

GROWTH OF SECOND PAINLEVÉ TRANSCENDENTS

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ABSTRACT. Let w be a transcendental solution to the second Painlevé equation. We prove that the order of w is either 3 or $3/2$, and we determine a criterion indicating which of these cases occurs in terms of the limiting behavior of a well known auxiliary function outside an exceptional set.

1. INTRODUCTION

Painlevé differential equations have been a topic of extensive research activity since around 1900, and during the last two decades in particular. Despite this activity, and of the wide-ranged applications of Painlevé equations into physics, some of the mathematical reasoning has appeared to be partially defective and some of the gaps have been filled only recently. In particular, the meromorphic nature of the solutions, usually called Painlevé transcendents, has been rigorously settled during the last decade, see [6]–[9] and [14]. Another aspect of the field, still somewhat defectively understood, is the growth and value distribution of transcendental solutions to Painlevé equations. Here transcendence is to be understood in the sense of the classical function theory, i.e., as non-rational solutions. As an example, it is well-known and rigorously proved recently, see [10], [13], and [16], that all transcendental solutions to the second Painlevé equation

$$(1.0.1) \quad w'' = 2w^3 + zw + \alpha,$$

where α is a complex parameter, are of order $\rho(w)$ such that $3/2 \leq \rho(w) \leq 3$. However, it has remained open, whether all transcendental solutions to (1.0.1) are either of order $\rho(w) = 3/2$, or of order $\rho(w) = 3$. The truth of this claim seems plausible, not the least due to extensive investigations related to the asymptotic behavior of second Painlevé transcendents, see, e.g., [1], [2], and [11]. In this paper, a rigorous function-theoretic proof is given for the following result:

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Theorem 1.1. *Let w be an arbitrary transcendental, hence non-rational meromorphic solution to the second Painlevé equation (1.0.1). Then w is either of order $\rho(w) = 3/2$, or of order $\rho(w) = 3$.*

The proof below is, essentially, a combination of two ideas, namely of the re-scaling device successfully applied by Steinmetz in [15] and of a careful geometric analysis of the solutions. Before proceeding to the proof, we would like to point out an interesting intrinsic property of w separating these two growth alternatives. Indeed, recalling the first integral

$$(1.0.2) \quad w'^2 = w^4 + zw^2 + 2\alpha w - U$$

of (1.0.1), where $U' = w^2$, we show that the case $\rho(w) = 3$ is at hand exactly when

$$(A) \quad \limsup_{z \rightarrow \infty, z \notin E} \left| \frac{U(z)}{z^2} + \frac{1}{4} \right| =: \delta_0 > 0 \quad \text{and} \quad \limsup_{z \rightarrow \infty, z \notin E} \left| \frac{U(z)}{z^2} \right| > 0,$$

while we have $\rho(w) = 3/2$ exactly when either

$$(B) \quad \lim_{z \rightarrow \infty, z \notin E} \left(\frac{U(z)}{z^2} + \frac{1}{4} \right) = 0. \quad \text{or}$$

$$(C) \quad \lim_{z \rightarrow \infty, z \notin E} \frac{U(z)}{z^2} = 0.$$

Here

$$E := \bigcup_{w(p)=\infty} U(p),$$

and $U(p)$ stands for a neighborhood of a pole of w at p determined by $|w(z)| > K\sqrt{|z|}$ for a given positive real number K . What remains open is whether a solution w of (1.0.1) such that $\rho(w) = 3/2$ necessarily is one of the Airy solutions, see [3], Section 21. The three alternatives (A), (B) and (C) above are conjectured to be dependent on the parameter α . However, our proof below does not make explicit use of this parameter.

Similarly as in the case of the second Painlevé equation, it has remained open, whether all transcendental solutions to the fourth Painlevé equation

$$2ww'' = (w')^2 + 3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta$$

are either of order $\rho(w) = 2$, or of order $\rho(w) = 4$. It seems to us that a similar reasoning to what will be presented applied in this paper could be used to settle the case of the fourth Painlevé equation as well.

2. PROOF OF THEOREM 1.1

For the convenience of the reader, we first propose to describe the essentials of the relatively long proof of Theorem 1.1. Indeed, the proof consists of four sections.

In the Section 3, rescaling arguments are effectively used to show that whenever a transcendental solution w is relatively small at a point z , say, then firstly, w has at least one pole close to z and secondly, a zero close to z as well. The approximate claim concerning the distance to the pole of w nearest to z then follows in two parts: Lemma 6.1 gives the required distance estimate under the assumption that $|w(z)| = K\sqrt{|z|}$ for a fixed multiplier $K > 0$ small enough, while Lemma 6.3 settles the more general case of $|w(z)| \leq K\sqrt{|z|}$, again with K sufficiently small. The approximate claim concerning the distance to a zero of w close to z then follows in Lemma 7.1.

To make the task of the reader easier, three technical lemmas are presented separately: Lemma 4.1 essentially presents an integral equation satisfied by w , obtained by integrating the second Painlevé equation (1.0.1) twice. Lemma 5.1 then points out that whenever $|w(z) - w(\zeta)| = O(\sqrt{|\zeta|})$ in a disc B about ζ , then a lower estimate for $|w(z) - w(\zeta)|$ follows in terms of $w'(\zeta), w(\zeta)$ and $|z - \zeta|$ on a circle about ζ included in B . The final technical Lemma 6.2 shows that the lower estimate, under certain conditions imposed upon z and ζ takes the form $|w(z) - w(\zeta)| > 2|w(\zeta)|$.

In Section 8, we proceed to proving that the sets E_1 close to the points where the solution w takes certain given values, appear indeed to be small enough to estimate certain error terms. These estimates are needed in the remaining two sections, where an elliptic integral, resp., an elementary integral are needed to obtain estimates for sufficiently many counting functions. Via these estimates, the resulting order estimates complete the proof.

In Section 10, we proceed under the assumption that

$$(A) \limsup_{z \rightarrow \infty, z \notin E} \left| \frac{U(z)}{z^2} + \frac{1}{4} \right| =: \delta_0 > 0, \limsup_{z \rightarrow \infty, z \notin E} \left| \frac{U(z)}{z^2} \right| > 0.$$

In this case, if $z \in B(z_0, \varepsilon|z_0|)$, then we obtain that

$$(1 + o(1))(z - z_0)\sqrt{z_0} = \int_{w(z_0)/\sqrt{z_0}}^{w(z)/\sqrt{z_0}} \frac{dw/\sqrt{z_0}}{\sqrt{(w/\sqrt{z_0} + 1/2)^2 - A}}.$$

Here the right hand side is an elliptic integral, and taking the integral over its period ω , it is not difficult to conclude that $(1 + o(1))|z -$

$z_0|\sqrt{|z_0|} = |\omega|$. We then proceed to conclude that $w/\sqrt{z_0}$ takes all values from a disc $B(0, K)$ approximately μ times in $B(z_0, \varepsilon|z_0|)$, where μ counts the number of periodicity parallelograms. Taking now a sequence of points z_0 tending to infinity, we conclude that $\rho(w) \geq 3$, hence $\rho(w) = 3$.

In the final section the case is treated, when

$$(B) \lim_{z \rightarrow \infty, z \notin E} \left(\frac{U(z)}{z^2} + \frac{1}{4} \right) = 0 \text{ or}$$

$$(C) \lim_{z \rightarrow \infty, z \notin E} \frac{U(z)}{z^2} = 0.$$

Now the elliptic integral in the previous section reduces, resulting in a trigonometric situation. Elementary reasoning taking into account the periodicity sequences shows that $\rho(w) \leq 3/2$, hence $\rho(w) = 3/2$.

3. UPPER BOUNDS FOR THE DERIVATIVE WHEN THE SOLUTION IS SMALL

Let w be a transcendental solution of (1.0.1). Following Steinmetz [15], p. 47, we write

$$(3.0.3) \quad r(z) = \min \left\{ \frac{1}{|w(z)|}, \frac{1}{\sqrt{|w'(z)|}}, \frac{1}{\sqrt{|z|}} \right\},$$

so that if $w(z) = 0$ and $|w'(z)| \geq |z|$, then

$$r(z) = \frac{1}{\sqrt{|w'(z)|}}.$$

Steinmetz [17], Proposition 3.5, p. 79, obtained the following results.

Lemma 3.1. *Let w be a transcendental solution of (1.0.1). Then there are a small positive number $\kappa_0 \in (0, 1)$ and large positive numbers $\kappa_1 \geq 1$, $\kappa_2 \geq 1$ depending on w such that if the poles of w of modulus ≥ 1 are labelled as p_n , then we have*

$$(3.0.4) \quad |w(z)| \leq \kappa_1 \sqrt{|z|}$$

outside the union of disks

$$(3.0.5) \quad P(\kappa_0) = \cup_n B(p_n, \kappa_0/\sqrt{|p_n|}).$$

Furthermore,

$$(3.0.6) \quad |w'(z)| \leq \kappa_2 |z|$$

whenever $|z| \geq 1$ and z is outside $P(\kappa_0)$. We may in fact choose κ_0 to be any sufficiently small positive number, and then the conclusion holds with κ_1, κ_2 depending also on κ_0 .

Note that with the definition (3.0.3), $r(p_n) = 0$. Steinmetz defined separately $r(p) = 1/\sqrt{|p|}$ for each pole p of w , and we will also use this special definition of $r(p)$.

We prove the following more extensive lemma.

Lemma 3.2. *Let w be a transcendental solution of (1.0.1). There are positive numbers $L_1 \geq 1$ and $\theta_1 \leq 1$ depending on w such that whenever $z \neq 0$ and $|w(z)| \leq \theta_1 \sqrt{|z|}$, we have*

$$(3.0.7) \quad |w'(z)| \leq L_1 |z|.$$

3.1. Proof of Lemma 3.2. To get a contradiction, suppose that Lemma 3.2 is false. Then there is no θ_1 , no matter how small, for which there exists a finite L_1 such that (3.0.7) holds. Then there is a sequence z_n such that $z_n \rightarrow \infty$, $|w(z_n)|/\sqrt{|z_n|} \rightarrow 0$, and $|w'(z_n)/z_n| \rightarrow \infty$. By Lemma 3.1, we may assume that there exists $\kappa'_0 \in (0, \kappa_0]$ such that all the z_n are in $P(\kappa'_0)$. We may further assume that $|z_n| > 1$, $|w(z_n)| < \sqrt{|z_n|}$, and $|w'(z_n)/z_n| > 1$ for all n . Following Steinmetz, we note that

$$(3.1.8) \quad r(z_n) = \frac{1}{\sqrt{|w'(z_n)|}}$$

for all n . Furthermore, $r_n^2 |z_n| = |z_n/w'(z_n)| \rightarrow 0$ and $r_n \rightarrow 0$ as $n \rightarrow \infty$.

For a point z' , let $p(z')$ denote any one of the poles of w nearest to z' . The distance $|z' - p(z')|$ is well defined and the possible variation in $|p(z')|$ is not large. Set

$$\kappa_3(\theta_1) = \inf \left\{ |z' - p(z')| \sqrt{|p(z')|} : |w(z')| \leq \theta_1 \sqrt{|z'|}, |z'| \geq 1 \right\}.$$

If $\kappa_3(\theta_1) > 0$ for some $\theta_1 > 0$, we may replace κ_0 by $\min\{\kappa_0, \kappa_3(\theta_1)\}$ and achieve the situation where all the points z_n are outside $P(\kappa_0)$, which will finish the proof. Let us therefore assume that $\kappa_3(\theta_1) = 0$ for all $\theta_1 > 0$. Then we may assume that in addition to all previous conditions, we have

$$\lim_{n \rightarrow \infty} |z_n - p(z_n)| \sqrt{|p(z_n)|} = 0.$$

For each n we choose $m = m(n)$ so that $z_n \in B(p_m, \kappa'_0/\sqrt{|p_m|})$. This is possible since all the z_n are in $P(\kappa'_0)$. The above argument shows that by passing to a subsequence of z_n , if necessary, which we do without changing notation, we may assume that

$$(3.1.9) \quad |z_n - p_{m(n)}| = o(r(p_{m(n)})).$$

By passing to a subsequence, if necessary, without changing notation, we may assume that the limits

$$\begin{aligned} y_0 &= \lim_{n \rightarrow \infty} r(z_n)w(z_n) = \lim_{n \rightarrow \infty} w(z_n)/\sqrt{|w'(z_n)|} = 0, \\ y'_0 &= \lim_{n \rightarrow \infty} r(z_n)^2 w'(z_n) = \lim_{n \rightarrow \infty} w'(z_n)/|w'(z_n)|, \\ a &= \lim_{n \rightarrow \infty} r(z_n)^2 z_n = \lim_{n \rightarrow \infty} z_n/|w'(z_n)| = 0 \end{aligned}$$

exist. Then $|y'_0| = 1$.

We set, for a complex variable t ,

$$y_n(t) = r(z_n)w(z_n + r(z_n)t)$$

so that

$$y_n'' = 2y_n^3 + r(z_n)z_n y_n + r(z_n)^3 t y_n + r(z_n)^3 \alpha.$$

By passing to a subsequence we may assume that the functions y_n converge locally uniformly in \mathbb{C} to a limit function $y = y(t)$ satisfying the differential equation

$$(3.1.10) \quad y''(t) = 2y(t)^3$$

together with the initial value conditions $y(0) = 0$, $y'(0) = y'_0 \neq 0$.

Write $p_{m(n)} = b_n$. Note that $r(b_n) = 1/\sqrt{|b_n|}$. By Lemma 3.1, we can find a point $q_n \in S(b_n, \kappa'_0 r(b_n))$ such that $|w(q_n)| \leq \kappa_1 \sqrt{|q_n|}$ and $|w'(q_n)| \leq \kappa_2 |q_n| \leq 2\kappa_2 |z_n|$. Thus

$$(3.1.11) \quad \frac{2}{\sqrt{|z_n|}} \geq \frac{1}{\sqrt{|q_n|}} \geq r(q_n) \geq \frac{1}{\max\{\kappa_1, \sqrt{\kappa_2}\}} \frac{1}{\sqrt{|q_n|}} = \frac{\kappa_5}{\sqrt{|q_n|}} \geq \frac{\kappa_5}{2\sqrt{|z_n|}},$$

say.

By passing to a further subsequence, we may assume that the limits

$$\begin{aligned} u_0 &= \lim_{n \rightarrow \infty} r(q_n)w(q_n), \\ u'_0 &= \lim_{n \rightarrow \infty} r(q_n)^2 w'(q_n), \\ b &= \lim_{n \rightarrow \infty} r(q_n)^2 q_n \end{aligned}$$

exist. Then $|u_0| \leq \kappa_1$, $|u'_0| \leq \kappa_2$, and $\kappa_5^2 \leq |b| \leq 1$. We may also assume that

$$r(q_n)w(q_n + r(q_n)t) \rightarrow u(t)$$

locally uniformly in \mathbb{C} , where u is meromorphic in \mathbb{C} , $u(0) = u_0$, $u'(0) = u'_0$, and

$$(3.1.12) \quad u''(t) = 2u(t)^3 + bu(t).$$

We have $q_n + r(q_n)t = b_n$ for $t = t_n = (b_n - q_n)/r(q_n)$. Since

$$|b_n - q_n| \leq \kappa'_0/\sqrt{|b_n|} \leq 2\kappa'_0/\sqrt{|q_n|} \leq 4\kappa'_0 r(q_n),$$

the sequence t_n is bounded, so by passing to a subsequence we may assume that $t_n \rightarrow \tau \in \mathbb{C}$. Since $w(b_n) = \infty$, it follows from uniform convergence in the disk $B(0, 2|\tau|)$ that $u(\tau) = \infty$.

We have $q_n + r(q_n)t = z_n$ for

$$t = v_n = (z_n - q_n)/r(q_n) = t_n + (z_n - b_n)/r(q_n) = t_n + o(1)$$

by (3.1.9) and (3.1.11). Since $|r(q_n)w(z_n)| \leq 2|w(z_n)|/\sqrt{|z_n|} \rightarrow 0$, it follows from uniform convergence in the disk $B(0, 2|\tau|)$ that $u(\tau) = 0$. This is a contradiction. This proves Lemma 3.2.

4. A REPRESENTATION FOR THE SOLUTION

We will need to perform several estimates for a solution w of (1.0.1). In that connection, we will repeatedly need the following representation of w .

