ON THE NORM OF THE BEURLING–AHLFORS TRANSFORMATION: SUMMARY OF CHANGES FROM FIRST VERSION (12/28/07) TO SECOND VERSION (2/19/08), AND COMMENTS ON DIMENSION 4

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Sections 12, 13, and 14 in the second version are new, and the numbering of the other sections has changed accordingly.

We have also added a comment on weakened martingale conditions (16) in Section 3.

In Theorem 1, it is stated now that the result holds not only for $p-$norms but for Burkholder's functions u. Of course, the method of proof is that one first obtains the result for the functions u. It seems to be the case that before this paper, while there were results for the $p-$norm of the Beurling–Ahlfors transformation (involving constants > 1), there were no end results for the functions u, which were only used as intermediate tools.

We summarize what is done in the added sections. We refer to the paper for certain formulas and for the terminology used.

When are there matching lengths for two affine mappings ? In Section 12, we determine when for two affine mappings $f(z) = az+b\overline{z}$ and $g(z) = cz+d\overline{z}$ there exists an affine mapping $F(z) = Az+B\overline{z}$ that has matching lengths with each of f and g and is such that for these two comparisons (of F to f and to g) the martingale conditions (11) are satisfied. The answer, given in Lemma 2, is that this is possible if, and only if, f and g themselves have matching lengths.

We note that using the weakened martingale conditions (16) instead of the martingale conditions (11) would not help in those situations where each of the two affine mappings takes the unit circle onto a circle but when the two image circles have different radii, this being an example that can usually be given when formal methods, independent of the values of the parameters, fail.

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A different triangulation. In Section 13, we consider a triangulation that would probably provide a good basis for trying to prove the $p-$norm conjecture by induction on the number of triangles. The orientation of the triangles is somewhat different from the one in Theorem 1. We prove that in the small case considered, the conjecture can still be proved, but a much greater number of rotations is required.

Nonetheless, we manage to do this by using only rotations independent of the parameter values. The squares of the sines and cosines of the rotation angles turn out to be related to the first few Fibonacci numbers. There is probably a natural reason for this in this particular case. However, it is not clear whether there is any general pattern involving the Fibonacci numbers; one should not conclude that from so little evidence.

Adding more parameters and more triangles. In Section 14, we consider what happens when we add first one and then two new parameters (and more triangles) to the triangulation discussed in Section 13. We are able to combine terms several times from the boundary towards the center by using appropriate rotations, independent of the parameter values. However, this comes to an end at a certain point.

We show that the obstacle is not a lack of area identities (which is one possible obstruction, as discussed in Section 11), but the fact that when weights (the total area of the triangles used so far) increase, the denominators of the terms used will have to decrease (which phenomenon is based on area identities but is different from the lack of such identities). This makes it more difficult to find matching lengths, and suggests that here we have come to a point where parameter values must be taken into account to proceed.

Induction on the number of triangles, with the flexibility that comes from allowing the areas of the triangles vary ? We then suggest (in subsections 14.3–14.5) that to follow an induction process based on the number of triangles, one should allow the *areas of the triangles* in the domain of definition of the function *vary* (in all the figures used in the paper, all initial triangles have one or two possible areas only).

Thus, while it is the case that when new parameters are added, the sum of $u-$terms as in (22) or (23) may increase, it might still be the case that the new, larger sum does not exceed the sum of $u-$terms for a smaller triangulation provided that the areas of *some* (perhaps only one or two) of the triangles in the smaller triangulation are allowed to be different from what they were at the beginning of the induction step.

So if we assume that the problem has been solved for a certain number of triangles with arbitrary areas (subject to them forming a plane domain, of course) then one is hoping to be able to solve the problem for one more parameter as well. The larger problem with usually two new triangles and one new parameter would, however, be reduced not to the case obtained by simply deleting the two new triangles, but by replacing certain four triangles (including two old ones, see subsection 14.5) by up to two triangles of different areas that will fit the geometric mapping properties of the function.

This suggestion (which, if successful, would resolve the problem that the sum of $u-$terms may increase when adding a new parameter) seems worth pursuing.

