Let f be as in the assumptions of Theorem 1.1. Then each z_j is real with $z_j \geq 1$ for $1 \leq j \leq n-2$. Writing $f'(z) = (1-z)\prod_{j=1}^{n-2}(1-z/z_j)$, we obtain $f(z) = \int_0^z (1-u)\prod_{j=1}^{n-2}(1-u/z_j)\,du$. Writing then $u = tz$, we get

$$\frac{f(z)}{z} = \int_0^1 (1-tz)\prod_{j=1}^{n-2}\left(1-\frac{tz}{z_j}\right)dt.$$

Hence

$$\left|\frac{f(1)}{1}\right| = \int_0^1 (1-t)\prod_{j=1}^{n-2}\left(1-\frac{t}{z_j}\right)dt \leq \int_0^1 (1-t)\,dt = \frac{1}{2},$$

and the inequality is strict if $n \geq 3$, while equality holds if $n = 2$. This completes the proof of Theorem 1.1. This proof is trivial and may be known, but it serves to establish notation.

3. Proof of Theorem 1.2

3.1. **Set-up.** Let f satisfy the assumptions of Theorem 1.2. In view of Theorem 1.1, we may assume that f has critical points on two distinct rays, so that the degree of f is at least 3. Assuming that $z_{n-1} = 1 \le |z_j|$ for $1 \le j \le n-2$, we may label the critical points as $z_j = r_j e^{i\theta}$ for $1 \le j \le k$, where $0 < \theta < 2\pi$, $1 \le k \le n-2$, and $1 \le u = r_1 \le r_2 \le \cdots \le r_k$; and assume that z_j is real with $z_j \ge 1$ for $k < j \le n-2$. Expressing f as in the proof of Theorem 1.1, we may write

$$\frac{f(z)}{z} = \int_0^1 (1-tz)\left(1 - \frac{tz}{ue^{i\theta}}\right) \prod_{j=2}^{k} \left(1 - \frac{tz}{r_j e^{i\theta}}\right) \prod_{j=k+1}^{n-2} \left(1 - \frac{tz}{z_j}\right) dt.$$

We will prove that $\min\{|f(1)|, |f(ue^{i\theta})|/u\} \le 2/3$ and in the process discover the cases of equality.

We define the real number A by

$$A = u\left(\sum_{j=2}^{k} \frac{1}{r_j} - \sum_{j=k+1}^{n-2} \frac{1}{z_j}\right).$$

If $n = 3$, then we must have $k = 1$, and the above sum has no terms; in this case we set $A = 0$. We have

$$\left|\frac{f(1)}{1}\right| \le \int_0^1 (1-t)\left(1 + \frac{t}{u}\right) \prod_{j=2}^{k} \left(1 + \frac{t}{r_j}\right) \prod_{j=k+1}^{n-2} \left(1 - \frac{t}{z_j}\right) dt,$$

noting that all the factors in the integrand are positive when $0 \le t < 1$; so that since $1 + x \le e^x$ for all real x, we have

$$\left|\frac{f(1)}{1}\right| \le \int_0^1 (1-t)\left(1 + \frac{t}{u}\right) e^{At/u}\, dt = h_1(u, A),$$

say. Similarly,

$$\left|\frac{f(ue^{i\theta})}{ue^{i\theta}}\right| \le \int_0^1 (1-t)\,(1+tu) \prod_{j=2}^{k} \left(1 - \frac{tu}{r_j}\right) \prod_{j=k+1}^{n-2} \left(1 + \frac{tu}{z_j}\right) dt,$$

so that

$$\left|\frac{f(ue^{i\theta})}{ue^{i\theta}}\right| \le \int_0^1 (1-t)\,(1+tu)\, e^{-At}\, dt = h_1(1/u, -A/u) = h_2(u, A),$$

say. We note that

$$h_2(u, 0) = 1/2 + u/6 \ge 2/3 \ge h_1(u, 0) = 1/2 + 1/(6u) \tag{3}$$

and $h_1(1, 0) = h_2(1, 0) = 2/3$.