Lemma 4.1. *Let w be a solution of (1.0.1). Suppose that $\zeta \in \mathbb{C}$ and that w has no poles in the disk $B(\zeta, R)$. For $r, |z - \zeta| \in [0, R)$, write*

$$(4.0.13) \quad v(z) = w(z) - w(\zeta)$$

and

$$(4.0.14) \quad v_1(r) = \max\{|v(z)| : |z - \zeta| \leq r\}.$$

Then, if $z \in B(\zeta, R)$, we have

$$(4.0.15) \quad \frac{w(z) - w(\zeta)}{z - \zeta} - w'(\zeta) - \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2}(z - \zeta) - \frac{w(\zeta)}{6}(z - \zeta)^2 = \int_{\zeta}^z \left(\int_{\zeta}^y X_1(t) dt \right) dy,$$

where, for $t \in B(\zeta, R)$,

$$(4.0.16) \quad X_1(t) = v(t) \{ \zeta + (t - \zeta) + 6w(\zeta)^2 + 6w(\zeta)v(t) + 2v(t)^2 \}.$$

Furthermore,

$$(4.0.17) \quad |X_1(t)| \leq v_1(|t - \zeta|) \{ |\zeta| + |t - \zeta| + 6|w(\zeta)|^2 + 6|w(\zeta)|v_1(|t - \zeta|) + 2v_1(|t - \zeta|)^2 \}.$$

4.1. Proof of Lemma 4.1. Let the assumptions of Lemma 4.1 be satisfied, and let v and v_1 be defined by (4.0.13) and (4.0.14). With $z = \zeta + s$, we can write (1.0.1) in the form

$$\begin{aligned} w''(z) &= \alpha + (\zeta + s)(w(\zeta) + v) + 2(w(\zeta) + v)^3 \\ &= (\alpha + \zeta w(\zeta) + 2w(\zeta)^3) + sw(\zeta) + \zeta v + D_1 v, \end{aligned}$$

where

$$D_1 = D_1(z) = s + 6w(\zeta)^2 + v(6w(\zeta)) + 2v^2.$$

We integrate along line segments. It follows that

$$\begin{aligned} (4.1.18) \quad w'(z) - w'(\zeta) &= \int_{\zeta}^z w''(t) dt \\ &= \int_{\zeta}^z ((\alpha + \zeta w(\zeta) + 2w(\zeta)^3) + (t - \zeta)w(\zeta) + D_1(t)v(t)) dt \\ &\quad + \zeta \int_{\zeta}^z v(t) dt \\ &= (\alpha + \zeta w(\zeta) + 2w(\zeta)^3)(z - \zeta) + \frac{w(\zeta)}{2}(z - \zeta)^2 \\ &\quad + \int_{\zeta}^z D_1(t)v(t) dt + \zeta \int_{\zeta}^z v(t) dt. \end{aligned}$$

Therefore

$$\begin{aligned} (4.1.19) \quad w(z) &= w(\zeta) + w'(\zeta)(z - \zeta) + \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2}(z - \zeta)^2 \\ &\quad + \frac{w(\zeta)}{6}(z - \zeta)^3 + \int_{\zeta}^z \left(\int_{\zeta}^y D_1(t)v(t) dt \right) dy \\ &\quad + \zeta \int_{\zeta}^z \left(\int_{\zeta}^y v(t) dt \right) dy \end{aligned}$$

This yields (4.0.15). This completes the proof of Lemma 4.1.

5. SIZE OF THE SOLUTION ON TWO CIRCLES

We now apply Lemma 4.1 as follows. We assume that on a circle centered at a point ζ , there is at least one point z' where $|w(z') - w(\zeta)|$ is large, while inside the corresponding disk, $|w(z) - w(\zeta)| \leq |w(z') - w(\zeta)|$. We show that at every point z on a smaller circle centered at ζ , the quantity $|w(z) - w(\zeta)|$ is still sufficiently large. This will show that a component of the set where $w(z)$ is close to $w(\zeta)$ is contained in this smaller disk. This fact will then be applied in several situations in the sequel.

If $w(\zeta) \neq \infty$, we denote by $p(\zeta)$ a pole of w whose distance from ζ is as small as possible. Even though $p(\zeta)$ need not be unique, the distance $|\zeta - p(\zeta)|$ is well defined.

Lemma 5.1. *Let w be a solution of (1.0.1) with at least one finite pole. Suppose that $\zeta \in \mathbb{C} \setminus \{0\}$, $w(\zeta) \neq \infty$, and write $R = |\zeta - p(\zeta)|$. Suppose that $C > 0$ and define $K_0 \geq 0$ by $|w(\zeta)| = K_0\sqrt{|\zeta|}$.*

Let $\rho < R$ be the largest radius such that for all $z \in B(\zeta, \rho)$, we have $|w(z) - w(\zeta)| \leq C\sqrt{|\zeta|}$. Then

$$(5.0.20) \quad \max \left\{ |w'(\zeta)|, \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} \right| \rho \right\} \\ \geq \frac{C\sqrt{|\zeta|}}{2\rho} - \frac{K_0}{12}\rho^2\sqrt{|\zeta|} - \frac{\rho^2 C\sqrt{|\zeta|}}{4} \{ \rho + (1 + 6K_0^2 + 6K_0C + 2C^2)|\zeta| \},$$

and if $\eta \in (0, 1)$, then for all $z \in S(\zeta, \eta\rho)$, we have

$$(5.0.21) \quad |w(z) - w(\zeta)| \geq \eta\rho \left\{ \left| w'(\zeta) + \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2}(z - \zeta) \right| \right. \\ \left. - \frac{K_0}{6}\eta^2\rho^2\sqrt{|\zeta|} - \frac{\eta^2\rho^2 C\sqrt{|\zeta|}}{2} \{ \eta\rho + (1 + 6K_0^2 + 6K_0C + 2C^2)|\zeta| \} \right\}.$$

5.1. Proof of Lemma 5.1. Let the assumptions of Lemma 5.1 be satisfied, and let v and v_1 be defined by (4.0.13) and (4.0.14). Then there is a point z' such that $|z' - \zeta| = \rho$ and $|w(z') - w(\zeta)| = C\sqrt{|\zeta|}$. Applying (4.0.15) with $z = z'$, we obtain

$$(5.1.22) \quad \left| \frac{w(z') - w(\zeta)}{z' - \zeta} - w'(\zeta) - \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2}(z' - \zeta) \right| \\ \leq \left| \frac{w(\zeta)}{6}(z' - \zeta)^2 \right| + \int_{\zeta}^{z'} \left(\int_{\zeta}^y |X_1(t)| dt \right) dy.$$

From (4.0.17) and our assumptions, we obtain, for $t \in B(\zeta, \rho)$,

$$(5.1.23) \quad |X_1(t)| \leq C\sqrt{|\zeta|} \{ |\zeta| + \rho + 6K_0^2|\zeta| + 6K_0C|\zeta| + 2C^2|\zeta| \} \\ = C\sqrt{|\zeta|} \{ \rho + (1 + 6K_0^2 + 6K_0C + 2C^2)|\zeta| \}.$$

Hence

$$\int_{\zeta}^{z'} \left(\int_{\zeta}^y |X_1(t)| dt \right) dy \leq \frac{1}{2}\rho^2 C\sqrt{|\zeta|} \{ \rho + (1 + 6K_0^2 + 6K_0C + 2C^2)|\zeta| \}.$$

Thus we have

$$\begin{aligned}
(5.1.24) \quad & 2 \max \left\{ |w'(\zeta)|, \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} \right| \rho \right\} \\
& \geq |w'(\zeta)| + \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} (z' - \zeta) \right| \\
& \geq \left| w'(\zeta) + \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} (z' - \zeta) \right| \\
& = \left| \left(\frac{w(z') - w(\zeta)}{z' - \zeta} - w'(\zeta) - \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} (z' - \zeta) \right) - \frac{w(z') - w(\zeta)}{z' - \zeta} \right| \\
& \geq \left| \frac{w(z') - w(\zeta)}{z' - \zeta} \right| - \left| \frac{w(z') - w(\zeta)}{z - \zeta} - w'(\zeta) - \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} (z' - \zeta) \right| \\
& \geq \frac{C\sqrt{|\zeta|}}{\rho} - \frac{K_0\sqrt{|\zeta|}}{6} \rho^2 - \frac{1}{2} \rho^2 C \sqrt{|\zeta|} \{ \rho + (1 + 6K_0^2 + 6K_0C + 2C^2) |\zeta| \} \\
& = \frac{C\sqrt{|\zeta|}}{\rho} - \sqrt{|\zeta|} \rho^2 \left\{ \frac{K_0}{6} + \frac{1}{2} C \{ \rho + (1 + 6K_0^2 + 6K_0C + 2C^2) |\zeta| \} \right\}.
\end{aligned}$$

This yields (5.0.20).

The inequality (5.1.22) follows immediately from (4.0.15) and (5.1.23).

6. DISTANCE TO THE NEAREST POLE FROM POINTS WHERE THE SOLUTION IS SMALL

Later on, it is necessary for us to know that for every transcendental solution w of (1.0.1), if $|w(z)|/\sqrt{|z|}$ is small, then the distance from z to the nearest pole of w is $O(1/\sqrt{|z|})$. We will prove such a result in this Section. First we need to deal with the case when $|w(z)|/\sqrt{|z|}$ is equal to a small but fixed positive number.

Lemma 6.1. *Let w be a transcendental solution of (1.0.1). Then there are a small positive number $\kappa_7 \in (0, 1)$ such that for each $\kappa_9 \in (0, \kappa_7]$, there is a large positive number $\kappa_8 \geq 1$ depending on w and κ_9 , such that if $|z| \geq 1$ and $|w(z)| = \kappa_9 \sqrt{|z|}$, then*

$$(6.0.25) \quad |z - p(z)| < \kappa_8 / \sqrt{|z|}.$$

6.1. Proof of Lemma 6.1. To get a contradiction, suppose that Lemma 6.1 is false. Then there is no κ_9 , no matter how small, for which there exists a finite κ_8 such that (6.0.25) holds. Then for each small $\kappa_9 > 0$, there is a sequence z_n such that $z_n \rightarrow \infty$, $|w(z_n)|/\sqrt{|z_n|} = \kappa_9$, and

$$(6.1.26) \quad |z_n - p(z_n)| \sqrt{|z_n|} \rightarrow \infty.$$

We may further assume that $|z_n| > 1$, and $|z_n - p(z_n)|\sqrt{|z_n|} > 1$ for all n .

Taking κ_7 , and hence κ_9 , small enough, by Lemma 3.2 we may assume that $|w'(z_n)| \leq L_1|z_n|$ for all n . Hence

$$(6.1.27) \quad \frac{1}{\sqrt{L_1}\sqrt{|z_n|}} \leq r(z_n) \leq \frac{1}{\sqrt{|z_n|}}$$

for all n .

Following Steinmetz, we note that by passing to a subsequence, we may assume that the limits

$$\begin{aligned} y_0 &= \lim_{n \rightarrow \infty} r(z_n)w(z_n) \neq 0, \\ y'_0 &= \lim_{n \rightarrow \infty} r(z_n)^2w'(z_n), \\ a &= \lim_{n \rightarrow \infty} r(z_n)^2z_n \end{aligned}$$

exist. Indeed, $\kappa_9/\sqrt{L_1} \leq |y_0| \leq \kappa_9$. Then by (6.1.27), we have $a \neq 0$. We have $y'_0 = 0$ if, and only if, $w'(z_n)/z_n \rightarrow 0$.

We set, for a complex variable t ,

$$y_n(t) = r(z_n)w(z_n + r(z_n)t)$$

so that

$$y_n'' = 2y_n^3 + r(z_n)z_n y_n + r(z_n)^3 t y_n + r(z_n)^3 \alpha.$$

By passing to a subsequence we may assume that the functions y_n converge locally uniformly in \mathbb{C} to a limit function $y = y(t)$ satisfying the differential equation

$$(6.1.28) \quad y''(t) = 2y(t)^3 + ay(t)$$

together with the initial value conditions $y(0) = y_0 \neq 0$, $y'(0) = y'_0$.

Suppose that $y(t)$ is a non-constant solution of (6.1.28). We employ the usual notation of the Nevanlinna theory. Now, since $y''/y = 2y^2 + a$, it follows from Nevanlinna's lemma on the logarithmic derivative that

$$\begin{aligned} 2m(r, y) &= m(r, y^2) = m(r, 2y^2 + a) + O(1) \\ &= m(r, y''/y) + O(1) = O(\log T(r, y) + \log r) \end{aligned}$$

as $r \rightarrow \infty$ outside a set of finite linear measure, which implies that $N(r, y) = T(r, y)(1 - o(1))$. Thus, if y is transcendental, then y has infinitely many poles. If y is rational (but non-constant), it still follows that y has at least one finite pole.

If $y'_0 \neq 0$, the solution $y(t)$ to (6.1.28) is a non-constant (elliptic) function with infinitely many poles.

We assume for a while that $y'_0 \neq 0$. Fix a large but arbitrary $R > 0$. For $|t| \leq R$, and for all n so large that $|z_n - p(z_n)| > 2R/\sqrt{|z_n|}$, we have

$$|(z_n + r(z_n)t) - z_n| \leq Rr(z_n) \leq R/\sqrt{|z_n|} < (1/2)|z_n - p(z_n)|$$

so that $y_n(t) \neq \infty$. Since y_n does not converge to the constant infinity, it follows that y has no poles in $|t| \leq R$. Since R is arbitrary, it follows that y is entire, a contradiction. To reach a contradiction in this way, we only need to know that y has at least one finite pole. Our remarks on y being an elliptic function or having infinitely many poles, are merely side-remarks, so we do not try to justify them.

Suppose then that $y'_0 = 0$. Then the unique solution to (6.1.28) with the given initial conditions is the constant $y(t) \equiv \pm\sqrt{-a/2}$ if $y_0^2 = -a/2$, and otherwise $y(t)$ is a non-constant (elliptic) function with infinitely many poles. In the latter case, we get a contradiction as above.

Suppose that $y(t)$ is constant. If there are infinitely many n for which $r(z_n) = 1/\sqrt{|z_n|}$, we can pass to a subsequence and assume that this is true for all large n . In that case, $|a| = 1$ while $|y_0| = \kappa_9$, so that we cannot have $y_0^2 = -a/2$ provided that $\kappa_9 \leq \kappa_7 < 1/\sqrt{2}$. Thus we may assume that we have $r(z_n) = 1/\sqrt{|w'(z_n)|} < 1/\sqrt{|z_n|}$ for all n . But then $|y'_0| = 1$, so $y'_0 \neq 0$, a contradiction with the assumption that $y'_0 = 0$.

So in all cases we get a contradiction. This completes the proof of Lemma 6.1.

6.2. Estimates on circles. At this point, it is useful to state and prove yet another lemma, since it will be used in several ways later.

Lemma 6.2. *Let w be a transcendental solution of (1.0.1). Let $\kappa = \kappa_7 \in (0, 1)$ be as in Lemma 6.1. Suppose that $C > 1$ and that C satisfies the further conditions to be mentioned later. Suppose that $|\zeta| \geq 1$ and $0 < |w(\zeta)| = \kappa\sqrt{|\zeta|}$, and that $|\zeta - p(\zeta)| \leq s_0/\sqrt{|\zeta|}$ for some $s_0 > 0$. Let $s_1/\sqrt{|\zeta|}$, where $0 < s_1 < s_0$, be the largest radius such that for all z in the disk $B(\zeta, s_1/\sqrt{|\zeta|})$, we have*

$$|w(z) - w(\zeta)| < C\kappa\sqrt{|\zeta|}.$$

Then, provided that $|\zeta| \geq Z_0$, where Z_0 is as below, we have

$$|w(z) - w(\zeta)| > 2|w(\zeta)|$$

for all z with $|z - \zeta| = \eta s_1 / \sqrt{|\zeta|}$, and the function w has at least one zero in the disk $B(\zeta, \eta s_1 / \sqrt{|\zeta|})$, where $\eta \in (0, 1)$ and η and Z_0 are obtained as follows.

If $s_1 \leq 0.7$, we may use any $C \geq 30$ and $\eta \in (0, 1/2]$ as long as $C\kappa \leq 0.01$ and $C\eta \geq 6$. For example, we may take $\eta = 1/2$ provided that $\kappa \leq 1/3000$ and $30 \leq C \leq 1/(100\kappa)$. Here we take

$$Z_0 = \max \left\{ 165000, \left(\frac{5000|\alpha|}{\kappa} \right)^{2/3} \right\}.$$

If $s_1 > 0.7$ and

$$(6.2.29) \quad |w'(\zeta)| \geq |1 + B_1| \frac{\kappa s_1 |\zeta|}{2},$$

where $B_1 = (\alpha + 2w(\zeta)^3)/(\zeta w(\zeta))$, we may use any $C \geq 30$ and $\eta \in (0, 1/2]$ as long as $C\kappa \leq 0.01$ and $C\eta \geq 21$. For example, we may take $\eta = 1/2$ provided that $\kappa \leq 1/4200$ and $42 \leq C \leq 1/(100\kappa)$. Here

$$Z_0 = \max \left\{ 5000C^3, s_0^{2/3} \right\}.$$

If $s_1 > 0.7$ and

$$|w'(\zeta)| < |1 + B_1| \frac{\kappa s_1 |\zeta|}{2},$$

so that

$$(6.2.30) \quad t_0 = \frac{|w'(\zeta)|}{|1 + B_1| \frac{\kappa |\zeta|}{2}} < s_1,$$

we assume that $C\kappa \leq 1/100$, and we set

$$(6.2.31) \quad Z_0 = 23000C^2 \max \{1, s_0^2\}.$$

Then (i) we may choose any $\eta \in (0, t_0/(2s_1)]$ if $t_0 > 0$ and if we choose $\eta C \geq 9s_1/t_0$ and (ii) we may choose any $\eta \in [2t_0/s_1, 0.9]$ if $t_0/s_1 < 0.45$ and if we choose $\eta^2 C \geq 9$.

