ON CRITICAL VALUES OF POLYNOMIALS WITH REAL CRITICAL POINTS

AIMO HINKKANEN AND ILGIZ KAYUMOV

ABSTRACT. Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the zeros of f' are real. We show that there is a zero ζ of f' such that $|f(\zeta)/\zeta| \leq 2/3$, and that this inequality can be taken to be strict unless f is of the form $f(z) = z + cz^3$.

1. INTRODUCTION AND RESULTS

Smale [12], in connection with his investigations on effective algorithms that solve polynomial systems, formulated the following conjecture. Let f be a polynomial degree of $n \geq 2$ normalized by $f(0) = 0$ and $f'(0) = 1$. Then there exists a critical point ζ of f (that is, a zero of f') such that

$$(1) \quad \left| \frac{f(\zeta)}{\zeta} \right| < 1.$$

Smale proved this result with the constant 4 on the right hand side. At this time no constant < 4 has been obtained that would be valid for the class of all polynomials. It is only known that for $n \leq 4$, the inequality

$$(2) \quad \left| \frac{f(\zeta)}{\zeta} \right| \leq 1 - \frac{1}{n}$$

holds for some critical point ζ of f , with strict inequality unless $f(z) = z + cz^n$ (this result was proven by J.-C. Sikorav, see [13]). The same conclusion is supported by numerical experiments for $n \leq 10$, obtained by Sendov and Marinov, see [8].

There have been several serious attempts to solve Smale's problem, but no considerable improvements of the constant 4 have been obtained.

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Beardon, Minda and Ng [1] reduced the constant 4 to $4^{1-1/n}$, which was slightly improved in the papers [2], [7]. At this time the best upper estimate belongs to Crane (see [4], [5]). In the case when the critical points of f have equal modulus or the values of f at the critical points have equal modulus, Sheil-Small ([10], pp. 361–362) used the method of Córdova and Rusheweyh [3] to establish (1). Dubinin [6] obtained (2) when the critical points have equal modulus. Tischler [13] proved (2) when the non-zero zeros of f have equal modulus.

In the case when the polynomial f has only real zeros, Palais proved (1) (see [11], p. 159) while Tischler [13] obtained (2). If f has only real zeros, it follows from Rolle's theorem that f' has only real zeros. Since the converse does not hold, the case when f has only real critical points is more general than that when f has only real zeros.

Suppose that the polynomial f has only real critical points. Sheil-Small ([10], p. 368) obtained (1) in this case with the constant $e - 2$ on the right hand side. Rahman and Schmeisser ([9], p. 217) obtained the slightly better result (here $n \geq 3$)

$$\left| \frac{f(\zeta)}{\zeta} \right| \leq \frac{n-2}{n} \left(\left(\frac{n-1}{n-2} \right)^{n-1} - 2 \right) < e - 2.$$

We will obtain the sharp result when f has only real critical points (Corollary 1.3 below), as part of the appropriate special case of the following more general conjecture.

Conjecture. We conjecture that if f is a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$ such that the critical points of f are contained in the union of $k \geq 1$ rays from zero to infinity, then there is a critical point ζ of f such that $|f(\zeta)/\zeta| \leq 1 - 1/(k+1)$. Furthermore, we suggest that ζ can be chosen so that $|f(\zeta)/\zeta| < 1 - 1/(k+1)$ unless f is of the form $f(z) = z + cz^{k+1}$ for some non-zero complex constant c , in which case we have $|f(\zeta)/\zeta| = 1 - 1/(k+1)$ for all critical points ζ of f .

In this paper we prove this conjecture when $k = 1$ and $k = 2$.

Theorem 1.1. *Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the critical points of f lie on a single ray from zero to infinity. Then there is a critical point ζ of f such that $|f(\zeta)/\zeta| < 1/2$, unless f is of the form $f(z) = z + cz^2$ for some non-zero complex constant c , in which case we have $|f(\zeta)/\zeta| = 1/2$ for the only critical point ζ of f .*

Our main result is the following.

