ORDER OF GROWTH OF PAINLEVÉ TRANSCENDENTS

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1. Introduction

The growth of transcendental solutions of the Painlevé differential equations have been recently considered in several papers by using complex analytic reasoning. In particular, this concerns the first Painlevé differential equation

$$f'' = 6f^2 + z, \tag{1.1}$$

as well as the second and fourth Painlevé equations

$$f'' = 2f^3 + zf + \alpha, \qquad \alpha \in \mathbb{C} \tag{1.2}$$

and

$$2ff'' = (f')^2 + 3f^4 + 8zf^3 + 4(z^2 - \alpha)f^2 + 2\beta, \qquad \alpha, \beta \in \mathbb{C}. \tag{1.3}$$

It is well known that all solutions of these differential equations are meromorphic functions in the complex plane, see, e.g., [2]. One of the recent results about their growth proved by complex analytic methods is the following.

Theorem 1.1. *All transcendental solutions of (1.2), resp. of (1.3), are of order of growth at least* $3/2$*, resp.* 2*.*

Three independent proofs of this theorem, by different methods, appeared recently, see [6], [4] and [7]. This paper aims to show how the proof offered in [4] may be substantially shortened by making use of an idea originally proposed by J. Langley [5] for the case of (1.2).

2. Short proof of Theorem 1.1

A key tool in the proof offered in [4] is a lemma due to G. Gundersen in [3]. We recall this lemma for the convenience of the reader:

Lemma 2.1. *Let* F *be a transcendental meromorphic function in the complex plane, of finite order* ρ*, and suppose that* $\varepsilon > 0$*. Then there exists a set* $E \subset [1, \infty)$ *of finite logarithmic measure such that whenever* $|z| \in [1, \infty)$*, then*

$$\left| \frac{F^{(j)}(z)}{F(z)} \right| \leq |z|^{j(\rho - 1) + \varepsilon}$$

for $j = 1, 2$*. Moreover, the same estimates hold on all radii* $z = re^{i\theta}$*,* $r \geq R(\theta) > 1$ *outside of a set* $E_1 \subset [0, 2\pi)$ *of zero linear measure.*

This material is based upon work supported by the National Science Foundation under Grants No. 0200752 and 0457291.

The second author has been partially supported by the Academy of Finland grant 210245. Both authors were also supported by the Väisälä Fund of the Finnish Academy of Science and Letters.

Case of (1.2). Suppose that f is a transcendental solution of (1.2) of order $\rho < 3/2$. Since the case $\alpha = 0$ results in an immediate contradiction, see [4], p. 647, we may assume that $\alpha \neq 0$. We now write (1.2) in the form

$$\frac{f''}{f} = 2f^2 + z + \frac{\alpha}{f}. \tag{2.1}$$

By Lemma 2.1, we have $|f''/f| = O(|z|^{1-\delta})$ for some $\delta > 0$ on almost all circles of radius r, centered at the origin. If there is now a sequence of such permitted circles, with their radii tending to infinity, such that each circle in this sequence admits at least one point z such that

$$|z|^{-1+\varepsilon} \le |f(z)| \le |z|^{\frac{1}{2}-\varepsilon}$$

for some $\varepsilon > 0$, an immediate contradiction to (2.1) follows. Therefore, we must have a sequence of circles such that either $|f(z)| > |z|^{\frac{1}{2}-\varepsilon}$ or $|f(z)| < |z|^{-1+\varepsilon}$ holds on all of these circles. In the first of these two possibilities, (2.1) implies that

$$2f^2 = (-1 + o(1))z$$

and we obtain a contradiction exactly as in [4]. In the remaining final case, see [4] again, we may write (1.2) in the form

$$\frac{\alpha}{f} + z = \frac{f''}{f} - 2f^2 =: \frac{z}{f} y(z). \tag{2.2}$$

But then

$$|y(z)| \le \left|\frac{f}{z}\right| \left|\frac{f''}{f}\right| + \frac{2}{|z|}|f|^3 = O(|z|^{-1-\gamma}) \tag{2.3}$$

for some $\gamma > 0$, provided that $\varepsilon > 0$ is small enough. Writing now

$$f(z) = -\frac{\alpha}{z} + y(z)$$

and integrating we obtain

$$-\alpha = \sum_{|z_j|<r} \operatorname{Res}(f, z_j) - \frac{1}{2\pi i} \int_{|\zeta|=r} y(\zeta)\, d\zeta,$$

where z_j stands for the poles of f in $|z| < r$. By (2.3), the last integral tends to zero as $r \to \infty$. Since the residues of f are integers, and actually ± 1, we conclude that α has to be an integer. To arrive at the final contradiction, see [6], we now apply the Bäcklund transformation

$$f_{\alpha-1} := -f + \frac{\alpha - \frac{1}{2}}{f' - f^2 - \frac{z}{2}}$$

or its inverse

$$f_{\alpha+1} := -f - \frac{\alpha + \frac{1}{2}}{f' + f^2 + \frac{z}{2}}$$

to obtain a solution $f_{\alpha-1}$ of (1.2) with parameter $\alpha - 1$, resp. $f_{\alpha+1}$ with parameter $\alpha+1$. Clearly, the Bäcklund transformations are order preserving. Hence, applying one of these transformations finitely many times, we obtain a transcendental solution f_0 of (1.2) with parameter $\alpha = 0$ and of order $< 3/2$, a contradiction.