6.3. Proof of Lemma 6.2. Suppose that $|w(z) - w(\zeta)| > 2|w(\zeta)|$ for all z with $|z - \zeta| = \eta s_1 / \sqrt{|\zeta|}$. Then for these z , we have

$$|w(z)| \geq |w(z) - w(\zeta)| - |w(\zeta)| > |w(\zeta)| = |(w(z) - w(\zeta)) - w(z)|$$

so that we may apply Rouché's Theorem to the functions $w(z)$ and $w(z) - w(\zeta)$ in the disk $B(\zeta, \eta s_1 / \sqrt{|\zeta|})$. We see that in this disk, $w(z)$ and $w(z) - w(\zeta)$ have the same number of zeros, hence at least one zero, since $w(z) - w(\zeta)$ vanishes when $z = \zeta$. We conclude that w has at least one zero in $B(\zeta, \eta s_1 / \sqrt{|\zeta|})$. It remains for us to obtain the lower bound $|w(z) - w(\zeta)| > 2|w(\zeta)|$ on $S(\zeta, \eta s_1 / \sqrt{|\zeta|})$.

Note first that

$$\alpha + \zeta w(\zeta) + 2w(\zeta)^3 = \zeta w(\zeta)(1 + B_1).$$

We have

$$|B_1| \leq 2\kappa^2 + \frac{|\alpha|}{\kappa|\zeta|^{3/2}} \leq 1/100$$

provided that $\kappa \leq 1/100$ and $|\zeta| \geq (|\alpha|/(0.0098\kappa))^{2/3}$. The second inequality holds in all cases as soon as $|\zeta| \geq Z_0$.

We first apply Lemma 4.1, so that $0 < K_0 = \kappa$ and $\rho = s_1/\sqrt{|\zeta|}$. Note that the number that was denoted by C in Lemma 4.1 is now denoted by $C\kappa$. Now (5.0.20) implies that

$$(6.3.32) \quad \max \left\{ |w'(\zeta)|, \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} \right| \frac{s_1}{\sqrt{|\zeta|}} \right\} \\ \geq \frac{C\kappa|\zeta|}{2s_1} - \frac{\kappa s_1^2}{12\sqrt{|\zeta|}} - \frac{C\kappa s_1^2}{4\sqrt{|\zeta|}} \left\{ \frac{s_1}{\sqrt{|\zeta|}} + (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2\kappa^2)|\zeta| \right\}.$$

Suppose that $s_1 \leq 0.7$. Using the fact that $|\zeta| \geq Z_0$, $C\kappa \leq 1/100$, and $\kappa < 1$, we obtain

$$\left(\frac{\kappa s_1^2}{12\sqrt{|\zeta|}} + \frac{C\kappa s_1^2}{4\sqrt{|\zeta|}} \left\{ \frac{s_1}{\sqrt{|\zeta|}} + (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2\kappa^2)|\zeta| \right\} \right) \bigg/ \frac{C\kappa|\zeta|}{2s_1} \\ \leq \frac{1}{6C\kappa Z_0^{3/2}} + \frac{1}{2\sqrt{Z_0}} \left\{ \frac{1}{Z_0^{3/2}} + 7 + 6/100 + 1/5000 \right\} \leq 1/100.$$

Thus

$$\max \left\{ |w'(\zeta)|, \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} \right| \frac{s_1}{\sqrt{|\zeta|}} \right\} \geq \frac{0.99C\kappa|\zeta|}{2s_1}.$$

Suppose that

$$\max \left\{ |w'(\zeta)|, \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} \right| \frac{s_1}{\sqrt{|\zeta|}} \right\} = \left| \frac{\alpha + \zeta w(\zeta) + 2w(\zeta)^3}{2} \right| \frac{s_1}{\sqrt{|\zeta|}}.$$

Since $|B_1| \leq 1/100$, $\kappa \leq 1/100$, and $C > 1$, we then obtain

$$\frac{1.01s_1^2}{0.99} \geq C > 1,$$

which is a contradiction since $s_1^2 \leq 0.7^2 < 1/2$. Thus

$$|w'(\zeta)| \geq \frac{0.99C\kappa|\zeta|}{2s_1}.$$

From (5.0.21) we now deduce that for any $\eta \in (0, 1)$ and for all z with $|z - \zeta| = \eta s_1 / \sqrt{|\zeta|}$, we have

$$\begin{aligned} |w(z) - w(\zeta)| &\geq \eta \frac{s_1}{\sqrt{|\zeta|}} \left\{ \left| w'(\zeta) - (1 + B_1) \frac{\zeta w(\zeta)}{2} (z - \zeta) \right| \right. \\ &\quad \left. - \frac{\kappa \eta^2 s_1^2}{6\sqrt{|\zeta|}} - \frac{\eta^2 s_1^2 C \kappa}{2\sqrt{|\zeta|}} \left\{ \frac{\eta s_1}{\sqrt{|\zeta|}} + (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2 \kappa^2) |\zeta| \right\} \right\} \\ &\geq \eta \frac{s_1}{\sqrt{|\zeta|}} \frac{0.99 C \kappa |\zeta|}{2s_1} - \eta \frac{s_1}{\sqrt{|\zeta|}} 1.01 \eta \frac{s_1}{\sqrt{|\zeta|}} \frac{\kappa |\zeta|^{3/2}}{2} - \frac{\kappa \eta^2 s_1^2}{6\sqrt{|\zeta|}} \\ &\quad - \frac{\eta^2 s_1^2 C \kappa}{2\sqrt{|\zeta|}} \left\{ \frac{\eta s_1}{\sqrt{|\zeta|}} + (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2 \kappa^2) |\zeta| \right\} \\ &= A_1 - A_2, \end{aligned}$$

say, where

$$A_1 = \eta \frac{s_1}{\sqrt{|\zeta|}} \frac{0.99 C \kappa |\zeta|}{2s_1} = \frac{0.99 \eta C \kappa \sqrt{|\zeta|}}{2}.$$

We have

$$\begin{aligned} \frac{A_2}{\eta A_1} &= \frac{1.01 s_1^2}{0.99 C} + \frac{s_1^2}{2.97 C |\zeta|} + \frac{s_1^2}{0.99 |\zeta|} \frac{\eta s_1}{\sqrt{|\zeta|}} \\ &\quad + \frac{s_1^2}{0.99} (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2 \kappa^2) < 0.52 \end{aligned}$$

since $s_1 \leq 0.7$, $C \geq 30$, $C\kappa \leq 1/100$, provided that $|\zeta| \geq 5000$, which is true for $|\zeta| \geq Z_0$. We deduce that

$$|w(z) - w(\zeta)| \geq \frac{0.99 \eta C \kappa \sqrt{|\zeta|}}{2} (1 - 0.52 \eta).$$

Thus

$$|w(z) - w(\zeta)| > 2\kappa \sqrt{|\zeta|} = 2|w(\zeta)|$$

provided that

$$\eta C (1 - 0.52 \eta) > 4/0.99,$$

and this holds if $\eta \leq 1/2$ and $C\eta \geq 5.5$.

Assume next that $s_1 > 0.7$. Suppose first that

$$|w'(\zeta)| \geq |1 + B_1| \frac{\kappa s_1 |\zeta|}{2}.$$

Next, we first consider the case when $s_1^2 \geq C/2$.

From (5.0.21) we obtain, whenever $|z - \zeta| = \eta s_1 / \sqrt{|\zeta|}$, that (recall that C in (5.0.21) must be replaced by $C\kappa$)

$$|w(z) - w(\zeta)| \geq |1 + B_1| \frac{\eta \kappa s_1^2 \sqrt{|\zeta|}}{2} - \frac{(1.01)\kappa \eta^2 s_1^2}{2} \sqrt{|\zeta|} - A_3,$$

where

$$\begin{aligned} A_3 &= \frac{\eta s_1}{\sqrt{|\zeta|}} \left\{ \frac{\kappa \eta^2 s_1^2}{6\sqrt{|\zeta|}} + \frac{\eta^2 s_1^2 C \kappa}{2\sqrt{|\zeta|}} \left\{ \frac{\eta s_1}{\sqrt{|\zeta|}} + (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2 \kappa^2) |\zeta| \right\} \right\} \\ &\leq \frac{\eta^3 s_1^3 \kappa}{6|\zeta|} + \frac{\eta^3 s_1^3 C \kappa}{2|\zeta|} \left(\frac{\eta s_1}{\sqrt{|\zeta|}} + 1.0014 |\zeta| \right) \end{aligned}$$

so that

$$\begin{aligned} &\left(\frac{(1.01)\kappa \eta^2 s_1^2}{2} \sqrt{|\zeta|} + A_3 \right) / \frac{\eta \kappa s_1^2 \sqrt{|\zeta|}}{2} \\ &\leq 1.01\eta + \frac{\eta^2 s_1}{3|\zeta|^{3/2}} + \frac{\eta^3 s_1^2 C}{|\zeta|^2} + \frac{1.0014 \eta^2 s_1 C}{\sqrt{|\zeta|}} \leq 1.02\eta \end{aligned}$$

if $|\zeta| \geq Z_0$, since also $C \leq 2s_1^2 < 2s_0^2$, by our assumption.

Hence

$$|w(z) - w(\zeta)| \geq \frac{|w(\zeta)| s_1}{2} (0.99 - 1.02\eta) \eta s_1 > 2|w(\zeta)|$$

if $\eta s_1^2 (0.99 - 1.02\eta) > 4$.

Recall that $s_1^2 \geq C/2$. If we choose $\eta \leq 1/2$, it suffices to have $\eta C > 8/(0.99 - 0.51) \approx 16.7$. This condition is satisfied by the assumptions of Lemma 6.2.

If $s_1^2 < C/2$, then, instead of the above, we use the fact that $|w'(\zeta)| \geq (C\kappa|\zeta|)/(4s_1)$. For if this is not true, let us first note (recall that C in (5.0.20) must be replaced by $C\kappa$) that the right hand side of (5.0.20) is

$$\geq \frac{C\kappa|\zeta|}{2s_1} - A_4,$$

where

$$A_4 = \frac{\kappa}{12\sqrt{|\zeta|}} s_1^2 + \frac{s_1^2 C \kappa}{4\sqrt{|\zeta|}} \left\{ \frac{s_1}{\sqrt{|\zeta|}} + (1 + 6\kappa^2 + 6\kappa^2 C + 2C^2 \kappa^2) |\zeta| \right\}$$

so that

$$A_4 / \frac{C\kappa|\zeta|}{2s_1} \leq \frac{s_1^3}{6C|\zeta|^{3/2}} + \frac{s_1^4}{2|\zeta|^2} + \frac{1.0014}{2\sqrt{|\zeta|}} s_1^3 < 2 \frac{s_1^3}{\sqrt{|\zeta|}} < 2 \frac{(C/2)^{3/2}}{\sqrt{|\zeta|}}$$

if $|\zeta| \geq \max\{1, s_0^{2/3}\}$, which is true for $|\zeta| \geq Z_0$. Thus for $|\zeta| \geq Z_0$ the right hand side of (5.0.20) is

$$\geq \frac{C\kappa|\zeta|}{2s_1}0.99.$$

Now, if $|w'(\zeta)| \geq (C\kappa|\zeta|)/(4s_1)$ does not hold, then in view of (5.0.20) having to be true and the right hand side of (5.0.20) having the above lower bound, it follows that $|w'(\zeta)|$ cannot be the larger term on the left hand side of (5.0.20), so considering the other term on the left hand side of (5.0.20), we deduce, also using $|1 + B_1| \leq 1.01$, that

$$1.01 \frac{s_1\kappa|\zeta|}{2} \geq \frac{C\kappa|\zeta|}{2s_1}0.99,$$

which contradicts $s_1^2 < C/2$.

Consequently, we must have $|w'(\zeta)| \geq (C\kappa|\zeta|)/(4s_1)$, and hence we have by (5.0.21) for $|z - \zeta| = \eta s_1 / \sqrt{|\zeta|}$,

$$\left| \frac{w(z) - w(\zeta)}{z - \zeta} \right| \geq \frac{0.99C\kappa|\zeta|}{2s_1} - \frac{(1.01)\kappa\eta s_1}{2}|\zeta| - A_3/(\eta s_1/\sqrt{|\zeta|}),$$

where A_3 is as above, and hence (for $|\zeta| \geq Z_0$ and since $s_1^2 < C/2$),

$$\begin{aligned} |w(z) - w(\zeta)| &\geq \left(\frac{0.99C\eta}{2} - \frac{1.01}{2}\eta^2 s_1^2 - \frac{\eta^2 s_1^2}{200} \right) |w(\zeta)| \\ &\geq (0.99 - 0.51\eta) \frac{C\eta}{2} |w(\zeta)|. \end{aligned}$$

This will be $> 2|w(\zeta)|$ if $\eta \leq 1/2$ and $C\eta > 4/(0.99 - 0.51/2) \approx 5.442$.

Thus we may now assume that $s_1 > 0.7$ and

$$|w'(\zeta)| < |1 + B_1| \frac{\kappa s_1 |\zeta|}{2}.$$

so that by (5.0.20), we necessarily have

$$|1 + B_1| \frac{\kappa s_1 |\zeta|}{2} \geq \frac{0.99C\kappa}{2s_1} |\zeta|$$

and hence

$$(6.3.33) \quad 1.01s_1^2 \geq 0.99C.$$

Let t_0 be as defined in (6.2.30).

—Suppose that $t_0 < s_1$. Suppose that $t_0 > 0$ and that $0 < \eta \leq t_0/(2s_1) < 1/2$. Now we obtain for $|z - \zeta| = \eta s_1/\sqrt{|\zeta|}$,

$$\begin{aligned}
& \left| \frac{w(z) - w(\zeta)}{z - \zeta} \right| \geq \\
& |w'(\zeta)| - \frac{|1 + B_1| \kappa \eta s_1}{2} |\zeta| - \frac{A_3 \sqrt{|\zeta|}}{\eta s_1} \\
& > t_0 |1 + B_1| \frac{\kappa |\zeta|}{2} - \frac{|1 + B_1| \kappa \eta s_1}{2} |\zeta| - \frac{s_1 |w(\zeta)| \sqrt{|\zeta|} \eta^2}{100} \\
(6.3.34) \quad & = \frac{\kappa s_1 |\zeta|}{2} \left(\left(\frac{t_0}{s_1} - \eta \right) |1 + B_1| - \frac{\eta^2}{50} \right) \\
& > 0.1955 s_1 \kappa |\zeta|
\end{aligned}$$

and hence

$$\begin{aligned}
& \left| \frac{w(z) - w(\zeta)}{z - \zeta} \right| \geq \\
& \frac{\kappa s_1 |\zeta|}{2} \left(\left(\frac{t_0}{s_1} - \eta \right) |1 + B_1| - \frac{\eta^2}{50} \right) \\
& \geq \frac{\kappa s_1 |\zeta|}{2} \frac{t_0}{s_1} \left(\frac{0.99}{2} - \frac{1}{2000} \right) \\
& \geq 0.24725 \kappa s_1 |\zeta| \frac{t_0}{s_1}
\end{aligned}$$

so that, also by (6.3.33),

$$\frac{|w(z) - w(\zeta)|}{2|w(\zeta)|} > \frac{0.24725}{2} \eta s_1^2 \frac{t_0}{s_1} > 0.12 \eta C \frac{t_0}{s_1} > 1$$

provided that $\eta C \geq 9s_1/t_0$.