For each fixed $u \geq 1$, $h_1(u, A)$ is a strictly increasing positive continuous function of A with

$$\lim_{A\to-\infty} h_1(u, A) = 0, \qquad \lim_{A\to+\infty} h_1(u, A) = +\infty,$$

while $h_2(u, A)$ is a strictly decreasing positive continuous function of A with

$$\lim_{A\to-\infty} h_2(u, A) = +\infty, \qquad \lim_{A\to+\infty} h_2(u, A) = 0.$$

Hence there is a unique $A = A_0(u)$ such that $h_1(u, A_0(u)) = h_2(u, A_0(u))$. Clearly $\min\{h_1(u, A), h_2(u, A)\} \leq h_1(u, A_0(u))$ for all real A. Since $h_1(u, 0) \leq h_2(u, 0)$ with equality only when $u = 1$, we have $A_0(u) > 0$ if $u > 1$, and $A_0(1) = 0$. For $A < 0$, we have $h_1(u, A) < h_1(u, 0) \leq 2/3$.

3.2. **Estimates.** It suffices for us to prove that

$$\min\{h_1(u, A), h_2(u, A)\} \leq 2/3 \tag{4}$$

whenever $u \geq 1$ and A is real and then to find the cases of equality. We saw above that we may assume that $A \geq 0$.

Suppose first that $0 \leq A \leq 1$. Then, in view of the inequalities

$$e^x \leq 1 + x + (e - 2)x^2, \qquad e^{-x} \leq 1 - x + x^2/2,$$

valid for $0 \leq x \leq 1$, it is enough to show that one of the numbers

$$h_3(u, A) = \int_0^1 (1 - t)(1 + tu)(1 - At + A^2t^2/2)\, dt,$$

$$h_4(u, A) = \int_0^1 (1 - t)(1 + t/u)(1 + At/u + (e - 2)A^2t^2/u^2)\, dt \tag{5}$$

is less than or equal to $2/3$.

Suppose that $h_3(u, A) \geq 2/3$. Since h_3 is an increasing function of u, this implies that

$$u \geq \frac{5(4 + 4A - A^2)}{20 - 10A + 3A^2} = u_1 \geq 1,$$

say, which we substitute into (5). Since h_4 is a decreasing function of u, we have

$$h_4(u, A) \leq h_4(u_1, A) = \frac{P(A)}{7500(4 + 4A - A^2)^3} = h_5(A),$$

say, where

$$\begin{aligned} P(A) &= 320000 + 960000A + 504000A^2 - 140000A^3 - 115600A^4 \\ &\quad +32500A^5 + 3460A^6 - 2655A^7 + 288A^8 + 64000A^2e \\ &\quad -36000A^3e + 800A^4e + 12200A^5e - 6280A^6e + 1590A^7e \\ &\quad -144A^8e. \end{aligned}$$

Since $h_5(A)$ is a rational function, its maximum on the interval $[0,1]$ can be easily found. It is exactly $2/3$ and is attained for $A=0$ only.

Next consider the case $A \ge 1$. Suppose that

$$h_2(u,A) = \int_0^1 (1-t)(1+tu)e^{-At}\,dt \ge 2/3,$$

which implies that

$$u \ge \frac{A(-3+3e^A-3Ae^A+2A^2e^A)}{3(2+A-2e^A+Ae^A)}.$$

It is not difficult to show that for $A \ge 1$ we have

$$\frac{A}{u} \le \frac{3(2+A-2e^A+Ae^A)}{-3+3e^A-3Ae^A+2A^2e^A} \le \frac{2}{5}, \tag{6}$$

so that $u \ge 5/2$. Hence

$$\begin{aligned} h_1(u,A) &= \int_0^1 (1-t)(1+t/u)e^{At/u}\,dt \\ &\le \int_0^1 (1-t)(1+2t/5)e^{2t/5}\,dt = \frac{5}{4}(5-3e^{2/5}) < 2/3. \end{aligned}$$

This proves that $\min\{h_1(u,A), h_2(u,A)\} < 2/3$ when $A \ge 1$, completing the proof of (4) in all cases.