Theorem 1.2. *Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the critical points of f lie in the union of two rays from zero to infinity. Then there is a critical point ζ of f such that $|f(\zeta)/\zeta| < 2/3$, unless f is of the form $f(z) = z + cz^3$ for some non-zero complex constant c , in which case we have $|f(\zeta)/\zeta| = 2/3$ for both critical points ζ of f .*

Taking the two rays so that they form a straight line, we immediately obtain the following consequence of Theorem 1.2.

Corollary 1.3. *Let f be a polynomial of degree at least 2 with $f(0) = 0$ and $f'(0) = 1$. Suppose that all the zeros of f' are real. If f is not of the form $f(z) = z + cz^3$ for a negative real constant c , then there is a zero ζ of f' such that $|f(\zeta)/\zeta| < 2/3$. If f is of the form $f(z) = z + cz^3$ for a negative real constant c , then for both zeros ζ of f' we have $|f(\zeta)/\zeta| = 2/3$.*

Thus Corollary 1.3 applies, in particular, when f has only real zeros.

This result is surprising because the extremal polynomial has the fixed degree 3. As a general principle, one would think that one could get a larger value for some $|f(\zeta)/\zeta|$ by increasing the degree of f .

2. PROOF OF THEOREM 1.1

Suppose that f is of degree $n \geq 2$, and let the critical points of f be denoted by z_j for $1 \leq j \leq n-1$, with due count of multiplicity. Without loss of generality, we may replace f by the function $f(az)/a$ for any non-zero complex number a . Therefore we may assume that $z_{n-1} = 1$ and that $|z_j| \geq 1$ for $1 \leq j \leq n-2$. We will use this same observation throughout the paper.

Let f be as in the assumptions of Theorem 1.1. Then each z_j is real with $z_j \geq 1$ for $1 \leq j \leq n-2$. Writing $f'(z) = (1-z) \prod_{j=1}^{n-2} (1-z/z_j)$, we obtain $f(z) = \int_0^z (1-u) \prod_{j=1}^{n-2} (1-u/z_j) du$. Writing then $u = tz$, we get

$$\frac{f(z)}{z} = \int_0^1 (1-tz) \prod_{j=1}^{n-2} \left(1 - \frac{tz}{z_j}\right) dt.$$

Hence

$$\left| \frac{f(1)}{1} \right| = \int_0^1 (1-t) \prod_{j=1}^{n-2} \left(1 - \frac{t}{z_j}\right) dt \leq \int_0^1 (1-t) dt = \frac{1}{2},$$

and the inequality is strict if $n \geq 3$, while equality holds if $n = 2$. This completes the proof of Theorem 1.1. This proof is trivial and may be known, but it serves to establish notation.

3. PROOF OF THEOREM 1.2

3.1. Set-up. Let f satisfy the assumptions of Theorem 1.2. In view of Theorem 1.1, we may assume that f has critical points on two distinct rays, so that the degree of f is at least 3. Assuming that $z_{n-1} = 1 \leq |z_j|$ for $1 \leq j \leq n-2$, we may label the critical points as $z_j = r_j e^{i\theta}$ for $1 \leq j \leq k$, where $0 < \theta < 2\pi$, $1 \leq k \leq n-2$, and $1 \leq u = r_1 \leq r_2 \leq \dots \leq r_k$; and assume that z_j is real with $z_j \geq 1$ for $k < j \leq n-2$. Expressing f as in the proof of Theorem 1.1, we may write

$$\frac{f(z)}{z} = \int_0^1 (1-tz) \left(1 - \frac{tz}{ue^{i\theta}}\right) \prod_{j=2}^k \left(1 - \frac{tz}{r_j e^{i\theta}}\right) \prod_{j=k+1}^{n-2} \left(1 - \frac{tz}{z_j}\right) dt.$$

We will prove that $\min\{|f(1)|, |f(ue^{i\theta})|/u\} \leq 2/3$ and in the process discover the cases of equality.