Then suppose that $s_1 > 0.7$ and that $t_0/s_1 < 0.45$, and that $\eta \in [2t_0/s_1, 0.9]$. This time we obtain for $|z - \zeta| = \eta s_1/\sqrt{|\zeta|}$,

$$\begin{aligned}
& \frac{|w(z) - w(\zeta)|}{2|w(\zeta)|} > \frac{\eta s_1^2}{4} \left(\left(\eta - \frac{t_0}{s_1} \right) |1 + B_1| - 2\eta^2/100 \right) \\
& \geq (\eta - (t_0/s_1)) \frac{\eta C}{8.08} |1 + B_1| > \eta \frac{0.99C}{4.04} (0.05 \cdot 0.99 - 0.95^2 \cdot 0.02) > 1
\end{aligned}$$

and hence

$$\begin{aligned}
& \frac{|w(z) - w(\zeta)|}{2|w(\zeta)|} > \frac{\eta s_1^2}{4} \left(\left(\eta - \frac{t_0}{s_1} \right) |1 + B_1| - \frac{2\eta^2}{100} \right) \\
& \geq \frac{\eta^2 0.99C}{4.04} \left(\frac{0.99}{2} - \frac{\eta}{50} \right) \geq 0.116 \eta^2 C > 1
\end{aligned}$$

provided that $\eta^2 C \geq 9$.

This completes the proof of Lemma 6.2.

6.4. Distance to the nearest pole from points where the solution is arbitrarily small. We have seen that the nearest pole of w is not too far away from a point z where $|w(z)|/\sqrt{|z|}$ is small but fixed, and the maximum distance to the nearest pole may, as far as we know, exhibit a dependence on exactly how small $|w(z)|/\sqrt{|z|}$ is. We now wish to show that there is no such dependence.

Lemma 6.3. *Let w be a transcendental solution of (1.0.1). Then there are a small positive number $\kappa_7 \in (0, 1)$ and a large positive number $\kappa_8 \geq 1$ depending on w such that if $|z| \geq 1$ and $|w(z)| \leq \kappa_7 \sqrt{|z|}$, then*

$$(6.4.35) \quad |z - p(z)| < \kappa_8 / \sqrt{|z|}.$$

Proof of Lemma 6.3. Let $\kappa_7 > 0$ be as in Lemma 6.1. For the rest of the proof of this lemma, write κ for κ_7 .

Suppose that $|z_0| \geq 1$ and $|w(z_0)| < \kappa \sqrt{|z_0|}$. (If $|w(z_0)| = \kappa \sqrt{|z_0|}$, the conclusion follows from Lemma 6.1.)

It may be noted that the conclusion of Lemma 6.3 is not valid for a rational solution of (1.0.1) since such a solution w has only finitely many poles while $w(z) \rightarrow 0$ as $z \rightarrow \infty$.

We first make essential use of the assumption that w be a transcendental solution of (1.0.1). In this case it is known (e.g., [3], Lemma 2.4.2, p. 39, with $f = w$, $n = 2$, $P(z, f) = 2w$) that w has a small Nevanlinna proximity function, forcing w to have infinitely many poles. Let p be a pole of w with $|p| \geq |z_0|$, not necessarily the pole of w nearest to z_0 . Join z_0 to p by a path γ such that $|z| \geq |z_0|$ for all $z \in \gamma$.

If there is no p for which this can be achieved by taking γ to be the line segment from z_0 to p , then we can take γ to be the radial segment $[z_0, |p|z_0/|z_0|]$ followed by an arc of the circle $S(0, |p|)$ from $|p|z_0/|z_0|$ to p .

On γ , when moving from z_0 to p , we find the last point ζ such that $|w(\zeta)| = \kappa \sqrt{|\zeta|}$ and such that on the part of γ from z_0 to ζ , we have $|w(z)| < \kappa \sqrt{|z|}$.

Since κ is fixed, there is, by Lemma 6.1, a fixed $s_0 > 0$ such that $|\zeta - p(\zeta)| \leq s_0 / \sqrt{|\zeta|}$. We may have $|p(\zeta)| < |z_0|$. Let s_1 be as in Lemma 6.1.

Suppose that $|z_0 - \zeta| \geq \eta s_1 / \sqrt{|\zeta|}$. The part of the path γ from z_0 to ζ then contains a point z on the circle $S(\zeta, \eta s_1 / \sqrt{|\zeta|})$, at which we must have $|w(z) - w(\zeta)| < \kappa \sqrt{|\zeta|} = |w(\zeta)|$ by the definition of ζ .

We now obtain a contradiction by applying Lemma 6.2 to the point ζ . We only have to assume that κ is so small that Lemma 6.1 and Lemma 6.2 are both applicable. This means that in addition to the implicit condition arising from Lemma 6.1, we choose $\kappa \leq 1/26000$. This choice of κ next gives rise to a number s_0 . Further, we assume that $C = 137$ and

$$|z_0| \geq \max \left\{ 165000, \left(\frac{5000|\alpha|}{\kappa} \right)^{2/3}, 5000C^3, 23000C^2 \max \{1, s_0^2\} \right\}.$$

Then Lemma 6.2 implies that for all $z \in S(\zeta, \eta s_1/\sqrt{|\zeta|})$, we have $|w(z) - w(\zeta)| > 2|w(\zeta)|$. This is a contradiction.

It follows that $|z_0 - \zeta| < \eta s_1/\sqrt{|\zeta|} < s_0/\sqrt{|\zeta|}$. Thus we obtain the upper bound

$$|z_0 - p(z_0)| \leq |z_0 - p(\zeta)| < 2s_0/\sqrt{|\zeta|} \leq 2s_0/\sqrt{|z_0|}$$

since $|\zeta| \geq |z_0|$, for a certain fixed s_0 , which proves Lemma 6.3 with $\kappa_8 = 2s_0$.

7. DISTANCE TO THE NEAREST ZERO FROM POINTS WHERE THE SOLUTION IS SMALL

We continue to consider a given transcendental solution w of (1.0.1). Following Steinmetz, if the zeros of w of modulus ≥ 1 are labelled as q_n , and if $\kappa > 0$, we write

$$Q(\kappa) = \cup_n B(q_n, \kappa/\sqrt{|q_n|}).$$

In the definition of Steinmetz, one uses $r(q_n)$ instead of $1/\sqrt{|q_n|}$ in the radius of the disks that form $Q(\kappa)$. In view of Lemma 3.2, it does not amount to a great change to use $1/\sqrt{|q_n|}$. We prove the following result.

Lemma 7.1. *There is a number $\kappa_3 \in (0, 1)$ such that if $0 < \kappa \leq \kappa_3$, then there are numbers $R_0 > 2$ and $\kappa_4 \in (0, 1)$ depending on w and on κ such that if $|z| > R_0$ and $|w(z)| < \kappa\sqrt{|z|}$, then*

$$z \in Q(\kappa_4).$$

Furthermore, we may choose κ_4 as a function of κ so that

$$\lim_{\kappa \rightarrow 0} \kappa_4 = 0.$$

Remark 1. A comment in the paper N. Steinmetz, Value distribution of the Painlevé transcendents, Israel Journal of Math. 128 (2002), 29–52, contains a claim that implies Lemma 7.1 (it refers to Proposition 2 in that paper, which is for the first Painlevé equation, and claims

that the proof is the same for (1.0.1)). However, in the subsequent paper from 2005 cited above, this claim is effectively retracted as it is replaced by the conclusion, in Proposition 3.2(a) of this more recent paper, that we only have

$$\sqrt{|z|} = O(|w(z) - \sqrt{z}|)$$

outside the set corresponding to $Q(\kappa_4)$. But this does not prevent $|w(z)|$ from being very small outside $Q(\kappa_4)$ even if the zeros of w , by definition, are contained in $Q(\kappa_4)$. This suggests that a different method of proof must be considered.

Remark 2. In Lemma 7.1, it is essential that κ_3 is small enough. Namely, by Lemma 3.1, the set $\{z \in \mathbb{C} : |w(z)| < \kappa_2 \sqrt{|z|}\}$ has an unbounded component, which cannot be covered by the countable union of small disjoint disks.

Remark 3. In Lemma 7.1, it is to be expected that $R_0 \rightarrow \infty$ as $\kappa \rightarrow 0$.

7.1. Proof of Lemma 7.1. Let w be a given transcendental solution of (1.0.1). We assume that the previous lemmas have been applied to this w , yielding certain numbers as discussed below. Suppose that the number $\kappa_7 \in (0, 1)$ has been chosen to be small enough to satisfy the conditions of Lemmas 6.3 and 6.1. Suppose that $\varepsilon \in (0, 1)$. We need to show that there are numbers $\kappa \in (0, \kappa_7]$ and $R_0 > 2$ such that if $|z_0| \geq R_0$ and $|w(z_0)| < \kappa|z_0|$, then there is a zero q of w such that $|z_0 - q| < \varepsilon/\sqrt{|q|}$. This then means that we may take $\kappa_4 = \varepsilon$ in the statement of Lemma 7.1, and since $\varepsilon \in (0, 1)$ is arbitrary, this will then prove Lemma 7.1.

Let $\kappa_8 > 1$ be the number that arises from Lemma 6.3. Define $\eta_0 = \varepsilon/(4\kappa_8) < 1/4$. We shall show that we may take

$$\kappa = \min\{\kappa_7, \eta_0^2/7200\}$$

and, with $C = 72/\eta_0^2$,

$$R_1 = \max\left\{165000, \left(\frac{5000|\alpha|}{\kappa}\right)^{2/3}, 5000C^3, 23000C^2\kappa_8^2\right\},$$

$$R_0 = R_1 + \kappa_8/\sqrt{R_1}.$$

Choose a point z_0 with $|z_0| \geq R_0$ and $|w(z_0)| < \kappa|z_0|$. By Lemma 6.3, the pole $p(z_0)$ of w nearest to z_0 satisfies $|z_0 - p(z_0)| < \kappa_8/\sqrt{|z_0|}$. When we move along the line segment from z_0 to $p(z_0)$, let ζ be the first point z we encounter such that $|w(z)| = \kappa_7\sqrt{|z|}$. Thus $|w(\zeta)| = \kappa_7\sqrt{|\zeta|}$. We clearly have $|\zeta| \geq R_1$. We apply Lemma 6.2 to ζ , choosing C as above

and choosing η as follows. Our choice for C leads to a certain number s_1 with $s_1 < \kappa_8$.

We next explain how we choose η in the various cases when applying Lemma 6.2 and verify that the required conditions are satisfied for this choice of η . In all cases below, it is clear that our choice of R_1 satisfies $R_1 \geq Z_0$ (note that $s_0 \leq \kappa_8$). The condition $C\kappa \leq 1/100$ is always satisfied.

If $s_1 \leq 0.7$, we take $\eta = \eta_0$. Then our choice of C is valid provided that $C = 72/\eta_0^2 \geq 30$ and $C\eta = 72/\eta_0 \geq 6$. These conditions are satisfied.

If $s_1 > 0.7$ and (6.2.29) holds, we take $\eta = \eta_0$. Then our choice of C is valid provided that $C = 72/\eta_0^2 \geq 30$ and $C\eta = 72/\eta_0 \geq 21$. These conditions are satisfied.

If $s_1 > 0.7$ and (6.2.29) does not hold, so that $t_0 < s_1$, suppose first that $2t_0/s_1 \leq \eta_0$. Then we take $\eta = \eta_0$. Our choice of C is valid provided that $C\eta^2 = 72 \geq 9$. This condition is satisfied.

Suppose then that $s_1 > 0.7$, that (6.2.29) does not hold, and that $2t_0/s_1 > \eta_0$. Then we set $\eta = \eta_0/4$. Our choice of C is valid provided that $C\eta = 18/\eta_0 \geq 9s_1/t_0$. This condition is satisfied.

In each case, we took $\eta = \eta_0$ or $\eta = \eta_0/4$. Now Lemma 6.2 implies that

$$|z_0 - \zeta| < \eta s_1 / \sqrt{|\zeta|} < \eta_0 \kappa_8 / \sqrt{|\zeta|}$$

and that there is a zero q of w with $|q - \zeta| < \eta s_1 / \sqrt{|\zeta|}$. Thus

$$|z_0 - q| < 2\eta_0 \kappa_8 / \sqrt{|\zeta|}.$$

We have $|q| < |\zeta| + 1/\sqrt{|\zeta|} < 2|\zeta|$, hence

$$|z_0 - q| < 4\eta_0 \kappa_8 / \sqrt{|q|} < \varepsilon / \sqrt{|q|},$$

as required. This completes the proof of Lemma 7.1.

8. SIZE OF THE SET WHERE THE SOLUTION IS CLOSE TO A GIVEN VALUE

Let w be a transcendental solution of (1.0.1). In the proof of our main result, we will need to know that for certain values A of w , if $w(z_1) = A$ and $\varepsilon_1 > 0$ is small, the component of the set $\{z : |w(z) - A| < \varepsilon_1 \sqrt{|z_1|}\}$ containing z_1 is a subset of a disk of the form $B(z_1, \varepsilon_3 / \sqrt{|z_1|})$ for a suitable choice of ε_3 . Furthermore, we need to know that each point z with $|w(z) - A| < \varepsilon_1 \sqrt{|z|}$ is contained in such a component and hence in such a disk. This then shows that if we remove from the plane a small exceptional set, contained in the union of such disks, we will be

able to avoid the value A for w by a definite amount. This will be necessary for us to obtain estimates for certain error terms.

If $|A|/\sqrt{|z_1|}$ is small, we are essentially avoiding the set where w takes rather small values, and the size of that set has been estimated already. Hence we assume in this Section that $|A|/\sqrt{|z_1|}$ is not too close to zero.

Let us begin with a point z_0 with a sufficiently large modulus. We consider the following at most four distinct numbers

$$(8.0.36) \quad \pm \sqrt{z_0 \left(-\frac{1}{2} \pm \sqrt{\frac{U(z_0)}{z_0^2} + \frac{1}{4}} \right)},$$

and we denote any one of these four values by A . If we choose z_0 so that $|w(z_0)| \leq K\sqrt{|z_0|}$ for a fixed positive constant $K > 3$, then for each of the four choices of A , we have $|A| \leq K_1\sqrt{|z_0|}$ where K_1 depends only on w and on K .

Note that

$$(8.0.37) \quad \left| \frac{A^2}{z_0} + \frac{1}{2} \right|^2 = \left| \frac{U(z_0)}{z_0^2} + \frac{1}{4} \right|.$$

In the other direction, if we fix a large K and limit our attention to the set where $|w(z)| \leq K\sqrt{|z|}$, and if $|A|/\sqrt{|z_0|} - \varepsilon_1 > K$, then we will have $|w(z) - A| \geq \varepsilon_1\sqrt{|z|}$ for all z that we consider. Therefore we may assume that the value A that we consider satisfies $|A| \leq (K + \varepsilon_1)\sqrt{|z_0|}$.

We choose any $\varepsilon, \varepsilon_j \in (0, 1/100)$. Thus $K + \varepsilon_1 < K + 1$. We set $K_1 = (K + 2\varepsilon_1)/\sqrt{1 - \varepsilon} < 2K - 1$. By a result of Steinmetz [17], Proposition 3.5, p. 79, there is a number $K_2 > 1$ such that $|U(z)| \leq K_2|z|^2$ whenever $|w(z)| \leq 2K\sqrt{|z|}$. We set $K_3 = (2K)^2 + 2K_2$, $K_4 = 2\pi K_3$, and $K_5 = 4K_4 + 2K_2$. Then we take $K_6 = 79K^3 + 2|\alpha|$ and set $K_7 = 1 + \sqrt{K_5 + K_6}$. We further write $L = 10^{10}K_7$ and $K_9 = K\varepsilon_3 + L\varepsilon_3 + L^3\varepsilon_3(3 + 36K^2)$.

So if w and K are given, this determines K_2 through K_7 , and L and K_9 .

Lemma 8.1. *Let w be a transcendental solution of (1.0.1). Let $\varepsilon \in (1/100)$, $\varepsilon_0 \in (1/100)$ and $\varepsilon_2 \in (1/100)$ be small positive numbers, and let $K > 3$ be a large positive number. Suppose that $\varepsilon_0 \leq \kappa_7$, where κ_7 is as in Lemma 6.3. Suppose that ε and $\varepsilon_1 \in (1/100)$ are such that $2\varepsilon_1 < (1 - \varepsilon)\varepsilon_0$, $4K\varepsilon_1 < \varepsilon_2$, $\varepsilon \leq \varepsilon_1$, and*

$$\varepsilon_3 = \frac{10^6 K_7 \sqrt{\varepsilon_1}}{\varepsilon_0 \varepsilon_2} < 1/100.$$

Define $L = 10^{10}K_7$ and

$$Z_1 = \max \left\{ \frac{1}{\varepsilon_3}, \frac{512 \cdot 10^{12} K_9^2 L^6}{\varepsilon_0^2 \varepsilon_2^2 \varepsilon_1^4} (1 + |\alpha|), \frac{16 \cdot 10^8 K_7^2 (1 + |\alpha|)^2}{\varepsilon_1 \varepsilon_0^2 \varepsilon_2^2} \right\}.$$

Suppose that $|z_0| \geq Z_1$ and $|w(z_0)| \leq K\sqrt{|z_0|}$, and let A be any one of the four numbers given by (8.0.36). Suppose that

$$\varepsilon_0 \sqrt{|z_0|} \leq |A| \leq (K + \varepsilon_1) \sqrt{|z_0|}$$

and that

$$\left| \frac{A^2}{z_0} + \frac{1}{2} \right| \geq \varepsilon_2.$$

Suppose that $z_1 \in B(z_0, \varepsilon|z_0|)$ and that $|w(z_1) - A| < \varepsilon_1 \sqrt{|z_0|}$.