3.3. **Cases of equality.** If $\min\{h_1(u,A), h_2(u,A)\} = 2/3$, then by (4) and the monotonicity properties of h_1 and h_2, this can only be when $A = A_0(u)$; on the other hand, the analysis above shows that this can occur only when $A=0$, so we must have $A = A_0(u) = 0$ and thus $u=1$. Furthermore, to have the equality $\min\{|f(1)|, |f(ue^{i\theta})|/u\} = 2/3$, we must have, for all $t \in (0,1)$, equality in at least one of the estimates

$$\prod_{j=2}^{k}\left(1+\frac{t}{r_j}\right)\prod_{j=k+1}^{n-2}\left(1-\frac{t}{z_j}\right) \le e^{At} = 1$$

and

$$\prod_{j=2}^{k}\left(1-\frac{tu}{r_j}\right)\prod_{j=k+1}^{n-2}\left|1-\frac{tue^{i\theta}}{z_j}\right| \le e^{-Atu} = 1,$$

which is not possible unless the products are empty. This means that $n = 3$. With $n = 3$ and $u = 1$, we now get

$$\left|\frac{f(1)}{1}\right| = \left|\int_0^1 (1-t)\left(1 - \frac{t}{e^{i\theta}}\right)\, dt\right| = \left|\frac{1}{2} - \frac{1}{6e^{i\theta}}\right|$$

and

$$\left|\frac{f(ue^{i\theta})}{ue^{i\theta}}\right| = \left|\int_0^1 (1-t)\left(1 - te^{i\theta}\right)\, dt\right| = \left|\frac{1}{2} - \frac{e^{i\theta}}{6}\right|,$$

which are equal and are $< 2/3$ unless $e^{i\theta} = -1$, so that the critical points of f are then ± 1. In this case the equality $|f(\zeta)/\zeta| = 2/3$ indeed holds for both critical points ζ of f. Thus $f(z) = z - (1/3)z^3$, and since this f is obtained after a normalization, the original f is of the form $f(z) = z + cz^3$ for some non-zero complex constant c.

This completes the proof of Theorem 1.2.

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References

[1] A.F. Beardon, D. Minda, T.W. Ng, Smale's mean value conjecture and the hyperbolic metric, Math. Ann. 322:4 (2002), 623–632.

[2] A. Conte, E. Fujikawa and N. Lakic, Smale's mean value conjecture and the coefficients of univalent functions, Proc. Amer. Math. Soc. 135 (2007), 3295–3300.

[3] A. Córdova and S. Ruscheweyh, Subordination of polynomials, Rocky Mt. J. Math. 21:1 (1991), 159–170.

[4] E. Crane, Extremal polynomials in Smale's mean value conjecture, Comput. Methods Function Theory 6:1 (2006), 145–163.

[5] E. Crane, A bound for Smale's mean value conjecture for complex polynomials, Bull. London Math. Soc. 39 (2007), 781–791.

[6] V.N. Dubinin, Inequalities for critical values of polynomials, Sb. Math. 197 (2006), 1167–1176.

[7] E. Fujikawa and T. Sugawa, Geometric function theory and Smale's mean value conjecture, Proc. Japan Academy 82, Ser. A (2006), 97–100.

[8] P. Marinov and B. Sendov, Verification of the Smale's Mean Value Conjecture for $n \leq 10$, C.R. de l'Academie Bulgare de Sciences, 60:11 (2007), 1151–1156.

[9] Q.I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, 2003.

[10] T. Sheil-Small, Complex Polynomials, Cambridge University Press, 2002.

[11] M. Shub and S. Smale, Computational Complexity: On the Geometry of Polynomials and a Theory of Cost: II, SIAM J. Comput. 15:1 (1986), 145–161.

[12] S. Smale, The fundamental theorem of algebra and complexity theory, Bull. Amer. Math. Soc. (N. S.) 4 (1981), 1–36.

[13] D. Tischler, Critical points and values of complex polynomials, J. Complexity 5 (1989), 438–456.

University of Illinois at Urbana–Champaign, Department of Mathematics, 1409 West Green Street, Urbana, IL 61801 USA

Universitetskaya 17, Institute of Mathematics and Mechanics, Kazan State University, 420 008 Kazan, Russia
E-mail address: aimo@illinois.edu, Ilgis.Kayumov@ksu.ru