Case of (1.3). We first remark that all fourth Painlevé transcendents possess infinitely many poles. This immediately follows by applying the standard Clunie lemma [1] to conclude that $m(r,f)=O(\log r)$ from (1.3). Here $m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$ is Nevanlinna's proximity function. Moreover, all residues of f are integers and in fact ±1. Suppose now that f is a solution of (1.3) of order $\rho=\rho(f)<2$. As shown in [4], p. 650, we may construct a sequence of circles, permitted by Lemma 2.1, with their radii tending to infinity so that just one of the following three possibilities holds on all of these circles for some $\delta>0$:

$$f(z)=-2z+O(|z|^{1-\delta}), \tag{2.4}$$

$$f(z)=-\frac{2}{3}z+O(|z|^{1-\delta}), \tag{2.5}$$

$$f(z)=\frac{A}{z}+O(|z|^{-1-\delta}), \qquad A^2=-\frac{\beta}{2}\neq 0. \tag{2.6}$$

Actually, just one of these alternatives holds in a connected set E tending to infinity, for $|z|$ large enough, see [4], p. 651. Moreover, by [4], pp. 651–652, it suffices to consider the cases (2.5) and (2.6) only. Defining now a meromorphic function W as

$$W:=-\frac{(f')^2+2\beta}{4f}+\frac{1}{4}f^3+zf^2+(z^2-\alpha)f, \tag{2.7}$$

we know that $W'=f^2+2zf$ and that there exist entire functions u,v such that $u'=-uW$, $v=uf$, see, e.g., [2], pp. 121–122.

Consider first the case (2.6). Supposing first that $\alpha\neq 2A$, a simple consideration implies that

$$f(z)=\frac{A}{z}-\frac{\beta(\alpha-2A)}{4Az^3}+y(z), \tag{2.8}$$

where $|y(z)|=O(|z|^{-3-\delta})$, see [4], p. 652. Substituting this expression into $W'=f^2+2zf$ we obtain

$$W'(z)=2A-\frac{A(A-\alpha)}{z^2}+s(z),$$

where $|s(z)|=O(|z|^{-2-\delta})$. By integration,

$$W(z)=2Az+\gamma+\frac{A(A-\alpha)}{z}+t(z),$$

where γ is the constant of integration and $|t(z)|=O(|z|^{-1-\delta})$. Integrating around a permitted circle similarly as in the case of (1.2), we get

$$\sum_{|z_j|<r}\operatorname{Res}(W,z_j)=A(A-\alpha)+\frac{1}{2\pi i}\int_{|\zeta|=r}t(\zeta)\,d\zeta.$$

Since $W=-\frac{u'}{u}$, we have $\operatorname{Res}(W,z_j)\in-\mathbb{N}$. Since the last integral tends to zero as r tends to infinity, there are at most finitely many poles of W, hence of f in the complex plane, contradicting the Clunie argument above.

Similarly, if $\alpha=2A$, we have

$$f(z)=\frac{A}{z}+\frac{3\beta(2+\beta)}{32Az^5}+O(|z|^{-5-\delta}), \tag{2.9}$$

see [4], p. 652. Substituting (2.9) into $W' = f^2 + 2zf$ and integrating we get

$$W(z) = 2Az + \gamma + \frac{1}{2}\frac{\beta}{z} + O(|z|^{-1-\delta}).$$

Integrating around a permitted circle as in the preceding case we now obtain

$$\sum_{|z_j|<r} \mathrm{Res}(W, z_j) = \frac{\beta}{2} + \frac{1}{2\pi i}\int_{|z|=r} O(|z|^{-1-\delta})\, dz,$$

resulting in a contradiction as before.

As for the remaining case (2.5), we first recall that

$$f(z) = -\frac{2}{3}z + \frac{\alpha}{z} - \frac{1 + 3\alpha^2 + \frac{9}{2}\beta}{4z^3} + O(|z|^{-3-\delta}),$$

see [4], p. 653. Proceeding as in the preceding case (2.6), we conclude that

$$W(z) = -\frac{8}{27}z^3 + \frac{2}{3}\alpha z + \gamma + \left(\frac{1}{6} - \frac{1}{2}\alpha^2 + \frac{3}{4}\beta\right)\frac{1}{z} + O(|z|^{-1-\delta})$$

and

$$\sum_{|z_j|<r} \mathrm{Res}(W, z_j) = \frac{1}{6} - \frac{1}{2}\alpha^2 + \frac{3}{4}\beta + \frac{1}{2\pi i}\int_{|z|=r} O(|z|^{-1-\delta})\, dz,$$

and a similar contradiction again follows.

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