385 Differential Equations and Orthogonal Functions
Exam 2-Solutions

1.
a)
Homogeneous DE:

$$y^{(5)} - 5y^{(4)} + 9y^{(3)} - 5y'' = 0,$$

Solution:

$$y_c(x) = c_1 + c_2x + c_3e^x + e^{2x}(c_4\cos x + c_5\sin x).$$

b)

$$y_p(x) = Ax^2 + Bx^3 + Cxe^x.$$

2.
A theorem in the textbook says that n linearly independent solutions of an $n-$th order DE generate a general solution. Therefore, we have to check whether y_1, y_2 and y_3 are linearly independent. We can use the Wronskian and get

$$W = 2\sin x\cos x(2(\sin^2 x - \cos^2 x) - (-2\sin x\cos x)2(-\sin^2 x + \cos^2 x)) = 0,$$

so y_1, y_2 and y_3 are linearly dependent and do not generate a general solution.
Another way to find this out is to find a linear combination of y_1, y_2 and y_3 that is equal to 0. With $c_1 = -3, c_2 = 1, c_3 = 1$ we have

$$c_1y_1 + c_2y_2 + c_3y_3 = -3 + \sin^2 x + \cos^2 x + 2 = -3 + 1 + 2 = 0,$$

again this implies that y_1, y_2 and y_3 are linearly dependent.

3.
Characteristic Equation:

$$r^2 - 1 = 0;$$

Roots: 1,-1.

$$y(x) = c_1 e^{-x} + c_2 e^x.$$

Imposition of the initial conditions yields

$$0 = c_1 + c_2, 0 = c_1 e^{-1} + c - 2e,$$

so

$$c_2 = -c_1$$

and

$$0 = c_1(e^{-1} + e).$$

We obtain

$$c_1 = 0 = c_2,$$

and the only solution of this endpoint value problem is the trivial function $y(x) = 0$. Hence, $\lambda = -1$ is not an eigenvalue.

4.
Characteristic Equation:

$$r^2 - 4r + 4 = (r-2)^2 = 0;$$

Roots: 2,2, and

$$y_c(x) = c_1 e^{2x} + c_2 x e^{2x}.$$

We set $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$. The Wronskian of y_1, y_2 is

$$W(y_1, y_2, cx) = e^{2x}(e^{2x} + 2xe^{2x}) - xe^{2x}2e^{2x} = e^{4x}.$$

We can plug into the formula in the textbook:

$$\begin{aligned} y_p(x) &= -y_1(x)\int \tfrac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \tfrac{y_1(x)f(x)}{W(x)}dx \\ &= -e^{2x}\int \tfrac{xe^{2x}e^{2x}}{e^{4x}}dx + xe^{2x}\int \tfrac{e^{2x}e^{2x}}{e^{4x}}dx \\ &= -e^{2x}\int xdx + xe^{2x}\int 1dx \\ &= -e^{2x}\tfrac{x^2}{2} + xe^{2x}x \\ &= \tfrac{x^2}{2}e^{2x}. \end{aligned}$$

5.

a)

Characteristic Equation:

$$\frac{1}{2}r^2 + r + 50 = 0;$$

Roots: $-1 \pm i\sqrt{99}$.
$x_c(t) = e^{-t}(c_1 \cos \sqrt{99}t + c_2 \sin \sqrt{99}t)$.

b)

The body approaches the equilibrium position, as for all damped forced motions. Mathematically, this can be seen by

$$\lim_{t\to\infty} x(t) = 0.$$

6.

a)

Characteristic Equation:

$$r^2 + 100 = 0;$$

Roots: ± 10.
$x_c(t) = c_1 \cos 10t + c_2 \sin 10t$.
$x_p(t) = A\cos 10t + B\sin 10t$ does not work as a trial solution, since we have duplications with y_c (both terms are duplicated).

We have to multiply the trial solution by t to get rid of the duplications and obtain

$$x_p(t) = At\cos 10t + Bt\sin 10t.$$

Substituting this in the DE gives

$$\begin{aligned} x_p''(t) + 100x_p(t) &= -20A\sin 10t + 20B\cos 10t - 100Bt\sin 10t - 100At\cos 10 \\ &= -20A\sin 10t + 20B\cos 10t. \end{aligned}$$

This expression has to equal $\cos 10t$, and the values for A and B that achieve that are $A = 0, B = \frac{1}{20}$.
Therefore, the position function is

$$x(t) = x_c(t) + x_p(t) = c_1\cos 10t + c_2\sin 10t + \frac{1}{20}t\sin 10t.$$

b)

Yes, this is an example for resonance. The criteria to check this is $w_0 = \omega$, where ω is the factor in the argument of the cos in the external force (=10 in this problem) and $\omega_0 = \sqrt{\frac{k}{m}} = 10$. Another way to see this is the appearance of the factor t in the position function, which makes the poition function become large for large t.

c)

The main feature of resonance are the amplitudes that become arbitrarily large (cf. Section 3.6 in the textbook).