

MAJORIZATION OF ANALYTIC FUNCTIONS

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let G be an open set in the complex plane \mathbb{C} with at least two finite boundary points. All notions of boundary and closure will be taken with respect to the finite plane \mathbb{C} . Let $f : \overline{G} \rightarrow \mathbb{C}$ be a continuous function that is analytic in G . We study the relationship between majorants of the modulus of continuity of f in \overline{G} and ∂G .

Let μ be a non-decreasing non-negative function defined for $t \geq 0$ such that

$$(1.1) \quad \mu(2t) \leq 2\mu(t)$$

for all $t \geq 0$. Such a function μ is called a **majorant**. We consider results of the type that $|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)$ for suitable choices of z_1 and z_2 , for some majorant μ , where C denotes a constant with $C \geq 1$.

Note that the constant function $\mu \equiv 0$ is a majorant, and that for all other majorants, we have $\mu(t) > 0$ for all $t > 0$. Of course, we are allowed to have $\mu(0) > 0$.

The following result was proved by the author in [5], which also contains an extensive survey of previous results.

Theorem 1. *If G is an open set with only bounded components, if f is analytic in G and continuous in \overline{G} , if μ is a majorant, and if*

$$(1.2) \quad |f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$$

for all $z_1, z_2 \in \partial G$, then

$$(1.3) \quad |f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)$$

for all $z_1, z_2 \in \overline{G}$ with $C = 3456$.

If (1.2) holds for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, then (1.3) holds for this z_1 and for all $z_2 \in \overline{G}$ with $C = 3456$.

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Apart from the best value of the constant C , this result completed a long line of research for analytic functions in open sets with only bounded components. The question remains of what can be said if the components of G are allowed to be unbounded. For Hölder-continuous functions this has been settled, in the presence of suitable growth conditions, with $C = 1$ on the right hand side of (1.3), by Gehring, Hayman, and the author [2], and for majorants μ for which $\log \mu(e^t)$ is a concave function of t , with $C = 1$, by the author [3].

In this paper we address this question by proving the following result.

Theorem 2. *Let G be an open set in the complex plane, let f be a function analytic in G and continuous in \overline{G} , and let μ be a majorant. Suppose that in each unbounded component U of G we have*

$$(1.4) \quad \liminf_{r \rightarrow \infty} \frac{\max\{|f(z)| : z \in \overline{U}, |z| = r\}}{r} = 0.$$

Firstly, if (1.2) holds for all $z_1, z_2 \in \partial G$, then (1.3) holds for all $z_1, z_2 \in \overline{G}$ with $C = 3456$.

Secondly, if (1.2) holds for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, then (1.3) holds for this z_1 and for all $z_2 \in \overline{G}$ with $C = 3456$.

We remark that some regularity condition such as (1.1) needs to be imposed on μ . Namely, an example due to Smith and Stegenga [6] shows that when we consider this situation for only one fixed $z_1 \in \partial G$, then the growth of the non-negative non-decreasing function

$$\mu_1(t) = \sup\{|f(z_1) - f(z_2)| : z_2 \in \partial G, |z_1 - z_2| \leq t\}$$

can be sufficiently irregular to prevent the validity of a result such as Theorem 1 for μ_1 instead of a majorant μ . Indeed, suppose that for any large $a > 0$, we choose a conformal mapping f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto the bounded domain which is the interior of the closure of $S_1 \cup S_2 \cup S_3$, where

$$S_1 = \{x + iy : |x| < 1, |y| < 1\},$$

$$S_2 = \{x + iy : 1 < x < 2, |y| < b\},$$

$$S_3 = \{x + iy : 2 < x < 2 + 2a, |y| < a\},$$

with $f(-1) = -1$, $f(0) = 0$, $f(1) = 2 + 2a$. Then, if we choose $a > 0$ to be large, and choose $b > 0$ to be small enough, depending on a , it follows that for some $t_0 \in (0, 2)$ (close to 2 if a is large), we have $|f(z) - f(-1)| \leq \sqrt{9 + b^2} < 4$ whenever $z \in \partial \mathbb{D}$ and $|z + 1| \leq t_0$, while $|f(t_0 - 1) - f(-1)| \geq a + 3$.

The case when μ is a positive constant in Theorems 1 and 2 corresponds to the classical maximum modulus principle for analytic functions. Thus Theorems 1 and 2 and their predecessors can be considered to be generalizations of the maximum principle.

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2. PRELIMINARY RESULTS

In order to avoid the repetition of certain arguments, we first note that the proofs given in [5] do, in fact, yield the following result.

