Math 285 — Final practice solutions

Problem 1: This is a constant coefficients equation, so we look for a solution of the form $y = \mathrm{e}^{rx}$. The characteristic equation for r is:

$$r^5 - 4\,r^4 + 4\,r^3 = 0$$

which has solutions $r = 0$ (three times), and $r = 2$ (twice). The general solution is therefore:

$$y(x) = A + B\,x + C\,x^2 + D\,\mathrm{e}^{2x} + E\,x\,\mathrm{e}^{2x}$$

Notice that any time there is a multiple root of the characteristic equation we need to multiply by x to get independent solutions.

Problem 2: We first rewrite the equation as

$$y'' = \frac{1}{x^2}$$

which shows that it can just be integrated twice to get the general solution

$$y = -\ln x + A\,x + B$$

We then apply the initial conditions:

$$y(1) = A + B = 0$$
$$y'(1) = -1 + A = 0$$

Therefore $A = 1$ and $B = -1$ and the solution is

$$y = -\ln x + x - 1$$

Problem 3: These are all equations that you should know how to solve by heart (as mentioned at the beginning of the semester):

$$\text{a)}\;\; y = A\,\mathrm{e}^{2x} + B\,\mathrm{e}^{-2x}$$

$$\text{b)}\;\; y = B\,\mathrm{e}^{-\cos x}$$

$$\text{c)}\;\; y = A\,\cos{(\sqrt{2}x)} + B\,\sin{(\sqrt{2}x)}$$

If you don't remember, a) and c) are constant coefficients equations, so after setting $y = \mathrm{e}^{rx}$ you would solve the characteristic equation for r and get $r = \pm 2$ for a) and $r = \pm\mathrm{i}\sqrt{2}$ for c). Equation b) is separable:

$$\frac{\mathrm{d}y}{y} = \sin{(x)}\,\mathrm{d}x$$

which leads to

$$\ln{(y)} = -\cos{(x)} + A; \qquad y(x) = B\,\mathrm{e}^{-\cos{(x)}}$$

Problem 4: Following the hint given, we can use the substitutions

$$y(x) = \mathrm{e}^{x}\,g(x); \qquad y'(x) = \mathrm{e}^{x}\,g(x) + \mathrm{e}^{x}\,g'(x);$$
$$y''(x) = \mathrm{e}^{x}\,g(x) + 2\,\mathrm{e}^{x}\,g'(x) + \mathrm{e}^{x}\,g''(x)$$

in the equation for $y(x)$ to obtain the equation for $g(x)$ after some simplifications:

$$g'' + (\lambda - 1)\,g = 0$$

It's easier to make the additional substitution $\mu = \lambda - 1$. The boundary conditions are now $g(0) = g(2) = 0$ since e^{x} can't be 0 for any finite x. Therefore we have

$$g'' + \mu\,g = 0$$
$$g(0) = g(2) = 0$$

This is a typical eigenvalue problem that has solutions:

$$\mu_n = \frac{n^2\pi^2}{4}; \qquad g_n(x) = \sin\left(\frac{n\pi x}{2}\right)$$

We substitute this $g(x)$ and μ back into $y(x)$ and λ to find the solutions sought:

$$\lambda = 1 + \frac{n^2\pi^2}{4}; \qquad y(x) = \mathrm{e}^x \sin\left(\frac{n\pi x}{2}\right)$$

with $n = 1, 2, \ldots$

Problem 5: The Fourier series of a constant is the same constant. So let's disregard 2 for now and just look at the Fourier series of $2\,t^2$. This function is even, so we know that we don't neet to calcualte the B_n coefficients. The Fourier series coefficient calculation for A_0 and A_n give (notice that $L = 1$):

$$A_0 = \frac{1}{1}\int_{-1}^{1} 2\,t^2\,\mathrm{d}t = \frac{4}{3}$$

$$A_n = \frac{1}{1}\int_{-1}^{1} 2\,t^2\,\cos(n\pi t)\,\mathrm{d}t = \frac{8(-1)^n}{n^2\pi^2}$$