We define the real number A by

$$A = u \left(\sum_{j=2}^k \frac{1}{r_j} - \sum_{j=k+1}^{n-2} \frac{1}{z_j} \right).$$

If $n = 3$, then we must have $k = 1$, and the above sum has no terms; in this case we set $A = 0$. We have

$$\left| \frac{f(1)}{1} \right| \leq \int_0^1 (1-t) \left(1 + \frac{t}{u}\right) \prod_{j=2}^k \left(1 + \frac{t}{r_j}\right) \prod_{j=k+1}^{n-2} \left(1 - \frac{t}{z_j}\right) dt,$$

noting that all the factors in the integrand are positive when $0 \leq t < 1$; so that since $1+x \leq e^x$ for all real x , we have

$$\left| \frac{f(1)}{1} \right| \leq \int_0^1 (1-t) \left(1 + \frac{t}{u}\right) e^{At/u} dt = h_1(u, A),$$

say. Similarly,

$$\left| \frac{f(ue^{i\theta})}{ue^{i\theta}} \right| \leq \int_0^1 (1-t) (1+tu) \prod_{j=2}^k \left(1 - \frac{tu}{r_j}\right) \prod_{j=k+1}^{n-2} \left(1 + \frac{tu}{z_j}\right) dt,$$

so that

$$\left| \frac{f(ue^{i\theta})}{ue^{i\theta}} \right| \leq \int_0^1 (1-t) (1+tu) e^{-At} dt = h_1(1/u, -A/u) = h_2(u, A),$$

say. We note that

$$(3) \quad h_2(u, 0) = 1/2 + u/6 \geq 2/3 \geq h_1(u, 0) = 1/2 + 1/(6u)$$

and $h_1(1, 0) = h_2(1, 0) = 2/3$.

For each fixed $u \geq 1$, $h_1(u, A)$ is a strictly increasing positive continuous function of A with

$$\lim_{A \rightarrow -\infty} h_1(u, A) = 0, \quad \lim_{A \rightarrow +\infty} h_1(u, A) = +\infty,$$

while $h_2(u, A)$ is a strictly decreasing positive continuous function of A with

$$\lim_{A \rightarrow -\infty} h_2(u, A) = +\infty, \quad \lim_{A \rightarrow +\infty} h_2(u, A) = 0.$$

Hence there is a unique $A = A_0(u)$ such that $h_1(u, A_0(u)) = h_2(u, A_0(u))$. Clearly $\min\{h_1(u, A), h_2(u, A)\} \leq h_1(u, A_0(u))$ for all real A . Since $h_1(u, 0) \leq h_2(u, 0)$ with equality only when $u = 1$, we have $A_0(u) > 0$ if $u > 1$, and $A_0(1) = 0$. For $A < 0$, we have $h_1(u, A) < h_1(u, 0) \leq 2/3$.

3.2. Estimates. It suffices for us to prove that

$$(4) \quad \min\{h_1(u, A), h_2(u, A)\} \leq 2/3$$

whenever $u \geq 1$ and A is real and then to find the cases of equality. We saw above that we may assume that $A \geq 0$.

Suppose first that $0 \leq A \leq 1$. Then, in view of the inequalities

$$e^x \leq 1 + x + (e - 2)x^2, \quad e^{-x} \leq 1 - x + x^2/2,$$

valid for $0 \leq x \leq 1$, it is enough to show that one of the numbers

$$h_3(u, A) = \int_0^1 (1 - t)(1 + tu)(1 - At + A^2 t^2/2) dt,$$

$$(5) \quad h_4(u, A) = \int_0^1 (1 - t)(1 + t/u)(1 + At/u + (e - 2)A^2 t^2/u^2) dt$$

is less than or equal to $2/3$.

Suppose that $h_3(u, A) \geq 2/3$. Since h_3 is an increasing function of u , this implies that

$$u \geq \frac{5(4 + 4A - A^2)}{20 - 10A + 3A^2} = u_1 \geq 1,$$

say, which we substitute into (5). Since h_4 is a decreasing function of u , we have

$$h_4(u, A) \leq h_4(u_1, A) = \frac{P(A)}{7500(4 + 4A - A^2)^3} = h_5(A),$$

say, where

$$\begin{aligned} P(A) = & 320000 + 960000A + 504000A^2 - 140000A^3 - 115600A^4 \\ & + 32500A^5 + 3460A^6 - 2655A^7 + 288A^8 + 64000A^2e \\ & - 36000A^3e + 800A^4e + 12200A^5e - 6280A^6e + 1590A^7e \\ & - 144A^8e. \end{aligned}$$

Since $h_5(A)$ is a rational function, its maximum on the interval $[0, 1]$ can be easily found. It is exactly $2/3$ and is attained for $A = 0$ only.