Theorem 3. *Suppose that G is an open set with only bounded components and that f is analytic in G and continuous in \overline{G} , and let μ be a majorant of the form*

$$(2.1) \quad \mu(t) = A \max\{t_0, t\}$$

for some positive constants A and t_0 . Suppose that $z_1 \notin G$. If

$$(2.2) \quad |f(z)| \leq \mu(|z - z_1|)$$

for all $z \in \partial G$, then

$$(2.3) \quad |f(z)| \leq C\mu(|z - z_1|)$$

for all $z \in \overline{G}$ with $C = 1728$.

To understand this, see Lemma 1.8 in [5], and note that it is mentioned in the proof of [5], Theorem 1.11 that for majorants of the form (2.1), we obtain the desired conclusion with $C = 1728$. In [5], we always took $A = 1$ in (2.1), but this is immaterial since both f and μ could be multiplied by the same positive constant.

The proof of [5], Theorem 1.11 is also based on the approximation result [4], Theorem 2, p. 309. Thus one needs to check that that result can be used to obtain Theorem 3. However, this is routine.

The following lemma was proved in [3], Lemma 2, pp. 249–250. It is an extension of a result of Tamrazov ([7], Lemma 4.1, p. 156) who had assumed that ∂G is bounded and that f is bounded.

Lemma 4. *Let G be an open set in \mathbb{C} with at least one finite boundary point. Let f be analytic in G and continuous in \overline{G} , and suppose that*

$$(2.4) \quad f(z) = o(|z|^2)$$

as $z \rightarrow \infty$ in each unbounded component of G . Then, for every positive t , we have

$$\begin{aligned} & \sup\{|f(z_1) - f(z_2)| : |z_1 - z_2| \leq t, z_1, z_2 \in \overline{G}\} \\ &= \sup\{|f(z_1) - f(z_2)| : |z_1 - z_2| \leq t, z_1 \in \partial G, z_2 \in \overline{G}\}. \end{aligned}$$

3. PROOF OF THEOREM 2

A remark on [5]. It is worth discussing first a special case. We may assume that $f(z_1) = 0$. Suppose that for some $t_0 > 0$ we have $\mu(t_0) = 0$. Then (1.1) implies that $\mu(2^n t_0) = 0$ for all $n \geq 1$, and since μ is non-decreasing and non-negative, it follows that $\mu(t) = 0$ for all $t \geq 0$. This means that the assumption (1.2) states that $f(z) \equiv 0$ for all $z \in \partial G$. If U is any bounded component of G and if we assume that f is not identically zero in U of G , it follows from the fact that f is continuous in \overline{U} and that $f \equiv 0$ on ∂U , that $|f|$ attains its maximum inside U , which is a contradiction. Hence $f \equiv 0$ in U , and therefore $f \equiv 0$ in every bounded component of G .

In the proof of Lemma 1.8 in [5], we divided by $\mu(t_0)$ for a certain $t_0 > 0$, tacitly assuming that $\mu(t_0) > 0$. The above argument shows that such an assumption was justified. However we were then dealing with only bounded domains and open sets with only bounded components. If G can have an unbounded component U , we need to be more careful.

Proof of Theorem 2. Let G , f , and μ be as in the assumptions of Theorem 2. If we assume that (1.2) holds for all $z_1, z_2 \in \partial G$, then it follows from Lemma 4 that to prove (1.3) for all $z_1, z_2 \in \overline{G}$, we may assume that $z_1 \in \partial G$ and $z_2 \in G$. Therefore this will follow if we can prove that (1.2) for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$ implies (1.3) for this fixed z_1 and for all $z_2 \in G$.

We note that our assumption (1.4) does not imply the requirement (2.4) in any obvious way. However, an examination of the proof of [3], Lemma 2, and the proof of its source, [2], Theorem 2, pp. 244–245, shows that Lemma 4 is valid if (2.4) is replaced by (1.4).

Therefore it only remains for us to assume that $z_1 \in \partial G$ and that (1.2) holds for all $z_2 \in \partial G$.

We will consider the case $\mu \equiv 0$ later. If $\mu \not\equiv 0$, then, as we have explained before, we have $\mu(t) > 0$ for all $t > 0$. So we assume now that this is the case.

Fix $w \in G$ and set $t_0 = |z_1 - w| > 0$. Define

$$\mu_0(t) = 2\mu(t_0) \quad \text{for } 0 \leq t \leq t_0$$

and

$$\mu_0(t) = 2\mu(t_0)t/t_0 \quad \text{for } t > t_0.$$

Then μ_0 is a majorant.

