Math 285 — Midterm 2 practice solutions

Problem 1: We know the Fourier Series of a constant is just the same constant, so the F.S. of 2 is just 2. We now calculate the F.S. of t, which is an odd function, so we only need to calculate the Fourier series coefficients b_n (done in class). The requested F.S. is just 2 plus the F.S. of t.

Problem 2: Using the substitution suggested by the problem and setting $\mu = \lambda - 1$ we get the boundary value problem for $g(x)$:

$$g'' + \mu\, g = 0; \qquad g(0) = g(2) = 0$$

We have solved a problem like this in class. The eigenvalues for μ are $n^2\pi^2/4$ with n positive integers, and the eigenfunctions are sine functions. Rewriting for λ and y we get:

$$\lambda_n = 1 + \frac{n^2\pi^2}{4}; \qquad y_n = \mathrm{e}^x \sin\left(\frac{n\pi x}{2}\right)$$

Problem 3: This is a damped forced oscillator. First we look for the complementary solution by setting $x_c = \mathrm{e}^{rt}$. The solutions for r are $-1 \pm \sqrt{6}\mathrm{i}$. Therefore the general form of x_c is

$$x_c = \mathrm{e}^{-t}\left[A\cos(\sqrt{6}\,t) + B\sin(\sqrt{6}\,t)\right]$$

Given the form of the forcing, we search for a particular solution of the form

$$x_p = a\cos(3t) + b\sin(3t)$$

Using this x_p in the inhomogeneous equation we get the values for a and b so that

$$x_p = -\frac{3}{10}\cos(3t) - \frac{1}{10}\sin(3t)$$

The general solution is given by $x_c + x_p$. As time goes to infinity, the solution x_c will go to zero, so that the solution will tend to just x_p, which does not depend on the initial conditions. Due to friction, the information from the initial conditions is lost as x_c goes to zero.

Problem 4: This is a linear constant coefficient equation, so we first look for the complementary solution in the form $y_c = \mathrm{e}^{rx}$. The solutions for r are 2 and 4. Therefore

$$y_c = A\mathrm{e}^{2x} + B\mathrm{e}^{4x}$$

For the particular solution we try

$$y_p = ax^2 + bx + c$$

Substituting in the inhomogeneous equation we get

$$y_p = x^2 + \frac{3}{2}x + 1$$

The general solution is $y_c + y_p$.

Problem 5: The complementary solution is given by

$$y_c = A\cos(3x) + B\sin(3x)$$

therefore we use the following substitution for y_p:

$$y_p = u_1(x)\cos(3x) + u_2(x)\sin(3x)$$

Following the variation of parameters procedure, we arrive at the following equations

$$u_1' = -\frac{1}{3}\sin^2(3x); \qquad u_2' = \frac{1}{3}\sin(3x)\cos(3x)$$

We integrate these using the double angle formulas and get

$$u_1 = -\frac{1}{6}x + \frac{1}{36}\sin(6x); \qquad u_2 = -\frac{1}{36}\cos(6x)$$

and finally, using the double angle formulas again:

$$y_p = \frac{1}{6}x\cos(3x) + \frac{1}{36}\sin(3x)$$

where the second term can be dropped because it is already included in y_c.