Then there are a radius $\rho \leq \varepsilon_3/\sqrt{|z_0|}$ and a point z' such that $w(z') = A$ and $|z_1 - z'| < \rho$. Furthermore, we have $|w(z) - A| > \varepsilon_1 \sqrt{|z_0|}$ for all z with $|z - z_1| = \rho$. Thus the component of the set $\{z : |w(z) - A| < \varepsilon_1 \sqrt{|z_0|}\}$ containing the point z_1 is a subset of the disk $B(z_1, \varepsilon_3/\sqrt{|z_0|})$.

8.1. Proof of Lemma 8.1. Let the assumptions of Lemma 8.1 be satisfied. Then

$$|w(z_1)| > |A| - \varepsilon_1 \sqrt{|z_0|} \geq (\varepsilon_0 - \varepsilon_1) \sqrt{|z_0|} > (1/2)\varepsilon_0 \max\{\sqrt{|z_0|}, \sqrt{|z_1|}\}.$$

Next, we have

$$|w(z_1)| < (K + 2\varepsilon_1) \sqrt{1 - \varepsilon}^{-1} \sqrt{|z_1|} = K_1 \sqrt{|z_1|}.$$

We may join z_0 to z_1 by a path γ of length $\leq \pi|z_1 - z_0|$ in $B(z_0, \varepsilon|z_0|)$ such that $|w(z)| \leq K_1 \sqrt{|z|}$ for all $z \in \gamma$. On γ , we have

$$\left(\frac{U}{z^2} \right)' = \frac{U'}{z^2} - \frac{2U}{z^3}$$

so that since $U' = w^2$, we have

$$\left| \left(\frac{U}{z^2} \right)' \right| \leq \frac{K_1^2}{|z|} + \frac{2K_2}{|z|} \leq \frac{K_3}{|z|}.$$

Hence, integrating along a path from z_0 to z_1 where $|w(z)| \leq K_1 \sqrt{|z|}$, we obtain

$$\left| \frac{U(z_1)}{z_1^2} - \frac{U(z_0)}{z_0^2} \right| \leq \int_{z_0}^{z_1} \left| \left(\frac{U(z)}{z^2} \right)' \right| |dz| \leq \frac{\pi|z_1 - z_0|K_3}{(1 - \varepsilon)|z_0|} = K_4 \frac{|z_1 - z_0|}{|z_0|} < K_4 \varepsilon.$$

We next note that from (8.0.36), we obtain

$$(A^2 + z_0/2)^2 = U(z_0) + z_0^2/4,$$

so that

$$A^4 + z_0 A^2 = U(z_0).$$

Write $w(z_1) = A + \theta\sqrt{|z_0|}$, where θ is a complex number with $|\theta| < \varepsilon_1$. Since

$$(w')^2 = w^4 + zw^2 + 2\alpha w - U,$$

we get

$$\begin{aligned} w'(z_1)^2 &= (A + \theta\sqrt{|z_0|})^4 + z_1(A + \theta\sqrt{|z_0|})^2 + 2\alpha(A + \theta\sqrt{|z_0|}) - U(z_1) \\ &= 4A^3\theta\sqrt{|z_0|} + 6A^2\theta^2|z_0| + 4A\theta^3|z_0|^{3/2} + \theta^4|z_0|^2 + 2\alpha A \\ &\quad + (z_1 - z_0)A^2 + 2\theta Az_1\sqrt{|z_0|} + z_1\theta^2|z_0| + 2\alpha\theta\sqrt{|z_0|} + U(z_0) - U(z_1). \end{aligned}$$

We have

$$\begin{aligned} |U(z_0) - U(z_1)| &= \left| z_1^2 \left(\frac{U(z_1)}{z_1^2} - \frac{U(z_0)}{z_0^2} \right) + \frac{U(z_0)}{z_0^2} (z_1 - z_0)(z_1 + z_0) \right| \\ &\leq (1 + \varepsilon)^2 |z_0|^2 K_4 \varepsilon + K_2 \varepsilon |z_0|^2 (1 + \varepsilon) \leq K_5 \varepsilon |z_0|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} &|4A^3\theta\sqrt{|z_0|} + 6A^2\theta^2|z_0| + 4A\theta^3|z_0|^{3/2} + \theta^4|z_0|^2 \\ &\quad + (z_1 - z_0)A^2 + 2\theta Az_1\sqrt{|z_0|} + z_1\theta^2|z_0| + 2\alpha\theta\sqrt{|z_0|}| \\ &\leq K_8 \max\{\varepsilon, \varepsilon_1\} |z_0|^2 \leq K_6 \max\{\varepsilon, \varepsilon_1\} |z_0|^2, \end{aligned}$$

where

$$\begin{aligned} K_8 &= 4(K_1 + \varepsilon_1)^3 + 6(K_1 + \varepsilon_1)^2 \varepsilon_1 + 4(K_1 + \varepsilon_1) \varepsilon_1^2 + \varepsilon_1^3 \\ &\quad + (K_1 + \varepsilon_1)^2 + 2(K_1 + \varepsilon_1)(1 + \varepsilon) + (1 + \varepsilon) \varepsilon_1 + 2|\alpha|. \end{aligned}$$

Since $\varepsilon \leq \varepsilon_1$, we conclude that

$$\begin{aligned} |w'(z_1)| &\leq (K_7 - 1)\sqrt{\varepsilon_1}|z_0| + \sqrt{2|\alpha||A|} \\ &\leq (K_7 - 1)\sqrt{\varepsilon_1}|z_0| + \sqrt{2|\alpha|2K\sqrt{|z_0|}} \leq K_7\sqrt{\varepsilon_1}|z_0| \end{aligned}$$

if $|z_0| \geq Z_1$.

We remark that this is an essential feature of the situation that we are considering: at the point z_1 , the function w takes a value of which we only know that it is not too large (in relation to the natural measure $\sqrt{|z_1|}$) but definitely $|w'(z_1)|$ is small (in relation to the natural measure $|z_1|$).

We apply Lemma 5.1 taking $\zeta = z_1$, $K_0 = |w(z_1)|/\sqrt{|z_1|}$, so that

$$\varepsilon_0/2 < K_0 < K_1 \leq 2K.$$

Recall that $L = 10^{10}K_7$. To get a contradiction, suppose that $|w(z) - w(z_1)| < L\varepsilon_1\sqrt{|z_1|}$ in the closed disk $\overline{B}(z_1, \varepsilon_3/\sqrt{|z_1|})$. Then, taking $C = L\varepsilon_1$ in Lemma 5.1, we deduce that $\rho > \varepsilon_3\sqrt{|z_1|}$, where ρ is the largest radius r such that we have $|w(z) - w(z_1)| < L\varepsilon_1\sqrt{|z_1|}$ in $B(z_1, r)$.

Thus we may choose η in Lemma 5.1 so that $\eta\rho = \eta_1\varepsilon_3/\sqrt{|z_1|}$, where $\eta_1 \in (0, 1]$ will be chosen later. In order to apply (5.0.21), we first estimate some terms on the right hand side of (5.0.21). We have

$$\begin{aligned} & \frac{K_0}{6}\eta^2\rho^2\sqrt{|\zeta|} + \frac{\eta^2\rho^2C\sqrt{|\zeta|}}{2} \{ \eta\rho + (1 + 6K_0^2 + 6K_0C + 2C^2)|\zeta| \} \\ & \leq \frac{2K}{6} \frac{\varepsilon_3^2}{\sqrt{|z_1|}^2} \sqrt{|z_1|} \\ & \quad + \frac{\frac{\varepsilon_3^2}{\sqrt{|z_1|}^2}L\varepsilon_3\sqrt{|z_1|}}{2} \left\{ \frac{\varepsilon_3}{\sqrt{|z_1|}} + (1 + 6(2K)^2 + 6(2K)L\varepsilon_3 + 2\varepsilon_3^2L^2)|z_1| \right\} \\ & \leq K_9\varepsilon_3^2\sqrt{|z_1|}, \end{aligned}$$

where, for $|z_1| \geq Z_1/2$,

$$K_9 = K\varepsilon_3 + L\varepsilon_3 + L^3\varepsilon_3(3 + 36K^2) \geq \frac{K}{3|z_1|} + \frac{L\varepsilon_3^2}{2|z_1|} + L^3\varepsilon_3(3 + 36K^2).$$

We write

$$t_0 = \frac{w'(z_1)}{z_1w(z_1) \left(\frac{w(z_1)^2}{z_1} + \frac{1}{2} \right)} \in \mathbb{C}.$$

We will consider two cases regarding $|t_0|/\rho$ later.

Thus (5.0.21) yields, for all $z \in S(z_1, \eta_1\varepsilon_3/\sqrt{|z_1|})$,

$$\begin{aligned} \frac{L\varepsilon_1|z_1|}{\eta_1\varepsilon_3} & \geq \frac{|w(z) - w(z_1)|}{|z - z_1|} \\ & \geq |z_1||w(z_1)| \left| \frac{w(z_1)^2}{z_0} + \frac{1}{2} \right| \left| t_0 - \eta_1 \frac{\varepsilon_3}{\sqrt{|z_1|}} \right| \\ & \quad - \frac{|\alpha|}{2} \frac{\varepsilon_3}{\sqrt{|z_1|}} - K_9\varepsilon_3^2\sqrt{|z_1|}. \end{aligned}$$

We have

$$\left| \frac{w(z_1)^2}{z_0} + \frac{1}{2} \right| \geq \left| \frac{A^2}{z_0} + \frac{1}{2} \right| - \frac{|(w(z_1) - A)(w(z_1) + A)|}{|z_0|} \geq \varepsilon_2 - 2K\varepsilon_1 > \frac{\varepsilon_2}{2}.$$

Thus

$$\frac{L\varepsilon_1}{\eta_1\varepsilon_3} \geq \frac{\varepsilon_0\varepsilon_2\varepsilon_3}{4} \left| t_0 \frac{\sqrt{|z_1|}}{\varepsilon_3} - \eta_1 \right| - \frac{|\alpha|}{2} \frac{\varepsilon_3}{|z_1|^{3/2}} - \frac{K_9\varepsilon_3^2}{\sqrt{|z_1|}}.$$

If $|t_0|\sqrt{|z_1|}/\varepsilon_3 \geq 2/3$, we choose $\eta_1 = 1/3$. If $|t_0|\sqrt{|z_1|}/\varepsilon_3 < 2/3$, we choose $\eta_1 = 1$. Thus $1/3 \leq \eta_1 \leq 1$ and $|t_0|\sqrt{|z_1|}/\varepsilon_3 - \eta_1 \geq 1/3$.

Since $|z_1| \geq Z_1/2$, we have

$$\frac{|\alpha|}{2} \frac{\varepsilon_3}{|z_1|^{3/2}} + \frac{K_9 \varepsilon_3^2}{\sqrt{|z_1|}} < 10^{-6} \frac{\varepsilon_0 \varepsilon_2 \varepsilon_1^2}{16} < 10^{-6} \frac{\varepsilon_0 \varepsilon_2 \varepsilon_3}{12}.$$

We conclude that

$$(8.1.38) \quad (1 - 10^{-6}) \frac{\varepsilon_0 \varepsilon_2 \varepsilon_3^2}{12} < 3L\varepsilon_1.$$

From the definition of ε_3 , we obtain

$$\frac{\varepsilon_0 \varepsilon_2 \varepsilon_3^2}{12} = \frac{10^{12} K_7^2 \varepsilon_1}{12 \varepsilon_0 \varepsilon_2}.$$

This shows that (8.1.38) cannot hold. This contradiction shows that the largest radius ρ such that we have $|w(z) - w(z_1)| < L\varepsilon_1 \sqrt{|z_1|}$ in $B(z_1, \rho)$, satisfies $\rho \leq \varepsilon_3 / \sqrt{|z_1|}$.

We now apply Lemma 5.1 to our $\rho \leq \varepsilon_3 / \sqrt{|z_1|}$ and a point $z \in S(z_1, \rho)$ such that $|w(z) - w(z_1)| = L\varepsilon_1 \sqrt{|z_1|}$. We note that (5.0.20) then yields, in view of the estimate we have obtained for the term involving α , for $|z_1| \geq Z_1/2$, that

$$\begin{aligned} & |z_1| |w(z_1)| \left| \frac{w(z_1)^2}{z_0} + \frac{1}{2} \right| \max \{ |t_0|, \rho \} \\ &= \max \left\{ |w'(z_1)|, |z_1| |w(z_1)| \left| \frac{w(z_1)^2}{z_0} + \frac{1}{2} \right| \rho \right\} \\ &\geq \frac{L\varepsilon_1 \sqrt{|z_1|}}{2\rho} - |z_1| 10^{-4} \varepsilon_0 \varepsilon_2 \varepsilon_1^2. \end{aligned}$$

We have

$$\frac{L\varepsilon_1 \sqrt{|z_1|}}{2\rho} - |z_1| 10^{-4} \varepsilon_0 \varepsilon_2 \varepsilon_1^2 \geq \frac{L\varepsilon_1 \sqrt{|z_1|}}{4\rho}$$

since this is equivalent to

$$|z_1| 10^{-4} \varepsilon_0 \varepsilon_2 \varepsilon_1^2 \leq \frac{L\varepsilon_1 \sqrt{|z_1|}}{4\rho},$$

that is,

$$\rho \sqrt{|z_1|} \leq \frac{10^4 L}{4\varepsilon_0 \varepsilon_2 \varepsilon_1},$$

which is true since $\rho \leq \varepsilon_3 / \sqrt{|z_1|}$ and $4\varepsilon_0 \varepsilon_2 \varepsilon_1 \varepsilon_3 < 10^4 L$. We deduce that

$$|z_1| |w(z_1)| \left| \frac{w(z_1)^2}{z_0} + \frac{1}{2} \right| \max \{ |t_0|, \rho \} \geq \frac{L\varepsilon_1 \sqrt{|z_1|}}{4\rho}.$$

Write

$$A' = \frac{|w(z_1)|}{\sqrt{|z_1|}} \left| \frac{w(z_1)^2}{z_0} + \frac{1}{2} \right|.$$

If $\rho \geq |t_0|$, this gives

$$\rho^2 |z_1| \geq \frac{L\varepsilon_1}{4A'}.$$

If $\rho < |t_0|$, we obtain

$$\frac{|t_0|}{\rho} \rho^2 |z_1| \geq \frac{L\varepsilon_1}{4A'}.$$

We now apply Lemma 5.1 taking in (5.0.21), $\eta = 1/4$ if $|t_0|/\rho \geq 1/2$ (note that we may have $|t_0|/\rho \geq 1$); and $\eta = 3/4$ if $|t_0|/\rho < 1/2$. Hence $|t_0 - (z - z_1)| \geq \rho/4$ when $|z - z_1| = \eta\rho$. For certain terms on the right hand side of (5.0.21), we may thus use the same upper bounds as before. This shows that for all $z \in S(z_1, \eta\rho)$, where $\eta\rho \leq 3\varepsilon_3/(4\sqrt{|z_1|})$, when $|z_1| \geq Z_1/2$, we have

$$\frac{|w(z) - w(z_1)|}{|z_1|\rho} \geq A' \sqrt{|z_1|} \left| \frac{t_0}{\rho} - \frac{z - z_1}{\rho} \right| \frac{\rho}{4} - 10^{-4} \varepsilon_0 \varepsilon_2 \varepsilon_1^2.$$

Suppose that $\rho \geq |t_0|$. Then we obtain

$$\begin{aligned} \frac{|w(z) - w(z_1)|}{\sqrt{|z_1|}} &\geq \frac{1}{16} \rho^2 |z_1| A' - 10^{-4} \rho \sqrt{|z_1|} \varepsilon_0 \varepsilon_2 \varepsilon_1^2 \\ &\geq \frac{L\varepsilon_1}{64} - 10^{-4} \varepsilon_3 \varepsilon_0 \varepsilon_2 \varepsilon_1^2 \geq \frac{L\varepsilon_1}{65} > 3\varepsilon_1. \end{aligned}$$

Suppose then that $\rho < |t_0|$. Now, with $\eta = 1/4$, we use the fact that

$$\left| \frac{t_0}{\rho} - \frac{z - z_1}{\rho} \right| \geq \frac{3}{4} \frac{|t_0|}{\rho}.$$

Thus we obtain

$$\begin{aligned} &\frac{|w(z) - w(z_1)|}{\sqrt{|z_1|}} \\ &\geq \frac{3}{16} \frac{|t_0|}{\rho} \rho^2 |z_1| A' - 10^{-4} \rho \sqrt{|z_1|} \varepsilon_0 \varepsilon_2 \varepsilon_1^2 \\ &\geq \frac{3L\varepsilon_1}{16} - 10^{-4} \varepsilon_3 \varepsilon_0 \varepsilon_2 \varepsilon_1^2 \geq \frac{L\varepsilon_1}{8} > 3\varepsilon_1. \end{aligned}$$

Therefore for all $z \in S(z_1, \eta\rho)$,

$$|w(z) - w(z_1)| > 2\varepsilon_1 \sqrt{|z_0|}.$$

We apply Rouché's Theorem, comparing the functions $w(z) - w(z_1)$ and $w(z) - A$ in the disk $B(z_1, \eta\rho)$, where $1/4 \leq \eta \leq 3/4$, and deduce

that in the disk $B(z_1, \eta\rho)$ there is a point z' where $w(z') = A$. Also, on $S(z_1, \eta\rho)$ we have

$$|w(z) - A| \geq |w(z) - w(z_1)| - |A - w(z_1)| > 2\varepsilon_1\sqrt{|z_1|} \geq \varepsilon_1\sqrt{|z_1|}$$

so that any component D of the set $\{z : |w(z) - A| < \varepsilon_1\sqrt{|z_1|}\}$ that intersects $B(z_1, \eta\rho)$ must be contained in $B(z_1, \eta\rho)$, and hence w maps such a component onto the set $B(A, \varepsilon_1\sqrt{|z_1|})$. Thus any such D contains a point that w maps to the point A . Letting z' be such a point in the component that contains z_1 , we find that $|z' - z_1| \leq \eta\rho < \varepsilon_3/\sqrt{|z_1|}$.