Suppose that $z \in \partial G$. If $|z - z_1| \leq t_0$, then by assumption (1.2), we have

$$|f(z) - f(z_1)| \leq \mu(|z - z_1|) \leq \mu(t_0) = \frac{1}{2}\mu_0(|z - z_1|) \leq \mu_0(|z - z_1|).$$

If $|z - z_1| > t_0$, choose a positive integer m so that

$$2^{m-1}t_0 < |z - z_1| \leq 2^m t_0.$$

Then

$$\begin{aligned} |f(z) - f(z_1)| &\leq \mu(|z - z_1|) \leq \mu(2^m t_0) \leq 2^m \mu(t_0) \\ &< 2\mu(t_0)|z - z_1|/t_0 = \mu_0(|z - z_1|). \end{aligned}$$

Thus we have $|f(z) - f(z_1)| \leq \mu_0(|z - z_1|)$ for all $z \in \partial G$.

Suppose that $R > t_0$. Let G_R denote that component of

$$G \cap \{z : |z| < R\}$$

containing the point w . Then

$$\partial G_R \subset \partial G \cup (G \cap \{z : |z| = R\}).$$

Note that $G_R \subset U$ where U is the component of G containing w , hence independent of R .

We wish to be able to choose R so that

$$(3.1) \quad |f(z) - f(z_1)| \leq \mu_0(|z - z_1|)$$

for all $z \in \partial G_R$. If U is bounded, then for all sufficiently large R , we have $\partial G_R \subset \partial G$, so that (3.1) holds. Therefore let us assume that for all sufficiently large R , we have $(\partial G_R) \setminus (\partial G) \neq \emptyset$, so that in particular, U is unbounded.

Set $A = 2\mu(t_0)/t_0 > 0$ and $B = A/3 > 0$. By our assumption (1.4), we can choose $R > t_0 + 2|z_1| + 2|f(z_1)|/B$ so that

$$(3.2) \quad |f(z)| < B|z|$$

whenever $z \in U$ and $|z| = R$. Hence (3.2) holds for all $z \in (\partial G_R) \setminus (\partial G)$. For these z , we have

$$\begin{aligned} |f(z) - f(z_1)| &\leq |f(z)| + |f(z_1)| < B|z| + |f(z_1)| \\ &\leq B|z - z_1| + (B|z_1| + |f(z_1)|) \end{aligned}$$

while $|z - z_1| \geq R - |z_1| > t_0$, so that

$$\begin{aligned} \mu_0(|z - z_1|) &= A|z - z_1| = B|z - z_1| + (A - B)|z - z_1| \\ &\geq B|z - z_1| + (A - B)(R - |z_1|) \\ &\geq B|z - z_1| + (B|z_1| + |f(z_1)|) \end{aligned}$$

since

$$\begin{aligned} B|z_1| + |f(z_1)| &\leq BR/2 + BR/2 = BR = (A - B)R/2 \\ &\leq (A - B)(R - |z_1|). \end{aligned}$$

Thus (3.1) holds for all $z \in \partial G_R$.

We now apply Theorem 3 to the restriction of $f - f(z_1)$, instead of f , to the bounded domain G_R , noting that $z_1 \notin G_R$ and that $\mu_0(t) = A \max\{t_0, t\}$ where $A = 2\mu(t_0)/t_0$ as before. Since $w \in G_R$, it follows that

$$\begin{aligned} |f(w) - f(z_1)| &\leq 1728\mu_0(|w - z_1|) = 1728\mu_0(t_0) \\ &= 3456\mu(t_0) = 3456\mu(|w - z_1|), \end{aligned}$$

as required. This completes the proof of Theorem 2 when the function $\mu(t)$ is not identically zero.

Suppose then that $\mu(t) \equiv 0$. Set $g(z) = f(z) - f(z_1)$. Thus our assumption is that $g \equiv 0$ on ∂G . We wish to conclude that $g \equiv 0$ in G . This is clear if each component of G is bounded, so let us assume that G has an unbounded component. Note that (1.4) holds with f replaced by g . There would now be many ways of obtaining the desired conclusion, but perhaps the following method is the one that is based on the earliest available sufficiently strong result. We refer to the paper of Fuchs [1]. Pick any real number $M > 0$. Since $|g(z)| \leq M$ for all $z \in \partial G$ and since (1.4) holds with f replaced by g , [1], Theorem 1, p. 285, applied to g/M , not in G but separately in each unbounded component of G , implies that $|g(z)| \leq M$ for all z in each unbounded component of G , and hence for all $z \in G$. Since $M > 0$ can be arbitrarily small in this conclusion, we see that $|g(z)| = 0$ and hence $g(z) = 0$ for all $z \in G$.

This completes the proof of Theorem 2.

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