Majorisation of analytic functions

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The earliest majorisation result for analytic functions is the maximum modulus principle. In the simplest setting, for a function f analytic in G and continuous in $\overline{G}$ for a bounded domain G in the complex plane $\mathbb{C}$, it states that if $|f(z)| \leq M$ for all $z \in \partial G$, where M is a positive constant, then $|f(z)| \leq M$ for all $z \in \overline{G}$. Note that under these assumptions, it is clear that $|f|$ is bounded in $\overline{G}$, and we are merely asking for the best upper bound. Many generalisations of the maximum modulus principle have been obtained. Some are based on relaxing the assumptions by allowing $|f|$ to be at least hypothetically unbounded, perhaps in an unbounded domain G, or in a bounded domain G with a set of potential singularities on the boundary. In those cases one assumes that $|f(z)|$ does not grow faster than at a prescribed rate, possibly depending on the shape of the domain, as z approaches infinity or one of the finite exceptional boundary points. This leads to Phragmén–Lindelöf type of theorems.

Another type of generalisation is based on replacing the constant M by a function of distance, and considering more carefully how f can vary from one point to another. This leads one to consider distances between function values at different points. Such problems are more delicate. We wish to avoid the extra problems arising from possible singularities on the boundary, so we consider this question in the following setting.

Let G is a bounded domain, let f be analytic in G and continuous in $\overline{G}$, and let $\mu(t)$ be a non-negative non-decreasing function defined for $t \geq 0$. Suppose that $z_1 \in \partial G$ and that

$$|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|) \tag{1}$$

for all $z_2 \in \partial G$. One can ask whether we then have

$$|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|) \tag{2}$$

for all $z_2 \in G$ for some absolute constant C.

One may note that by a fairly simple application of the the maximum modulus principle, the above result, if valid, implies that if (1) holds for all $z_1, z_2 \in \partial G$, then (2) holds for all $z_1, z_2 \in \overline{G}$.

Examples due to Smith and Stegenga [8] show that when we consider this situation for only one fixed $z_1 \in \partial G$, then the growth of the non-negative non-decreasing function

$$\mu_1(t) = \sup\{|f(z_1) - f(z_2)| : z_2 \in \partial G, |z_1 - z_2| \leq t\}$$

can be sufficiently irregular to prevent the validity of such a result in general. Indeed, for any large $a > 0$ there exists a conformal mapping f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto a bounded domain, which can be taken to be the union of three rectangles, such that for some $t_0 \in (0, 2)$ (close to 2 if a is large), we have $|f(z) - f(-1)| \leq 1$ whenever $z \in \partial\mathbb{D}$ and $|z+1| \leq t_0$, while $|f(t_0 - 1) - f(-1)| \geq a$.

The situation changes if we majorize the function μ_1, effectively a kind of modulus of continuity of f, by a sufficiently regular function. In the particular case of the unit disk, the situation also changes if we allow both z_1 and z_2 vary on $\partial\mathbb{D}$, for then the best μ_1 becomes a subadditive function of t.

Thus we say that a non-negative non-decreasing function $\mu(t)$ defined for $t \geq 0$ is a **majorant** if $\mu(2t) \leq 2\mu(t)$ for all $t \geq 0$. We ask whether (1) for all $z_2 \in \partial G$ implies (2) for all $z_2 \in G$ for an absolute constant C when μ is a majorant.

When considering Hölder continuity, we take $\mu(t) = t^\alpha$, where $0 < \alpha \leq 1$. In this case, Hardy and Littlewood [2] obtained the desired conclusion for some absolute constant C when $G = \mathbb{D}$, and Walsh and Sewell [7] proved the same with $C = 1$ when G is a Jordan domain. Gehring, Hayman and the speaker [1] extended this result, with $C = 1$, to arbitrary domains. The speaker [3] generalised this, with $C = 1$, to arbitrary domains when $\log \mu(e^t)$ is a concave function of t for real t.

For general majorants, it was proved that (1) implies (2) for some absolute constant C by Tamrazov [9] and by Rubel, Shields, and Taylor [6] when G is simply connected. In fact, Tamrazov [9] proved a more general result for domains whose boundary is uniformly thick in a suitable sense, with C depending on this thickness, and his result for simply connected domains is a special case of this.

A good representative of a doubly connected domain is the annulus $G = \{z \in \mathbb{C} : 1/R < |z| < R\}$ for some large $R > 1$. If we take $z_1 \in \partial G$ to be on the inner component of ∂G, we may assume, for all practical purposes (when R is very large), that $z_1 = 0$ and $f(0) = 0$. If we now take $\mu(t) = \max\{1, t\}$, our assumption (1) can be taken to be $|f(z)| \leq \max\{1, |z|\}$ for all $z \in \partial G$. As usual, write $M(r, f) = \max\{|f(z)| : |z| = r\}$. Then $M(1/R, f) \leq 1$ and $M(R, f) \leq R$. If we choose $z_2 = 1 \in G$ in (2), we see that we would like to show that $|f(1)| \leq C$ for an absolute constant C. Hadamard's three-circles theorem states that $\log M(r, f)$ is a convex function of $\log r$, which implies that

$$\log |f(1)| \leq \log M(1, f) \leq \frac{1}{2}(\log M(1/R, f) + \log M(R, f)) \leq \frac{1}{2} \log R,$$

which does not remain bounded by an absolute constant. This shows that there is still something to be proved here. Note that here the inner boundary component of ∂G is "thin" when viewed from $z_2 = 1 \in G$. This also exposes the fact that potential theoretic methods, based on using properties of $|f|$ alone (via the subharmonic function $\log |f|$), cannot yield the best result in all cases; the proofs in all earlier papers including [9], [6], [1], [3] were based on such methods. It will be necessary to find an effective way of using the argument of f, which has the special property that if G is bounded by finitely many disjoint Jordan curves, as we may assume by approximation (compare [4]), then $\arg f(z)$ changes by an integral multiple of 2π when z traverses any component of ∂G.

The first person to exploit the fact that one can do better for analytic functions than subharmonic functions by using the argument of f was Teichmüller, who in 1939 ([10], [11]) proved a sharp version of the three-circles theorem for annuli. In 1987, the speaker [4] extended Teichmüller's framework to multiply connected domains, but was able to prove a definite result only for doubly connected domains, obtaining $C = 1.63 \cdot 10^7$ for them.

The speaker has recently proved the following result [5], which shows that (1) implies (2) for an absolute constant C under all circumstances.

Theorem 1. *If G is a bounded domain, if f is analytic in G and continuous in $\overline{G}$, if μ is a majorant, and if (1) holds for all $z_1, z_2 \in \partial G$, then (2) holds for all $z_1, z_2 \in \overline{G}$ with $C = 3456$.*

If (1) holds for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, then (2) holds for this z_1 and for all $z_2 \in \overline{G}$ with $C = 3456$.

The proof of Theorem 1 is based on noting that by approximation, we may assume that G is bounded by finitely many disjoint analytic Jordan curves, and, by Runge's theorem, that f is rational with only one pole (ignoring multiplicities) at a preassigned point in each component of $\overline{\mathbb{C}} \setminus \overline{G}$. Then one combines local results based on the use of logarithmic capacity, already articulated in [9] and [4], with consideration of the level sets of suitable rational functions.

References

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