This completes the proof of Lemma 8.1.

9. UPPER ESTIMATES FOR THE NUMBER OF TIMES A VALUE IS TAKEN IN A DISK

Let w be a transcendental solution of (1.0.1). We need to know that in a disk of the form $B(z_0, \varepsilon|z_0|)$, where $\varepsilon > 0$ is small but fixed, while $z_0 \rightarrow \infty$ outside an exceptional set (so that $|w(z_0)| \leq K\sqrt{|z_0|}$ for some large fixed K), the function w does not take any value A too many times. We only consider values A such that also $|A| \leq K\sqrt{|z_0|}$.

To do this, we need an upper bound for the number of poles of w in $B(z_0, \varepsilon|z_0|)$ together with considerations from the Nevanlinna theory. There might be a great temptation to leave the proof to the reader, but we feel that for clarity, it is best to give the details.

The following result was obtained by Shimomura [12], Proposition 5.1, p. 266.

Lemma 9.1. *Let w be a transcendental solution of (1.0.1). Then there are fixed numbers $r_0 > 10$ and $C_1 > 10$ depending on w such that if σ is a pole of w with $|\sigma| > 2r_0$, then the disk $B(\sigma, C_1^{-1}|\sigma|^{-1/2})$ is disjoint from such disks for other poles of w .*

It is essential for us to obtain the estimate in the following lemma on the number of times that w can take a particular value in a disk whose radius is small if compared to the modulus of its center.

Lemma 9.2. *Let w be a transcendental solution of (1.0.1). Suppose that $K > 3$ is a fixed large number and that $1/K \leq |w(z_0)|\sqrt{|z_0|} \leq K$ and $\varepsilon_0 \in (0, 1)$. Then there are fixed numbers $r_0 > 10$ and $K_1 > 10$ depending on w and K such that if $0 < \varepsilon < 1/32$, $|z_0| > 4r_0$, and $|z_0| \geq \varepsilon_0^{-2}$, and if $A \in \mathbb{C}$ with $|A| \leq K\sqrt{|z_0|}$, $|w(z_0)| \geq \varepsilon_0\sqrt{|z_0|}$, and $|w(z_0) - A| \geq \varepsilon_0\sqrt{|z_0|}$, and if, in addition, $r \geq t_0$ where t_0 is such that $T(t, w) \leq t^4$ for all $t \geq t_0$, then in the disk $B(z_0, \varepsilon|z_0|)$, the function w*

takes the value A at most

$$\nu = K'\varepsilon^2|z_0|^3 + K'(1 + |z_0|\log|z_0|) + 2\log\frac{1}{\varepsilon_0}$$

times, with due count of multiplicity, where K' is a positive number that only depends on K , $|\alpha|$, and the number C_1 in Lemma 9.1.

9.1. A lemma from the Nevanlinna theory. We want to have an upper bound for $m(r, f''/f)$ for a suitable f related to the solution w of (1.0.1). If we were to obtain such an upper bound in the usual way by first writing $f''/f = (f''/f')(f'/f)$, then the upper bound would involve both $1/|f(0)|$ and $1/|f'(0)|$. Since we will take $f(z) = w(z + z_0)$ for variable points z_0 , the upper bound would require us to know that $|w'(z_0)|$ is not too close to zero. To avoid having to deal with such a problem, we prefer to be more careful when estimating $m(r, f''/f)$ so that only $1/|f(0)|$ remains in the bound but not $1/|f'(0)|$. This requires us to go back to basics. Hence we will need to prove the following result arising from the Nevanlinna theory.

Lemma 9.3. *Let f be a meromorphic function, not a polynomial of degree at most 1, in the closed disk $\overline{B}(0, R)$, where $0 < R < +\infty$. Suppose that $f(0) \notin \{0, \infty\}$. Then, if $0 < r < R$, we have*

$$(9.1.39) \quad m\left(r, \left(\frac{f'}{f}\right)'\right) \leq 7\log^+ T(R, f) + 7\log^+ R + 12\log^+ \frac{1}{R-r} \\ + 7\log^+ \log^+ \frac{1}{|f(0)|} + 2\log^+ \frac{1}{r} + 1 + \log 3 + 28\log 2$$

and

$$(9.1.40) \quad m\left(r, \frac{f''}{f}\right) \leq 15\log^+ T(R, f) + 17\log^+ R + 24\log^+ \frac{1}{R-r} \\ + 15\log^+ \log^+ \frac{1}{|f(0)|} + 4\log^+ \frac{1}{r} + 29 + \log 3 + 29\log 2 \\ < 15\log^+ T(R, f) + 17\log^+ R + 24\log^+ \frac{1}{R-r} \\ + 15\log^+ \log^+ \frac{1}{|f(0)|} + 4\log^+ \frac{1}{r} + 50.2.$$

Proof of Lemma 9.3. We follow the proof of Nevanlinna's lemma on the logarithmic derivative. Let the assumptions of Lemma 9.3 be satisfied. Suppose that $0 < |z| = r < \rho < R$. Suppose further that $f(z) \notin \{0, \infty\}$. We denote by a_μ the zeros and by b_ν the poles of f , with due count of multiplicity. By the differentiated Poisson–Jensen

formula ([4], p. 22), we have

$$(9.1.41) \quad \begin{aligned} \left(\frac{f'}{f}\right)'(z) &= \frac{2}{\pi} \int_0^{2\pi} \frac{\log |f(\rho e^{i\varphi})| \rho e^{i\varphi}}{(\rho e^{i\varphi} - z)^3} d\varphi \\ &+ \sum_{|b_\nu| < \rho} \left(\frac{1}{(b_\nu - z)^2} - \frac{\bar{b}_\nu^2}{(\rho^2 - \bar{b}_\nu z)^2} \right) \\ &- \sum_{|a_\mu| < \rho} \left(\frac{1}{(a_\mu - z)^2} - \frac{\bar{a}_\mu^2}{(\rho^2 - \bar{a}_\mu z)^2} \right). \end{aligned}$$

We have

$$(9.1.42) \quad \begin{aligned} &\left| \frac{2}{\pi} \int_0^{2\pi} \frac{\log |f(\rho e^{i\varphi})| \rho e^{i\varphi}}{(\rho e^{i\varphi} - z)^3} d\varphi \right| \\ &\leq \frac{4\rho}{(\rho - r)^3} \left\{ m(\rho, f) + m\left(\rho, \frac{1}{f}\right) \right\}. \end{aligned}$$

If f has no zeros or poles in $B(0, \rho)$, the remaining terms on the right hand side of (9.1.41) vanish. Suppose that this is not the case. We denote by $\delta(z)$ the distance from z to the nearest zero or pole of f in $B(0, \rho)$, and write

$$n = n(\rho, f) + n(\rho, 1/f).$$

We note that if a is any of the points a_μ or b_ν in $B(0, \rho)$, we have

$$|\rho^2 - \bar{a}z| \geq \rho^2 - \rho r = \rho(\rho - r), \quad \left| \frac{a}{\rho^2 - \bar{a}z} \right| \leq \frac{1}{\rho - r}, \quad \left| \frac{1}{a - z} \right| \leq \frac{1}{\delta(z)}.$$

This gives

$$(9.1.43) \quad \left| \left(\frac{f'}{f}\right)'(z) \right| \leq \frac{4\rho}{(\rho - r)^3} \left\{ m(\rho, f) + m\left(\rho, \frac{1}{f}\right) \right\} + \frac{n}{\delta(z)^2} + \frac{n}{(\rho - r)^3}.$$

Since $T(\rho, f) = T(\rho, 1/f) + \log |f(0)|$, we obtain

$$(9.1.44) \quad \left| \left(\frac{f'}{f}\right)'(z) \right| \leq \frac{4\rho}{(\rho - r)^3} \left\{ 2T(\rho, f) + \log^+ \frac{1}{|f(0)|} \right\} + \frac{n}{r^2} \frac{r^2}{\delta(z)^2} + \frac{n}{(\rho - r)^3}.$$

Thus

$$\begin{aligned} \log^+ \left| \left(\frac{f'}{f}\right)'(z) \right| &\leq \log^+ T(\rho, f) + 2 \log^+ n + \log^+ \rho + 2 \log^+ \frac{r}{\delta(z)} \\ &+ \log^+ \log^+ \frac{1}{|f(0)|} + 6 \log^+ \frac{1}{\rho - r} + 2 \log^+ \frac{1}{r} + \log 48. \end{aligned}$$

With $z = re^{i\theta}$, we integrate with respect to θ from 0 to 2π . In view of [4], Lemma 2.2, p. 35, we obtain

$$\begin{aligned} m\left(r, \left(\frac{f'}{f}\right)'\right) &\leq \log^+ T(\rho, f) + 6\log^+ n + \log^+ \rho \\ &+ \log^+ \log^+ \frac{1}{|f(0)|} + 6\log^+ \frac{1}{\rho - r} + 2\log^+ \frac{1}{r} + 1 + \log 48. \end{aligned}$$

As in [4], p. 37, we have

$$\log^+ n \leq \log^+ T(R, f) + \log^+ R + \log^+ \frac{1}{R - \rho} + \log^+ \log^+ \frac{1}{|f(0)|} + 2\log 2.$$

Hence

$$\begin{aligned} m\left(r, \left(\frac{f'}{f}\right)'\right) &\leq 7\log^+ T(R, f) + 7\log^+ R + 6\log^+ \frac{1}{R - \rho} \\ &+ 7\log^+ \log^+ \frac{1}{|f(0)|} + 6\log^+ \frac{1}{\rho - r} + 2\log^+ \frac{1}{r} + 1 + \log 3 + 16\log 2. \end{aligned}$$

Noting that we may choose $\rho = (R + r)/2$, we obtain (9.1.39).

By [4], Lemma 2.2, p. 35, we have

$$\begin{aligned} m\left(r, \frac{f'}{f}\right) &\leq 4\log^+ T(R, f) + 5\log^+ R + 6\log^+ \frac{1}{R - r} \\ &+ 4\log^+ \log^+ \frac{1}{|f(0)|} + \log^+ \frac{1}{r} + 14. \end{aligned}$$

Since

$$\frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2,$$

we obtain (9.1.40).

This completes the proof of Lemma 9.3.

9.2. Proof of Lemma 9.2. Pick any $s \in (0, \varepsilon]$ with $s|z_0|^{3/2} \geq 1$. Note that all poles of w are simple. Let the distinct poles of w in $B(z_0, s|z_0|)$ be σ_j . The area of the disk associated with σ_j as in Lemma 9.1 is

$$\geq \pi C_1^{-2} \{((1 + \varepsilon)|z_0|)^{-1/2}\}^2 \geq 0.8\pi C_1^{-2} |z_0|^{-1}.$$

Each such disk is contained in the disk $B(z_0, s|z_0| + C_1^{-1}(1 - \varepsilon)^{-1}|z_0|)^{-1/2}$, whose area is

$$\leq \pi s^2 |z_0|^2 \left(1 + \frac{1}{C_1(1 - \varepsilon)s|z_0|^{3/2}}\right)^2.$$

By Lemma 9.1, these small disks are disjoint, so that w has at most

$$2s^2 |z_0|^3 C_1^2$$

poles in $B(z_0, s|z_0|)$. So if $f(z) = w(z + z_0)$ and $w(z_0) \neq \infty$, we get

$$\begin{aligned} N(s|z_0|, f) &= N(1, f) + \int_1^{s|z_0|} \frac{n(t, f) dt}{t} \\ &\leq N(1, f) + |z_0|C_1^2(s^2|z_0|^2 - 1) < N(1, f) + C_1^2s^2|z_0|^3. \end{aligned}$$

We next apply Nevanlinna's first fundamental theorem ([4], Theorem 1.2, p. 5) to $f(z) = w(z + z_0)$. Assuming that $w(z_0) \neq A$, we obtain

$$\begin{aligned} (\log 2)n\left(r, \frac{1}{f-A}\right) &\leq \int_0^{2r} \frac{n(t, 1/(f-A)) dt}{t} = N\left(2r, \frac{1}{f-A}\right) \\ &\leq T\left(2r, \frac{1}{f-A}\right) \leq T(2r, f) - \log |w(z_0) - A| + \log^+ |A| + \log 2 \\ &= N(2r, f) + m(2r, f) - \log |w(z_0) - A| + \log^+ |A| + \log 2 \\ &\leq N(1, f) + 4C_1^2r^2|z_0| + m(2r, f) + \log \frac{2K}{\varepsilon_0} + \log |z_0|. \end{aligned}$$

We now have to estimate the terms on the right hand side. We first estimate $N(1, f)$. Let σ be a pole of w in $B(z_0, 1)$, so $\sigma \neq z_0$ by assumption. If there are no such poles then $N(1, f) = 0$. Otherwise, let first σ_0 be such a pole nearest to z_0 . By Lemmas 4.1 and 4.2 in [13], we have $|z_0 - \sigma_0| \geq C_2/(K\sqrt{|z_0|})$ for some positive absolute constant C_2 . Also, due to the presence of disjoint disks around the poles as described in Lemma 9.1, there are at most $C_1^2(|z_0| + 1)$ poles of w in $B(z_0, 1)$. Hence

$$N(1, f) = \sum_{w(a)=\infty, |a-z_0|<1} \log \frac{1}{|a-z_0|} \leq C_1^2(|z_0| + 1) \log \frac{K\sqrt{|z_0|}}{C_2}.$$

Then we consider the term $m(2r, f)$. Since

$$f^2 = \frac{1}{2} \frac{f''}{f} - \frac{z}{2} - \frac{\alpha}{2f},$$

we see that if $|z| = 2r$ and $|f(z)| = |w(z + z_0)| \geq 1$, then

$$2 \log^+ |f| \leq \log^+ \left| \frac{f''}{f} \right| + \log^+ r + \log^+ |\alpha| + \log 6.$$

Thus by Lemma 9.3, we have for $r \geq 3$,

$$\begin{aligned} &2m(2r, f) - (\log^+ r + \log^+ |\alpha| + \log 6) \\ &< 15 \log^+ T(4r, f) + 17 \log^+ r + 15 \log^+ \log^+ \frac{1}{|f(0)|} + 74. \end{aligned}$$

Hence

$$\begin{aligned} m(2r, f) &\leq 8 \log^+ T(4r, f) + 38 + \frac{1}{2} \log^+ |\alpha| \\ &\quad + 9 \log r + 8 \log^+ \log^+ \frac{1}{|w(z_0)|}. \end{aligned}$$

We use Cartan's identity ([4], Theorem 1.3, p. 8)

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N(r, f, e^{i\theta}) d\theta + \log^+ |f(0)|.$$

We write

$$\nu(r, f, t) = \int_0^{2\pi} n(r, f, te^{i\varphi}) d\varphi$$

and

$$A(r, f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{|f'(te^{i\varphi})|^2}{(1 + |f(te^{i\varphi})|^2)^2} t dt d\varphi.$$

