The question of whether countable compact Hausdorff spaces are Polish came up when having beer with a group of set theorists at a bar after an AMS meeting. Although the question perhaps wasn’t terribly interesting, I had fun finding the (positive) answer and the current note is the write-up of the proof.

To prove that a countable compact Hausdorff space \( X \) is Polish, it is enough to show that it is first-countable: indeed, the countability of \( X \) then implies that \( X \) is second-countable, allowing us to apply the Urysohn metrization theorem; thus \( X \) is compact metrizable and hence Polish. The proof below of the first-countability of \( X \) (somewhat surprisingly) goes through showing that \( X \) must be zero-dimensional.

**Proposition 1.** The cardinality of any compact Hausdorff perfect nonempty topological space \( X \) is at least continuum.

**Proof.** Using the perfectness and the normality of \( X \), construct a Cantor scheme \((U_s)_{s\in 2^{<\mathbb{N}}} \) such that

(i) \( U_s \) is nonempty open;
(ii) \( \overline{U_{s-i}} \subseteq U_s \), for \( i \in \{0, 1\} \).

For each \( x \in C \), \( \bigcap_n U_{x|_n} = \bigcap_n \overline{U_{x|_n}} \neq \emptyset \), by compactness. Hence, the Axiom of Choice gives an injection of \( C \) into \( X \).

From this we get the following corollary, which also follows from amenability of \( \mathbb{Z} \):

**Corollary 2.** There is no compact Hausdorff topology on \( \mathbb{Z} \) making the translation action of \( \mathbb{Z} \) on itself continuous.

**Proof.** Assume for contradiction that there is such a topology \( \tau \). If there is an isolated point, then all points are isolated, by the continuity of the translation action, which contradicts compactness. Thus \((\mathbb{Z}, \tau)\) is perfect, contradicting the above proposition.

Recall that a topological space \( X \) is called totally disconnected if every maximal connected component in \( X \) is a singleton.

**Proposition 3.** Any normal \( T_1 \) topological space \( X \) of cardinality less than continuum is totally disconnected.

**Proof.** Let \( Y \subseteq X \) be connected and assume for contradiction that there are distinct \( x, y \in Y \). Since \( X \) is \( T_1 \), the singletons \( \{x\}, \{y\} \) are closed. Thus, By Urysohn’s lemma, there is a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(y) = 1 \). But then \( f(Y) \) is connected and hence must contain \([0, 1]\), contradicting \( Y \) being less than continuum.
Recall that $X$ is called zero-dimensional if it admits a basis of clopen sets. Clearly, zero-dimensional implies totally disconnected, for $T_1$ spaces. The converse fails in general (even for metric spaces), but holds for locally compact Hausdorff spaces. The proof of this is not hard and can be found, for example, in [AT08] (Proposition 3.1.7). This and Proposition 3 together imply:

**Corollary 4.** Any locally compact normal $T_1$ space of cardinality less than continuum is zero-dimensional. In particular, any countable compact Hausdorff space is zero-dimensional.

**Lemma 5.** Any countable compact Hausdorff space $X$ is first-countable.

**Proof.** By Corollary 4, $X$ is zero-dimensional. Let $U$ be the collection of all open sets $V \subseteq X$ such that for all $x \in V$, there is a countable neighborhood base at $x$. Let $U$ be the union of all sets in $U$ and put $K = U^c$.

**Claim.** $K$ is perfect in the relative topology.

**Proof of Claim.** Assume for contradiction that there is $x \in K$ that is isolated in $K$. Hence there is a clopen neighborhood $V \subseteq X$ of $x$ such that $V \cap K = \{x\}$. Note that $x$ is not an isolated point in $X$ as otherwise $x \in U$. This implies in particular that $V$ is infinite ($X$ is Hausdorff). Enumerate $V \setminus \{x\} = \{x_n\}_{n \in \mathbb{N}}$ and inductively construct a decreasing sequence $(V_n)_{n \in \mathbb{N}}$ of clopen neighborhoods of $x$ such that

(i) $V_0 = V$;
(ii) $V_n \subseteq V$;
(iii) $x_n \notin V_{n+1}$.

Let $V_0 = V$ and assume $V_n$ is constructed. Note that $V_n$ is not a singleton as $x$ is not isolated in $X$, and let $k \in \mathbb{N}$ be the least such that $x_k \in V_n$. Take $V_{n+1} \subseteq V_n$ to be a clopen neighborhood of $x$ not containing $x_k$.

We show that the sets $V_n$ form a neighborhood base at $x$. First, put $U_n = V_n \setminus V_{n+1}$ and note that for every $m \in \mathbb{N}$,

$$V_m = \{x\} \cup \bigcup_{n \geq m} U_n. \quad (\ast)$$

Now let $W \subseteq X$ be a clopen neighborhood of $x$. Note that $W \supseteq U_n$, for all but finitely many $n \in \mathbb{N}$, as otherwise, by (i) and (\ast), $\{W \cap V\} \cup \{U_n \setminus W : n \in \mathbb{N}\}$ would be an infinite pairwise disjoint open cover of $V$, contradicting the compactness of $V$. By (\ast) again, this implies that $V_m \subseteq W$, for some $m \in \mathbb{N}$.

Thus $x$ has a countable neighborhood base and hence $V \in U$, contradicting $x \in K$. \hspace{1cm} \Box

By Proposition 1, $K$ has to be empty and hence $X$ is first-countable. \hspace{1cm} \Box

Corollary 4 and Lemma 5 imply the following theorem:

**Theorem 6.** Every countable compact Hausdorff space $X$ is Polish and zero-dimensional.

**Proof.** We only need to note that if a countable $X$ is first-countable, then it is second-countable. Hence by the Urysohn metrization theorem, $X$ is metrizable and thus Polish. \hspace{1cm} \Box

**References**