Math 415 — Midterm 3 practice solutions

Problem 1: To be linear, a transformation has to verify the two properties: i) $L(\boldsymbol{x}+\boldsymbol{x}_1) = L(\boldsymbol{x}) + L(\boldsymbol{x}_1)$ and ii) $L(\alpha\boldsymbol{x}) = \alpha L(\boldsymbol{x})$ (where $\boldsymbol{x}_1$ just means a different vector from $\boldsymbol{x}$). Let's check these for case a):

$$L\left(\boldsymbol{x}+\boldsymbol{x}_1\right) = L\begin{pmatrix} x+x_1 \\ y+y_1 \\ z+z_1 \end{pmatrix} = \begin{pmatrix} 2(x+x_1)+y+y_1 \\ 2(x+x_1)+z+z_1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2x+y \\ 2x+z \end{pmatrix} + \begin{pmatrix} 2x_1+y_1 \\ 2x_1+z_1 \end{pmatrix} = L\left(\boldsymbol{x}\right) + L\left(\boldsymbol{x}_1\right)$$

Also

$$L\left(\alpha\boldsymbol{x}\right) = L\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} = \begin{pmatrix} 2\alpha x+\alpha y \\ 2\alpha x+\alpha z \end{pmatrix} = \alpha\begin{pmatrix} 2x+y \\ 2x+z \end{pmatrix} = \alpha L\left(\boldsymbol{x}\right)$$

So they are both verified and this is a linear transformation. Note also that the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

is such that $L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ (i.e. $\boldsymbol{A}$ is the matrix representation of the given L) and matrix multiplication is linear. So the transformation is linear.

For case b):

$$L\left(\alpha\boldsymbol{x}\right) = L\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} = \begin{pmatrix} 2\alpha x \\ 2 \end{pmatrix} \neq \alpha\begin{pmatrix} 2x \\ 2 \end{pmatrix} = \alpha L\left(\boldsymbol{x}\right)$$

so this transformation is not linear. We could also have checked property i) and see that it doesn't work. You only need to check that one doesn't work, to show that it is not linear. One could be tempted to say there is no matrix $\boldsymbol{A}$ such that $L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, but the non-existence of something is hard to prove...

Problem 2: Given a matrix $\boldsymbol{A}$, if it is complete, one can put the eigenvalues of $\boldsymbol{A}$ in a diagonal matrix $\boldsymbol{\Lambda}$ and the corresponding eigenvectors as columns in a matrix $\boldsymbol{V}$ and have

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$$

This is called diagonalizing $\boldsymbol{A}$. So we just have to write

$$\boldsymbol{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}; \qquad \boldsymbol{V} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

and calculate

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1} = \frac{1}{3}\begin{pmatrix} 8 & -2 \\ -1 & 7 \end{pmatrix}$$

This should be the matrix we needed. You can verify this by calculating the eigenvalues and eigenvectors of $\boldsymbol{A}$.

Problem 3: To find the eigenvalues we need to solve the characteristic equation

$$det\,(\boldsymbol{A} - \lambda\boldsymbol{I}) = det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & 4-\lambda & 0 \\ 6 & 1 & 1-\lambda \end{pmatrix} = 0$$

where $\boldsymbol{A}$ is the given matrix. To calculate the determinant of this 3×3 matrix we can use formula 1.86 in the book, which you may have learned in another class (maybe high school?), or the cofactor expansion, which we just learned in class, on the second row, to get

$$(4-\lambda)[(2-\lambda)(1-\lambda) - 6] = 0$$
$$(4-\lambda)(\lambda+1)(\lambda-4) = 0$$

So the eigenvalues are $\lambda = 4$ (double root) and $\lambda = -1$. For $\lambda = 4$ we look for the eigenvectors by finding the nullspace of $(\boldsymbol{A} - 4\boldsymbol{I})$ which gives us the single eigenvector $(1, 0, 2)^T$ (so the matrix is not complete.) For $\lambda = -1$ we solve for the nullspace of $(\boldsymbol{A} - (-1)\boldsymbol{I})$ which gives us the eigenvector $(1, 0, -3)^T$.

Problem 4: To find the standard matrix representation of a linear transformation we can apply it to the standard basis in $\mathbb{R}^3$: $(1,0,0)^T, (0,1,0)^T$, and $(0,0,1)^T$. For S, the vector along the z-axis will be unchanged, since the z-axis is on the $x = y$ plane. The vectors on the x-axis and the y-axis will change place with each other, so that

$$S\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}; \qquad S\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \qquad S\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Therefore the matrix representation for S is

$$\boldsymbol{S} = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$$

For T, the vector on the x-axis will not move, while the vector on the y-axis will go on the z-axis, and the vector on the z-axis will go to the y-axis, but in the negative direction:

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}; \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

Therefore the matrix representation for T is

$$\boldsymbol{T} = \begin{pmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{pmatrix}$$

The two transformation don't commute, and it can be verified by showing that $\boldsymbol{ST} \neq \boldsymbol{TS}$.

Problem 5: If $\boldsymbol{A}$ is the standard matrix representation and $\boldsymbol{B}$ is the matrix representation in the basis given by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then they are related as

$$\boldsymbol{B} = \boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} \qquad \text{where} \qquad \boldsymbol{V} = \begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}$$

Inverting this we get

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{B}\boldsymbol{V}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$