Then

$$T_0(r, f) = \int_0^r A(t) \frac{dt}{t}$$

is the Ahlfors–Shimizu characteristic of f that satisfies ([4], p. 13)

$$|T(r, f) - T_0(r, f)| \leq \log^+ |f(0)| + \frac{1}{2} \log 2.$$

We obtain

$$T(r, f) = \int_0^r \nu(t, f, 1) \frac{dt}{t} + \log^+ |f(0)|.$$

By [5], we have

$$\nu(t, f, 1) \leq A(ct, f) + \frac{\log 2}{\log c}$$

whenever $c > 1$. Thus, taking $c = 2$, we obtain

$$\begin{aligned} T(r, f) &\leq T(1, f) + \log r + A(2r, f) \log r \\ &\leq T(1, f) + \log r + (\log r) A(|z_0| + 2r, w) \\ &\leq T(1, f) + \log r + \frac{\log r}{\log 2} \int_{2r}^{4r} A(t, w) \frac{dt}{t} \\ &\leq T(1, f) + \log r + \frac{\log r}{\log 2} T_0(|z_0| + 4r, w) \\ &\leq T(1, f) + \log r + \frac{\log r}{\log 2} T(|z_0| + 4r, w) + \frac{\log r}{\log 2} \left(\log^+ |w(0)| + \frac{1}{2} \log 2 \right). \end{aligned}$$

Without loss of generality we may assume that $w(0) \neq \infty$. We replace above r by $4r$. Since the order of w is at most 3, we have $T(r, w) \leq r^4$

for all large r . Thus when $|z_0|$ is large enough, we have (recall that $r \geq 3$)

$$\begin{aligned} \log^+ T(4r, f) &\leq \log^+ N(1, f) + \log^+ m(1, f) + 4 \log(|z_0| + 8r) \\ &\quad + 3 \log \log r + \log^+ \log^+ |w(0)| + \log \frac{10}{(\log 2)^2}. \end{aligned}$$

We already have an upper bound for $N(1, f)$ that we can use here. Going through the arguments above that led to an upper bound for $m(2r, f)$, but this time with radius 1 instead of $2r$, we get

$$m(1, f) \leq 8 \log^+ T(2, f) + 8 \log^+ \log^+ \frac{1}{|w(z_0)|} + \frac{\log^+ |\alpha|}{2} + \frac{\log 6}{2} + 37.$$

Using the fact that $T(2, f) \leq T(4r, f)$, we then obtain

$$\begin{aligned} \log^+ T(4r, f) &\leq \log^+ \log^+ T(4r, f) + \log^+ \log^+ K + \log |z_0| \\ &\quad + \log^+ \log^+ \frac{1}{C_2} + \log \log |z_0| + 2 \log^+ C_1 + \log^+ \log^+ \log^+ \frac{1}{|w(z_0)|} \\ &\quad + \log^+ \frac{\log^+ |\alpha|}{2} + 10.5, \end{aligned}$$

which implies (since $\log^+ x \leq x/2$) that

$$\begin{aligned} \log^+ T(4r, f) &\leq 2 \log^+ \log^+ K + 2 \log |z_0| \\ &\quad + 2 \log^+ \log^+ \frac{1}{C_2} + 2 \log \log |z_0| + 4 \log^+ C_1 + 2 \log^+ \log^+ \log^+ \frac{1}{|w(z_0)|} \\ &\quad + 2 \log^+ \frac{\log^+ |\alpha|}{2} + 21. \end{aligned}$$

Therefore

$$\begin{aligned} m(2r, f) &\leq 16 \log^+ \log^+ K + 16 \log |z_0| \\ &\quad + 16 \log^+ \log^+ \frac{1}{C_2} + 16 \log \log |z_0| + 32 \log^+ C_1 + 16 \log^+ \log^+ \log^+ \frac{1}{|w(z_0)|} \\ &\quad + 16 \log^+ \frac{\log^+ |\alpha|}{2} + 206 + \frac{1}{2} \log^+ |\alpha| + 9 \log r + 8 \log^+ \log^+ \frac{1}{|w(z_0)|}. \end{aligned}$$

It follows that

$$\begin{aligned}
(\log 2)n \left(r, \frac{1}{f-A} \right) &\leq C_1^2(|z_0| + 1) \log \frac{K\sqrt{|z_0|}}{C_2} + 4C_1^2 r^2 |z_0| \\
&+ 16 \log^+ \log^+ K + 16 \log |z_0| + 16 \log^+ \log^+ \frac{1}{C_2} + 16 \log \log |z_0| \\
&+ 32 \log^+ C_1 + 16 \log^+ \log^+ \log^+ \frac{1}{|w(z_0)|} + 16 \log^+ \frac{\log^+ |\alpha|}{2} \\
&+ 206 + \frac{1}{2} \log^+ |\alpha| + 9 \log r + 8 \log^+ \log^+ \frac{1}{|w(z_0)|} + \log \frac{2K}{\varepsilon_0} + \log |z_0|.
\end{aligned}$$

We will have $32r \leq |z_0|$. It is clear that for some positive constant C_3 depending only on $|\alpha|$, K , C_1 , and C_2 , this results in the estimate

$$\begin{aligned}
(9.2.45) \quad n \left(r, \frac{1}{f-A} \right) &\leq \frac{4}{\log 2} C_1^2 r^2 |z_0| + C_3(1 + |z_0| \log |z_0|) \\
&+ 35 \log^+ \log^+ \frac{1}{|w(z_0)|} + \frac{1}{\log 2} \log \frac{1}{\varepsilon_0}.
\end{aligned}$$

Now set $r = \varepsilon |z_0|$, where $0 < \varepsilon \leq 1/32$, and note that $|w(z_0)| \geq \varepsilon_0 \sqrt{|w(z_0)|}$. Then

$$\begin{aligned}
(9.2.46) \quad n \left(r, \frac{1}{f-A} \right) &\leq \frac{4}{\log 2} C_1^2 \varepsilon^2 |z_0|^3 + C_3(1 + |z_0| \log |z_0|) \\
&+ 35 \log^+ \log^+ \frac{1}{|w(z_0)|} + \frac{1}{\log 2} \log \frac{1}{\varepsilon_0} \\
&\leq \frac{4}{\log 2} C_1^2 \varepsilon^2 |z_0|^3 + C_3(1 + |z_0| \log |z_0|) \\
&+ 35 \log^+ \log^+ \frac{1}{\varepsilon_0 \sqrt{|z_0|}} + \frac{1}{\log 2} \log \frac{1}{\varepsilon_0}.
\end{aligned}$$

This completes the proof of Lemma 9.2.

10. INTEGRATING OUTSIDE AN EXCEPTIONAL SET: ELLIPTIC CASE

Let w be a transcendental solution of (1.0.1), and recall (1.0.2). Suppose that we are in case (A), so that

$$(10.0.47) \quad \limsup_{z \rightarrow \infty, z \notin E} \left| \frac{U(z)}{z^2} + \frac{1}{4} \right| =: \delta_0 > 0$$

and

$$(10.0.48) \quad \limsup_{z \rightarrow \infty, z \notin E} \left| \frac{U(z)}{z^2} \right| > 0.$$

Then there is a sequence of points $a_n \rightarrow \infty$ such that $a_n \notin E$ and

$$\left| \frac{U(a_n)}{a_n^2} + \frac{1}{4} \right| \rightarrow \delta_0.$$

If by passing to a subsequence of a_n without changing notation we can achieve

$$\limsup \left| \frac{U(a_n)}{a_n^2} \right| > 0,$$

then we assume that we have passed to such a subsequence.

Otherwise, we have $U(a_n)/a_n^2 \rightarrow 0$. However, since (10.0.48) holds and since the complement of the exceptional set E is connected and any two points outside E can be connected by a path that does not deviate too much from the line segment joining the two points, we can join a_n outside E to a point b_n with $|b_n| \geq |a_n|$ such that $\left| \frac{U(b_n)}{b_n^2} \right|$ is close to a value $b \in (0, 1/5]$ depending on w . This is because $U(z)/z^2$ is continuous and there are points outside E where $|U(z)/z^2|$ gets close to a fixed positive number. Since $|U(z)/z^2|$ is close to zero at the a_n , we may stop on such a path on our way from a_n onwards at a point b_n where $|U(z)/z^2|$ is positive but not too large even if $|U(z)/z^2|$ were to get very large at some other points. In this way, by passing to a further subsequence without changing notation if necessary, we find points b_n such that

$$\left| \frac{U(b_n)}{b_n^2} + \frac{1}{4} \right| \rightarrow \delta_1 > 0 \quad \text{and} \quad \left| \frac{U(b_n)}{b_n^2} \right| \rightarrow \delta_2 > 0,$$

that is, both sequences tend to strictly positive limits as $n \rightarrow \infty$.

We now fix n , write z_0 instead of b_n (or instead of a_n if it was not necessary to replace a_n by b_n), and consider what can be said about the behaviour of w in a disk $B(z_0, \varepsilon|z_0|)$, where $\varepsilon > 0$ is to be chosen in a moment.

Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ be small positive parameters whose relations we will fix later. We assume these parameters to be so small that the various assumptions of the lemmas that we will use will be satisfied. At the end, each of these parameters will be fixed. We take the view that after fixing all the parameters, we then perform all considerations only for points z_0 such that $|z_0|$ is large enough, depending on these choices. This is possible since z_0 represents a generic point in a sequence that tends to infinity.

Observe first that we may fix $K_1 > 0$ so that whenever $|z|$ is sufficiently large and $z \notin E$, then $|U(z)| \leq K_1|z|^2$ and $|w(z)| \leq K\sqrt{|z|}$. Moreover, again if $z \notin E$, we conclude from $(U/z^2)' = U'/z^2 - 2U/z^3$

that

$$|(U/z^2)'| \leq |w/z|^2 + 2K_1/|z| \leq |z|^{-1}(2K_1 + K^2).$$

If $\varepsilon > 0$ is small enough and $z \notin E$ in the disc $B(z_0, \varepsilon|z_0|)$, then $|z - z_0| < \varepsilon|z_0|$ and so

$$\left| \frac{U(z)}{z^2} - \frac{U(z_0)}{z_0^2} \right| \leq \int_{z_0}^z \left| \left(\frac{U(t)}{t^2} \right)' \right| |dt| \leq \frac{2K_1 + K^2}{(1 - \varepsilon)|z_0|} K_2 |z - z_0| < K_3 \varepsilon.$$

Here K_2 depends on the length of the path of integration from z_0 to z outside of the set E . Clearly, $K_3 = (1 - \varepsilon)^{-1} K_2 (2K_1 + K^2)$. Therefore, (1.0.2) can be written as $(w'/z)^2 = Q$, where

$$(10.0.49) Q(z) \equiv \left(\frac{w(z)^2}{z} + \frac{1}{2} \right)^2 - \left(\frac{U(z)}{z^2} + \frac{1}{4} - \frac{2\alpha w(z)}{z^2} \right)$$

$$(10.0.50) \quad = \left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 - \left(\frac{U(z_0)}{z_0^2} + \frac{1}{4} \right) + c_1,$$

where

$$(10.0.51) \quad c_1(z) = \frac{2\alpha w(z)}{z^2} - \left(\frac{U(z)}{z^2} - \frac{U(z_0)}{z_0^2} \right) + w(z)^2 \frac{z_0 - z}{z_0 z} + w(z)^4 \frac{(z_0 - z)(z_0 + z)}{z_0^2 z^2}.$$

Thus

$$(10.0.52) \quad |c_1| \leq \frac{2|\alpha||w(z)|}{|z_0|^{3/2}} (1 + 2\varepsilon) + K_3 \varepsilon + K^2 \varepsilon + 3K^4 \varepsilon \leq K_4 \varepsilon,$$

provided that $\varepsilon > 0$ has been suitably fixed first, and then z_0 has been chosen with $|z_0|$ large enough.

Denoting now

$$(10.0.53) \quad Q_1 := \left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 - \left(\frac{U(z_0)}{z_0^2} + \frac{1}{4} \right)$$

we see that

$$\sqrt{Q} = \sqrt{Q_1 + c_1} = \sqrt{Q_1} \sqrt{1 + c_1/Q_1}$$

and so

$$\frac{1}{\sqrt{Q}} = \frac{1}{\sqrt{Q_1}} \frac{1}{\sqrt{1 + c_1/Q_1}}.$$

Choose now $\varepsilon_1 > 0$ small enough and define

$$E_1 = \left(z : \left| \left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 - \left(\frac{U(z_0)}{z_0^2} + \frac{1}{4} \right) \right| < \varepsilon_1, \frac{|z - z_0|}{z_0} < \varepsilon \right).$$

In other words, in the disk $B(z_0, \varepsilon|z_0|)$, we consider small disk-like sets about the points ζ_0 such that

$$w(\zeta_0) = \pm \sqrt{z_0 \left(-\frac{1}{2} \pm \sqrt{\frac{U(z_0)}{z_0^2} + \frac{1}{4}} \right)},$$

i.e., about points where w takes certain four (fixed) values.

These values $w(\zeta_0)$ are what were denoted by A in Lemma 8.1. Let us denote these four values by A_1, A_2, A_3, A_4 , so that

$$\left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 - \left(\frac{U(z_0)}{z_0^2} + \frac{1}{4} \right) = \prod_{j=1}^4 \left(\frac{w(z) - A_j}{\sqrt{z_0}} \right).$$

Here either value for $\sqrt{z_0}$ can be used as long as we are consistent.

We consider the separation between any two A_j . If two of them were very close since the outer square root is close to zero, we would have

$$-\frac{1}{2} \pm \sqrt{\frac{U(z_0)}{z_0^2} + \frac{1}{4}}$$

close to zero, which means that $U(z_0)/z_0^2$ is close to zero. We have excluded this, and it follows that any two points like this are separated by at least $\eta_0 \sqrt{|z_0|}$, for a fixed $\eta_0 > 0$. If two points arising from different signs for $\sqrt{\frac{U(z_0)}{z_0^2} + \frac{1}{4}}$ are close to each other, then their squares are close to each other, so that the two values of $\sqrt{\frac{U(z_0)}{z_0^2} + \frac{1}{4}}$ are close to each other. But this is impossible since $\frac{U(z_0)}{z_0^2} + \frac{1}{4}$ is bounded away from zero. It follows that

$$\min\{|A_j - A_k| : 1 \leq j < k \leq 4\} \geq \eta_0 \sqrt{|z_0|},$$

for a fixed $\eta_0 > 0$, independent of the choice of z_0 in our sequence. Indeed η_0 depends only on δ_1 and δ_2 . It follows that if $z \in E_1$, then, if ε_1 is small enough in relation to η_0 , we have

$$|w(z) - A_j| < \frac{8\varepsilon_1 \sqrt{|z_0|}}{\eta_0^3}$$

for exactly one value of j with $1 \leq j \leq 4$. Here we assume that ε_1 is so small if compared to η_0 , that for all $k \neq j$ we then have $|w(z) - A_k| \geq (\eta_0/2) \sqrt{|z_0|}$.

Our assumptions imply further that for some ε_2 depending only on η_0 , we have $|A_j| \geq \varepsilon_2 \sqrt{|z_0|}$ for all j with $1 \leq j \leq 4$. Similarly, since $U(z)/z^2$ is bounded outside E , we have $|A_j| \geq K_0 \sqrt{|z_0|}$ for all j with $1 \leq j \leq 4$, for a fixed $K_0 > 1$.

Suppose that $w(z_0)$ itself is close to one of the values A_j , say $|w(z_0) - A_j| < \varepsilon_2 \sqrt{|z_0|}$. Then we modify z_0 before proceeding further, as follows. Move z_0 along a line segment towards the nearest pole of w . By Lemma 8.1, we need to move by a distance less than $\varepsilon_3/\sqrt{|z_0|}$ to get to a point z' such that $|w(z') - A_j| = \varepsilon_2 \sqrt{|z_0|}$. If ε_2 is small enough in relation to η_0 , then $|w(z') - A_k| \geq \varepsilon_2 \sqrt{|z_0|}$ for all k with $1 \leq k \leq 4$, so that we can replace z_0 by z' , without changing notation. We are staying outside E , and by the analysis above,

$$\left| \frac{U(z_0)}{z_0^2} - \frac{U(z')}{z'^2} \right| \leq K_3 \frac{|z_0 - z'|}{|z_0|} \leq K_3 \varepsilon_3 |z_0|^{-3/2},$$

so that our assumptions concerning δ_1 and δ_2 do not change.

