385 Differential Equations and Orthogonal Functions
Assignment 8-solutions

10.1.1
The Theorem 1 in 10.1 implies that all eigenvalues are nonnegative. If $\lambda = 0$, then the solution of the DE is

$$y(x) = Ax + B.$$

If $\lambda > 0$ and $\alpha = \sqrt{\lambda}$, then the equation

$$y'' + \alpha^2 y = 0$$

has general solution

$$y(x) = A\cos\alpha x + B\cos\alpha x.$$

The endpoint conditions give

$$B = 0 = \sin\alpha L,$$

so αL must be a multiple of $n\pi$. Positive eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2},$$

associated eigenfunctions are

$$y_n(x) = \cos\frac{n\pi x}{L}.$$

10.1.7
The coefficients c_n in Eq. (23) of this section is given by Formula (25) with $f(x) = r(x) = 1, a = 0, b = L$ and $y_n(x) = \sin\frac{\beta_n x}{L}$. Using the fact that $\tan\beta_n = -\frac{\beta_n}{hL}$, so

$$\frac{\sin\beta_n}{\beta_n} = -\frac{\cos\beta_n}{hL},$$

we find that

$$\int_0^L \sin^2\frac{\beta_n x}{L}dx = \int_0^L \frac{1}{2}(1-\cos\frac{2\beta_n x}{L})dx = \frac{1}{2}(x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L})|_0^L$$

$$= \frac{hL + \cos^2 \beta_n}{2h}$$

and

$$\int_0^L \sin \frac{\beta_n x}{L} dx = \frac{L(1 - \cos \beta_n}{\beta_n}.$$

The desired eigenfunction expansion is

$$1 = 2hL \sum_{n=1}^{\infty} \frac{1 - \cos \beta_n}{\beta_n (hL + \cos^2 \beta_n)} \sin \frac{\beta_n x}{L}.$$

10.1.11

If $\lambda = 0$, then the solution of the DE is

$$y(x) = Ax + B.$$

$y(0) = 0$ give s$B = 0$, so $y(x) = Ax$. Hence

$$hy(L) - y'(L) = A(hL - 1) = 0$$

if and only if $hL = 1$, in which case $\lambda_0 = 0$ has associated eigenfunction $y_0(x) = x$.

10.2.4

The substitution $u(x, y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0, Y'' - \alpha^2 Y = 0$$

with constant $\lambda = \alpha^2$. Due to example 5 the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, X(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda = \frac{\beta_n}{L}$ and eigenfunctions $X_n(x) = \sin \frac{\beta_n x}{L}$ with β_n being the positive roots of the equation $\tan x = -\frac{x}{hL}$. The bounded solution of

$$Y'' - \frac{\beta_n^2}{L^2} Y = 0$$

is

$$Y_n(y) = \exp(-\frac{\beta_n y}{L}),$$

so the resulting formal series solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{\beta_n x}{L} \exp(-\frac{\beta_n y}{L}).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int\limits_0^L f(x) \sin \frac{\beta_n x}{L} dx}{\int\limits_0^L \sin^2 \frac{\beta_n x}{L} dx} = \frac{4\beta_n}{L(2\beta_n - \sin 2\beta_n)} \int\limits_0^L \cos \frac{\beta_n x}{L} dx.$$