Next consider the case $A \geq 1$. Suppose that

$$h_2(u, A) = \int_0^1 (1-t)(1+tu)e^{-At} dt \geq 2/3,$$

which implies that

$$u \geq \frac{A(-3 + 3e^A - 3Ae^A + 2A^2e^A)}{3(2 + A - 2e^A + Ae^A)}.$$

It is not difficult to show that for $A \geq 1$ we have

$$(6) \quad \frac{A}{u} \leq \frac{3(2 + A - 2e^A + Ae^A)}{-3 + 3e^A - 3Ae^A + 2A^2e^A} \leq \frac{2}{5},$$

so that $u \geq 5/2$. Hence

$$\begin{aligned} h_1(u, A) &= \int_0^1 (1-t)(1+t/u)e^{At/u} dt \\ &\leq \int_0^1 (1-t)(1+2t/5)e^{2t/5} dt = \frac{5}{4}(5 - 3e^{2/5}) < 2/3. \end{aligned}$$

This proves that $\min\{h_1(u, A), h_2(u, A)\} < 2/3$ when $A \geq 1$, completing the proof of (4) in all cases.

3.3. Cases of equality. If $\min\{h_1(u, A), h_2(u, A)\} = 2/3$, then by (4) and the monotonicity properties of h_1 and h_2 , this can only be when $A = A_0(u)$; on the other hand, the analysis above shows that this can occur only when $A = 0$, so we must have $A = A_0(u) = 0$ and thus $u = 1$. Furthermore, to have the equality $\min\{|f(1)|, |f(ue^{i\theta})|/u\} = 2/3$, we must have, for all $t \in (0, 1)$, equality in at least one of the estimates

$$\prod_{j=2}^k \left(1 + \frac{t}{r_j}\right) \prod_{j=k+1}^{n-2} \left(1 - \frac{t}{z_j}\right) \leq e^{At} = 1$$

and

$$\prod_{j=2}^k \left(1 - \frac{tu}{r_j}\right) \prod_{j=k+1}^{n-2} \left|1 - \frac{tue^{i\theta}}{z_j}\right| \leq e^{-Atu} = 1,$$

which is not possible unless the products are empty. This means that $n = 3$. With $n = 3$ and $u = 1$, we now get

$$\left| \frac{f(1)}{1} \right| = \left| \int_0^1 (1-t) \left(1 - \frac{t}{e^{i\theta}} \right) dt \right| = \left| \frac{1}{2} - \frac{1}{6e^{i\theta}} \right|$$

and

$$\left| \frac{f(ue^{i\theta})}{ue^{i\theta}} \right| = \left| \int_0^1 (1-t) (1 - te^{i\theta}) dt \right| = \left| \frac{1}{2} - \frac{e^{i\theta}}{6} \right|,$$

which are equal and are $< 2/3$ unless $e^{i\theta} = -1$, so that the critical points of f are then ± 1 . In this case the equality $|f(\zeta)/\zeta| = 2/3$ indeed holds for both critical points ζ of f . Thus $f(z) = z - (1/3)z^3$, and since this f is obtained after a normalization, the original f is of the form $f(z) = z + cz^3$ for some non-zero complex constant c .

This completes the proof of Theorem 1.2.

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UNIVERSITY OF ILLINOIS AT URBANA–CHAMPAIGN, DEPARTMENT OF MATHEMATICS, 1409 WEST GREEN STREET, URBANA, IL 61801 USA

UNIVERSITetskAYA 17, INSTITUTE OF MATHEMATICS AND MECHANICS, KAZAN STATE UNIVERSITY, 420 008 KAZAN, RUSSIA

E-mail address: `aimo@illinois.edu`, `Ilgiz.Kayumov@ksu.ru`