After ensuring that z_0 is properly chosen as above, the situation now is addressed by Lemmas 8.1 and 9.2. Once all parameters have been fixed and $|z_0|$ is large enough, it follows that in the disk $B(z_0, \varepsilon|z_0|)$, there are at most $K'\varepsilon^2|z_0|^3$ points ζ_0 at which w takes any one of the four values A_j , for a suitable constant K' that can be chosen independent of ε and the other ε_j -type parameters (the price for this independence is that $|z_0|$ must be larger than a number that does depend also on ε). Furthermore, there are disjoint disks around these points ζ_0 , whose union contains E_1 , such that the total area of the disks does not exceed

$$K'\varepsilon^2|z_0|^3 \pi (\varepsilon_3/\sqrt{|z_0|})^2 = \pi K'\varepsilon^2 \varepsilon_3^2 |z_0|^2,$$

which is a small fraction of the area $\pi\varepsilon^2|z_0|^2$ of the disk $B(z_0, \varepsilon|z_0|)$. This is because K' is independent of ε_3 , and we can choose ε_3 sufficiently small. It follows that

$$B_1 \equiv B(z_0, \varepsilon|z_0|) \setminus (E \cup E_1)$$

is a path connected set whose any two points a, b can be connected in it by a path whose length is at most $\pi|a - b|$, and whose area is almost equal to that of the disk $B(z_0, \varepsilon|z_0|)$.

Suppose now that $z \in B_1$. We wish to estimate $w(z)$. In the integrals below, we integrate from z_0 to z along a path in B_1 whose length is not too large, as we just explained. Fix a choice for $\sqrt{z_0}$. Write $\tilde{w}(z) = w(z)/\sqrt{z_0}$. From (1.0.2), we obtain, with an appropriate local choice of the square root,

$$\begin{aligned} \frac{z^2 - z_0^2}{2} &= \int_{z_0}^z t dt = \int_{z_0}^z \frac{dw(t)}{\sqrt{Q(t)}} = \int_{z_0}^z \frac{dw(t)}{\sqrt{Q_1}} \frac{1}{\sqrt{1 + c_1/Q_1}} \\ &= \int_{z_0}^z \frac{dw(t)}{\sqrt{Q_1}} + \int_{z_0}^z \frac{dw(t)}{\sqrt{Q_1}} \left(\frac{1}{\sqrt{1 + c_1/Q_1}} - 1 \right). \end{aligned}$$

We can write

$$\frac{1}{\sqrt{Q_1}} \left(\frac{1}{\sqrt{1 + c_1/Q_1}} - 1 \right) = \frac{1}{\sqrt{Q_1}} \left(\frac{1 - \sqrt{1 + c_1/Q_1}}{\sqrt{1 + c_1/Q_1}} \right).$$

We have

$$\int_{z_0}^z \frac{dw(t)}{\sqrt{Q_1(t)}} = \int_{w(z_0)}^{w(z)} \frac{dw}{\sqrt{\prod_{j=1}^4 \left(\frac{w-A_j}{\sqrt{z_0}} \right)}},$$

which is an elliptic integral. With $A'_j = A_j/\sqrt{z_0}$, it reads

$$\sqrt{z_0} \int_{\tilde{w}(z_0)}^{\tilde{w}(z)} \frac{d\tilde{w}}{\sqrt{\prod_{j=1}^4 (\tilde{w} - A'_j)}} = \sqrt{z_0} I(\tilde{w}(z)),$$

say. Here the integration variable w follows the path of $w(t)$ as t moves from z_0 to z , and analogously for the variable \tilde{w} . In particular, we have $|\tilde{w}(t)| \leq K$ and $|\tilde{w} - A'_j| \geq \varepsilon_1$ for all j , throughout this path. There is an elliptic function $P(w)$ such that $P(I(w)) = w$ for all $w \in \mathbb{C}$. The function P depends on the four numbers A'_j and on the number $\tilde{w}(z_0)$. Due to the constraints on the A_j , the period parallelogram of P has area A' such that $1/C \leq A' \leq C$ for some $C > 1$ depending only on ε_2 and K_0 , hence independent of the point z_0 in our sequence.

On our path, we have

$$|c_1/Q_1| \leq K_4 \varepsilon / \varepsilon_1.$$

Now we come to the only really subtle point in the argument. We choose some small value for ε_1 . We may choose a small extra parameter $\eta_1 \in (0, 1/100]$. After that, we choose ε so small that $K_4 \varepsilon / \varepsilon_1 < \eta_1$. Here it is essential that the number K' was independent of ε and of ε_1 , so that we can still guarantee that the set E_1 is small.

After these choices, we note that

$$|1 - \sqrt{1 + c_1/Q_1}| \leq \eta_1, \quad |\sqrt{1 + c_1/Q_1}| \geq 1 - \eta_1,$$

so that

$$\left| \frac{1 - \sqrt{1 + c_1/Q_1}}{\sqrt{1 + c_1/Q_1}} \right| \leq \frac{\eta_1}{1 - \eta_1} < 2\eta_1.$$

We obtain

$$\frac{z^2 - z_0^2}{2\sqrt{z_0}} = I(\tilde{w}(z)) + \int_{z_0}^z \frac{d\tilde{w}(t)}{\sqrt{\prod_{j=1}^4 (\tilde{w}(t) - A'_j)}} Q_2(t),$$

where

$$|Q_2(t)| \leq 2\eta_1$$

for all t on our path.

On the set where $|\tilde{w}| \leq K$, we have $|w'(t)| \leq K_6|t|$ for some $K_6 > 1$. Thus

$$\begin{aligned} \left| \int_{z_0}^z \frac{d\tilde{w}(t)}{\sqrt{\prod_{j=1}^4 (\tilde{w}(t) - A'_j)}} Q_2(t) \right| &\leq \frac{\pi|z - z_0|K_6(1 + \varepsilon)\sqrt{|z_0|}2\eta_1}{\varepsilon_2^2} \\ &\leq \frac{4\pi\eta_1K_6}{\varepsilon_2^2}|z - z_0|\sqrt{|z_0|}. \end{aligned}$$

If $\eta_2 \in (0, 1/100]$ is another preassigned parameter, we may assume that η_1 was chosen so small that

$$\frac{4\pi\eta_1K_6}{\varepsilon_2^2} < \eta_2.$$

We now obtain

$$\tilde{w}(z) = P \left(\frac{z + z_0}{2\sqrt{z_0}}(z - z_0) + A_5|z - z_0|\sqrt{|z_0|} \right),$$

where the function A_5 satisfies $|A_5| \leq \eta_2$. Hence

$$\tilde{w}(z) = P \left(\frac{z + z_0}{2\sqrt{z_0}}(z - z_0) (1 + A_6) \right) = P(\sqrt{z_0}(z - z_0) (1 + A_7)),$$

where the functions A_6, A_7 satisfy $|A_6| \leq \eta_2/(1 - \varepsilon) < 2\eta_2$, $|A_7| \leq 2\eta_2 + \varepsilon$.

As z covers the disk $B(z_0, \varepsilon|z_0|)$, the quantity $\sqrt{z_0}(z - z_0)$, and hence also the quantity

$$Z = \sqrt{z_0}(z - z_0) (1 + A_7),$$

covers a set that is almost equal to a disk centered at the origin and of radius $\varepsilon|z_0|^{3/2}$ and hence of area $\pi\varepsilon^2|z_0|^3$. As z omits the set $E \cup E_1$, only a small fraction of this area can be omitted. Thus the quantity Z covers about

$$n_1 = \pi\varepsilon^2|z_0|^3/C$$

period parallelograms of P , covering, of course, only those parts of them where $|P| \leq K$. Now K is large, so values on the unit circle are taken by P at least about n_1 times in B_1 . By Cartan's identity, since ε is fixed, it follows that $T((1 + \varepsilon)|z_0|, w)$ is at least a fixed constant times $|z_0|^3$. Since we obtain this result for a sequence of points z_0 tending to infinity, it follows that the order of w is at least 3. This proves Theorem 1.1 in case A.

11. INTEGRATING OUTSIDE AN EXCEPTIONAL SET:
TRIGONOMETRIC CASE

11.1. **Case B.** We give a shorter proof since the details can be filled in in the same way as in Section 10.

We first consider the case

$$\lim_{z \rightarrow \infty, z \notin E} \left(\frac{U(z)}{z^2} + \frac{1}{4} \right) = 0.$$

We may proceed as in Section 10 up to

$$(11.1.54) \quad Q = \left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 - \left(\frac{U(z_0)}{z_0^2} + \frac{1}{4} \right) + c_1,$$

where

$$|c_1| \leq \frac{2|\alpha||w(z)|}{|z_0|^{3/2}}(1 + 2\varepsilon) + K_3\varepsilon + K^2\varepsilon + 3K^4\varepsilon \leq K_4\varepsilon,$$

provided that $\varepsilon > 0$ has been suitably fixed first, and then z_0 has been chosen large enough. Since $\lim_{z \rightarrow \infty, z \notin E} \left(\frac{U(z)}{z^2} + \frac{1}{4} \right) = 0$, (11.1.54) takes the form

$$(11.1.55) \quad Q = \left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 + c(z),$$

where we may assume that $|c(z)| \leq K\varepsilon$ for some $K > 0$ and $z \notin E$, again provided $|z|$ is large enough. Moreover, we may assume that the exceptional set E includes sufficiently small neighborhoods $V(q)$ of the zeros q of $w(z)$ to enable the argument below. Denoting now $\tilde{w}(z) := w(z)/\sqrt{z_0}$, $w_0 = w(z_0)$ we first get for $z \in B(z_0, \varepsilon|z_0|)$,

$$(11.1.56) \quad \int_{w_0}^w \frac{dw}{\sqrt{Q-c}} = \sqrt{z_0} \int_{\tilde{w}_0}^{\tilde{w}} \frac{d\tilde{w}}{\tilde{w}^2 + 1/2}.$$

Writing $\sqrt{Q} = \sqrt{Q-c} \sqrt{1+c/(Q-c)}$, we have $|c/(Q-c)| \leq K'\varepsilon$. Here we assume that $|w(z)| > A\sqrt{|z|}$ for a suitable small constant A for $z \notin E$. Therefore, $\sqrt{1+c/(Q-c)} = 1 + o(1)$ and so

$$(1 + o(1)) \int_{w_0}^w \frac{dw}{\sqrt{Q}} = \sqrt{z_0} \int_{\tilde{w}_0}^{\tilde{w}} \frac{d\tilde{w}}{\tilde{w}^2 + 1/2}.$$

Observing that $w'/z = \sqrt{Q}$, hence $\int_{w_0}^w \frac{dw}{\sqrt{Q}} = \int_{z_0}^z z dz$, and that $|z - z_0| < \varepsilon|z_0|$, we immediately obtain

$$(1 + o(1))(z - z_0)\sqrt{z_0} = \int_{\tilde{w}_0}^{\tilde{w}} \frac{d\tilde{w}}{\tilde{w}^2 + 1/2} = \sqrt{2} \int_{\tilde{w}_0}^{\tilde{w}} \frac{d(\tilde{w}\sqrt{2})}{(\tilde{w}\sqrt{2})^2 + 1}$$

$$= \sqrt{2}(\arctan(\sqrt{2}w(z)/\sqrt{z_0}) - \arctan(\sqrt{2}w(z_0)/\sqrt{z_0})).$$

Cover now $\{z : r < |z| < 2r\} \setminus E$ by a fixed number of discs $B(z_0, \varepsilon|z_0|)$, $\varepsilon > 0$ small enough. Indeed, we need at most $3/\varepsilon^2$ such discs. Let now β , say, be an arbitrary value on the unit circle. An elementary computation shows that there are at most $\sqrt{2}\pi^{-1}\varepsilon|z_0|^{3/2}(1+o(1))$ β -points of w in the disc $B(z_0, \varepsilon|z_0|)$, and these β -points are simple for z large enough. Since there are no such β -points in $E \cup E_1$ for z large enough, we see that there are at most

$$\frac{3\sqrt{2}}{\varepsilon}|z_0|^{3/2}(1+o(1)) \leq (2/\varepsilon)r^{3/2}$$

β -points of w in the annulus $\{z : r < |z| < 2r\}$ for r sufficiently large. Recalling the Cartan formula, we obtain that

$$T(r, w) = \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{i\theta}, w) d\theta + b,$$

b a constant, grows at most like $r^{3/2}$, meaning that $\rho(w) \leq 3/2$. Hence $\rho(w) = 3/2$.

11.2. Case C. Finally, suppose $w(z)$ is a transcendental solution of (1.0.1) such that

$$\lim_{z \rightarrow \infty, z \notin E} \frac{U(z)}{z^2} = 0.$$

We proceed to show that $\rho(w) = 3/2$ in this case as well. (In fact, we already needed to know this when we dealt with Case A.) Using the notations of the preceding case, we have

$$Q = \left(\frac{w(z)^2}{z_0} + \frac{1}{2} \right)^2 - \frac{1}{4} + c(z),$$

where the error term $c(z)$ satisfies $|c(z)| \leq K'\varepsilon$, hence $c(z) = 1 + o(1)$, using the same reasoning as above. Denoting this time $t(z) = w(z)/\sqrt{z_0}$, $t_0 = w(z_0)$, we obtain

$$(1+o(1)) \int_{w_0}^w \frac{dw}{\sqrt{Q}} = \sqrt{z_0} \int_{t_0}^t \frac{dt}{\sqrt{(t+1/2)^2 - 1/4}} = \sqrt{z_0} \int_{t_0}^t \frac{dt}{\sqrt{t^2 + t}}.$$

Since $w'/z = \sqrt{Q}$, we may integrate the right hand side and use the previous reasoning to the left hand side to obtain

$$(1+o(1))(z-z_0)\sqrt{z_0} = 2 \log(\sqrt{t} + \sqrt{1+t}) + O(1).$$

We may write this in the form

$$(11.2.57) \quad \sqrt{t} + \sqrt{1+t} = \exp\left(\frac{1}{2}(1+o(1))(z-z_0)\sqrt{z_0} + O(1)\right).$$

Covering again the set $\{z : r < |z| < 2r\} \setminus E$ by $3/\varepsilon^2$ discs $B(z_0, \varepsilon|z_0|)$ as in the previous case, taking a value β arbitrarily from the unit circle, and recalling the periodicity $2\pi i$ of the exponential function, we immediately see that there are at most $O(|z_0|^{3/2})$ β -points of w in such a disc $B(z_0, \varepsilon|z_0|)$. Since there are no β -points of $w(z)$ in the exceptional set for z_0 large enough, we deduce that $n(r, w) = O(r^{3/2})$. Making use of the Cartan formula again, we conclude that $w(z)$ would be of order $3/2$ at most, hence of order $\rho(w) = 3/2$.

REFERENCES

- [1] Clarkson, P., *Painlevé equations – nonlinear special functions*, In: *Orthogonal Polynomials and Special Functions*, 331–411, Lecture Notes Math. **1883**, Springer, Berlin, 2006.
- [2] Fokas, A., Its, A., Kapaev, A. and Novokshenov, V., *Painlevé Transcendents: The Riemann–Hilbert Approach*, American Mathematical Society, Providence, R.I., 2006.
- [3] Gromak, V., Laine, I. and Shimomura, S., *Painlevé Differential Equations in the Complex Plane*, W. de Gruyter, Berlin–New York, 2002.
- [4] Hayman, W.K., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
- [5] Hinkkanen, A., *On the averages of the counting function of a meromorphic function*, Ann. Acad. Sci. Fenn. Ser. AI Dissertationes 26 (1980).
- [6] Hinkkanen, A. and Laine, I., *Solutions of the first and second Painlevé equations are meromorphic*, J. Anal. Math. **79** (1999), 345–377.
- [7] Hinkkanen, A. and Laine, I., *Solutions of a modified third Painlevé equation are meromorphic*, J. Anal. Math. **85** (2001), 323–337.
- [8] Hinkkanen, A. and Laine, I., *Solutions of a modified fifth Painlevé equation are meromorphic*, Report Univ. Jyväskylä **83** (2001), 133–146.
- [9] Hinkkanen, A. and Laine, I., *The meromorphic nature of the sixth Painlevé transcendents*, J. Anal. Math. **85** (2004), 319–342.
- [10] Hinkkanen, A. and Laine, I., *Growth results for Painlevé transcendents*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 645–655.
- [11] Kitaev, A., *Elliptic asymptotics of the first and second Painlevé transcendents*, Russian Math. Surveys **49** (1994), 81–150.
- [12] Shimomura, S., *Growth of the first, the second and the fourth Painlevé transcendents*, Math. Proc. Camb. Phil. Soc. **134** (2003), 259–269.
- [13] Shimomura, S., *Lower estimates for the growth of the fourth and the second Painlevé transcendents*, Proc. Edinb. Math. Soc. **47** (2004), 231–249.
- [14] Steinmetz, S., *On Painlevé’s equations I, II and IV*, J. Anal. Math. **82** (2000), 363–377.
- [15] Steinmetz, N., *Value distribution of Painlevé transcendents*, Israel J. Math. **128** (2002), 29–52.

- [16] Steinmetz, N., *Boutroux's method vs. re-scaling lower estimates for the orders of growth of the second and fourth Painlevé transcendents*, Port. Math. **61** (2004), 369–374.
- [17] Steinmetz, N., *Global properties of the Painlevé transcendents, new results and open questions*, Ann. Acad. Sci. Fenn. A I Math. **30** (2005), 71–98.

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