On the Beurling–Ahlfors transformation in dimension 4**.** Here is a remark on a topic that is not yet in the paper. We have discussed in the paper the Beurling–Ahlfors transformation in dimension 3 in some detail. It turned out that the transformation defined by Iwaniec and Martin involving an $8-$tuple of complex-valued differential forms could be reduced to the transformation taking $\nabla\varphi + \nabla \times \mathbf{A}$ to $\nabla\varphi - \nabla \times \mathbf{A}$, where φ is a real-valued function in $\mathbb{R}^3$ and $\mathbf{A}$ is an $\mathbb{R}^3-$valued function in $\mathbb{R}^3$. Additionally, any reasonable function f in $\mathbb{R}^3$ (that tends to zero rapidly enough at infinity) can be represented in the form $f = \nabla\varphi + \nabla \times \mathbf{A}$ where φ and $\mathbf{A}$ are not unique. Indeed we may add a constant to φ and replace $\mathbf{A}$ by $\mathbf{A} + \nabla\psi$ for a real-valued function ψ in $\mathbb{R}^3$.

There turn out to be possibilities for similar reductions in higher dimensions. The computations get more complicated as the dimension n increases. Computations performed by the author when $n = 4$ suggest the following. We start with a tuple of $2^n = 16$ complex-valued differential forms in $\mathbb{R}^4$, and notice that it suffices to consider real-valued functions (for the same reason as in dimension 3) and that there are groups of functions with the same structure, so that only two groups remain to be considered; if the conjecture for the $p-$norms or the functions u can be proved for these only, then the same inequalities follow for the set of 16 forms, even if the forms were to be extended to be, say, $\mathbb{R}^m-$valued, for any m, or Hilbert-space valued, instead of complex-valued.

What are these two groups ? After re-indexing, one group reads as follows. We suggest that

$$\int_{\mathbb{R}^4} |\nabla\varphi + A|^p \leq (p^* - 1)^p \int_{\mathbb{R}^4} |\nabla\varphi - A|^p,$$

where $\varphi : \mathbb{R}^4 \to \mathbb{R}$, and $A : \mathbb{R}^4 \to \mathbb{R}^4$ depends on derivatives of 6 real-valued functions $f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}$ via (using the notation in the paper, so $f_{12}^2 = \partial f_{12}/\partial x_2$)

$A = (f_{12}^2 - f_{13}^3 + f_{14}^4, -f_{23}^1 - f_{13}^3 + f_{24}^4, -f_{13}^1 + f_{23}^2 + f_{34}^4, -f_{14}^1 + f_{24}^2 - f_{34}^3).$

The other group gives the conjecture that

$$\int_{\mathbb{R}^4} |B + C|^p \leq (p^* - 1)^p \int_{\mathbb{R}^4} |B - C|^p,$$

where B and C are certain $\mathbb{R}^6-$valued functions whose components (indexed by pairs (i, j) where $1 \leq i < j \leq 4$) depend on certain first-order partial derivatives of two $\mathbb{R}^4-$valued functions D and E. In hopefully obvious notation, $B_{ij} = \partial_i D_j - \partial_j D_i$ and if $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, then $C_{ij} = \pm\partial_k E_l \pm \partial_l E_k$ where the signs are determined in a suitable way.

For a general dimension $n \geq 4$, it should be the case that the problems on the $p-$norm of the Beurling–Ahlfors transformation in $\mathbb{R}^n$, as defined by Iwaniec and Martin involving an 2^n-tuple of complex-valued differential forms, can be reduced to the consideration of some smaller groups of tuples of real-valued functions, and furthermore there may be a connection to the representation of sufficiently nice functions in $\mathbb{R}^n$ by means of such expressions. In particular, among the resulting operators, there should be two differential operators D_1 and D_2 such that any nice f can be written as $f = D_1 f_1 + D_2 f_2$ for suitable functions f_1 and f_2 defined in $\mathbb{R}^n$ and taking values in $\mathbb{R}^{j_1}$ and $\mathbb{R}^{j_2}$, say, where j_1 and j_2 depend on n (corresponding to $f = \nabla\varphi + \nabla \times \mathbf{A}$ when $n = 3$, $j_1 = 1$, $j_2 = 3$) and one of the proper, simpler counterparts of the Beurling–Ahlfors transformation takes $D_1 f_1 + D_2 f_2$ to $D_1 f_1 - D_2 f_2$.

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