Adding back the constant 2 we have the following Fourier series:

$$\frac{8}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi t)$$

Problem 6: This is a typical diffusion equation problem. Using separation of variables: $y(x,t) = X(x)T(t)$, we get to the equations

$$2T_tX = TX_{xx}$$

$$\frac{2T_t}{T} = \frac{X_{xx}}{X} = -\lambda$$

Adding the boundary conditions, we have for X:

$$X_{xx} + \lambda X = 0; \qquad X_x(0) = X_x(3) = 0$$

The solutions are

$$\lambda_0 = 0; \qquad X_0 = 1$$
$$\lambda_n = \frac{n^2\pi^2}{9}; \qquad X_n = \cos\left(\frac{n\pi x}{3}\right)$$

The equation for T is

$$T_t = -\frac{\lambda}{2}T$$

and using the λ's found above we have

$$T_0 = 1$$

$$T_n = \mathrm{e}^{-\frac{n^2\pi^2 t}{18}}$$

Putting all of these together we get the general solution

$$y(x,t) = \frac{A_0}{2} + \sum_{n-1}^{\infty} A_n\, \mathrm{e}^{-\frac{n^2\pi^2 t}{18}} \cos\left(\frac{n\pi x}{3}\right)$$

We can now apply the initial condition:

$$y(x,0) = \frac{A_0}{2} + \sum_{n-1}^{\infty} A_n \cos\left(\frac{n\pi x}{3}\right) = g(x)$$

which leads to the formula for the coefficients

$$A_0 = \frac{2}{3}\int_0^3 g(x)\,\mathrm{d}x$$
$$A_n = \frac{2}{3}\int_0^3 g(x)\cos\left(\frac{n\pi x}{3}\right)\mathrm{d}x$$

As $t \to +\infty$, the solution reduces to $A_0/2$, which is the average of the initial condition. Since heat cannot escape the system, the total amount of heat in the system is unchanged, but it is now uniformly distributed, therefore y is a constant.

Problem 7: The equilibrium solutions are found by setting the r.h.s. to 0 and are $P = 0$ and $P = 4$ (twice). Sketching the slope field we observe that $P = 0$ is an unstable point, while $P = 4$ is neither stable nor unstable (It is stable below and unstable above). For the particular solutions, a look at the slope field (not shown, but easy to draw) indicates that a) will go to $-\infty$, b) to 4 and c) to $+\infty$.

Problem 8: For $a = 0$ the equation is

$$x'' + 16\,x = \sin(4t)$$

The complementary solution is

$$x_c(t) = A\cos(4t) + B\sin(4t)$$

Since the forcing (the inhomogeneous term on the r.h.s.) duplicates part of x_c, we look for a particular solution as $x_p = Ct\cos(4t) + Dt\sin(4t)$. Using this in the initial equation we get $C = -1/8$ and $D = 0$. Therefore the general solution for $a = 0$ is

$$x_c + x_p = A\cos(4t) + B\sin(4t) - \frac{1}{8}t\cos(4t)$$

Notice that the last term grows linearly in time. This is because the forcing has the same frequency as the system, so there is resonance.

For $a = 1$ there is no more resonance. The equation is now

$$x'' + x' + 16\,x = \sin(4t)$$

The complementary solution is found by setting the r.h.s. to 0 and trying $x_c = \mathrm{e}^{rt}$ to get

$$x_c = \mathrm{e}^{-\frac{t}{2}}\left[A\cos\left(\frac{\sqrt{63}t}{2}\right) + B\sin\left(\frac{\sqrt{63}t}{2}\right)\right]$$

Since there is no duplication (no resonance) we can just look for a particular solution as $x_p = C\cos(4t) + D\sin(4t)$ and get $C = -1/4$ and $D = 0$. The general solution for $a = 1$ is therefore

$$x_c + x_p = \mathrm{e}^{-\frac{t}{2}} \left[A\cos\left(\frac{\sqrt{63}t}{2}\right) + B\sin\left(\frac{\sqrt{63}t}{2}\right) \right] - \frac{1}{4}\cos(4t)$$

Friction (the term with a) has eliminated resonance (no more linear growth in time) and the information on the initial condition (A and B) is lost in time due to the decaying exponential.