385 Differential Equations and Orthogonal Functions
Practice Problems

The following notes **sketch** solutions to the problems
1.1.32-34
1.2.11-13, 28, 43,
1.3.12,13,
1.4.19-22,
1.5.21-24,
1.6.12, 13, 20, 31, 32,
2.1.12,
2.3.4,
3.1.33.
In an exam you would be required to give more details than are given here.

1.1.32-34
32. $\frac{dP}{dt} = k\sqrt{P}$.
33. $\frac{dv}{dt} = kv^2$.
34. $\frac{dv}{dt} = k(250 - v)$

1.2.11-13
11. If $a(t) = 50$, then $v(t) = \int 50dt = 50t + v_0 = 50t + 10$. Hence

$$x(t) = \int (50t + 10)dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$, then $v(t) = \int(-20)dt = -20t + v_0 = -20t -$

15. Hence

$$x(t) = \int (-20t - 15)dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$, then $v(t) = \int 3tdt = \frac{3}{2}t^2 + v_0 = \frac{3}{2}t^2 + 5$. Hence

$$x(t) = \int (\frac{3}{2}t^2 + 5)dt = \frac{3}{2}t^3 + 5t = \frac{1}{2}t^3 + 5t.$$

1.2.28

$v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground ($y = 0$) when $t = 4.77s$, with velocity $v = v(4.77) = -192.64$ ft/sec, an impact speed of about 131 mph.

1.2.43

The velocity and position functions for the spacecraft are $v_s(t) = 0.0098t$ and $x_S(t) = 0.0049t^2$, and the corresponding functions for the projectile are $v_p(t) = \frac{1}{10}c = 3 \times 10^7$ and $x_P = 3 \times 10^7 t$. The condition that $x_P = x_S$ when the spacecraft overtakes the projectile gives $0.0049t^2 = 3 \times 10^7 t$, whence

$$t = \frac{3 \times 10^7}{0.0049} \approx 6.12245 \times 10^9 \text{ sec}$$

$$\approx \frac{6.12245 \times 10^9}{(3600)(24)(365.25)} \approx 194 \text{ years.}$$

1.3.11

Both $f(x) = 2x^2y^2$ and $\frac{df}{dy} = 4x^2y$ are continuous in a neighborhood of (1,1), so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 1$.

1.3.13

Both $f(x) = y^{1/3}$ and $\frac{df}{dy} = \frac{1}{3}y^{-2/3}$ are continuous in a neighborhood of (0,1), so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

1.3.15

$f(x) = (x-y)^{1/2}$is not continuous at (2,2), because it is not even defined if $x > y$. Hence the theorem gives neither existence nor uniqueness in any neighborhood of $x = 2$.

1.4.19

$$\begin{aligned} & \frac{dy}{dx} && = ye^x \\ \Rightarrow\ & \int \frac{dy}{y} && = \int e^x dx + C \\ \Rightarrow\ & \ln y && = e^x + \ln C \\ \Rightarrow\ & y(x) && = C\exp(e^x) \\ & y(0) && = 2e \\ \Rightarrow\ & y(x) && = 2\exp(e^x). \end{aligned}$$

1.4.20

$$\begin{aligned} & \frac{dy}{dx} && = 3x^2(y^2+1) \\ \Rightarrow\ & \int \frac{dy}{1+y^2} && = \int 3x^2 dx + C \\ \Rightarrow\ & \tan^{-1} y && = x^3 + C \\ \Rightarrow\ & y(x) && = C\tan(x^3 + C) \\ & y(0) && = 1 \\ & C && = \tan^{-1} 1 = \pi/4 \\ \Rightarrow\ & y(x) && = \tan(x^3 + \pi/4). \end{aligned}$$

1.4.21

$$\begin{aligned}
&2y\tfrac{dy}{dx} &&= \tfrac{x}{\sqrt{x^2-16}}\\
\Rightarrow\ &\int 2y dy &&= \int \tfrac{x}{\sqrt{x^2-16}} dx + C\\
\Rightarrow\ &y^2 &&= \sqrt{x^2-16} + C\\
&y(5) &&= 2\\
&C &&= 1\\
\Rightarrow\ &y^2 &&= 1 + \sqrt{x^2-16}.
\end{aligned}$$

1.4.22

$$\begin{aligned}
&\tfrac{dy}{dx} &&= 4x^3y - y\\
\Rightarrow\ &\int \tfrac{dy}{y} &&= \int 4x^3 - 1dx\\
\Rightarrow\ &\ln y &&= x^4 - x + \ln C\\
&y(1) &&= -3\\
&C &&= -3\\
\Rightarrow\ &y(x) &&= -3\exp(x^4 - x).
\end{aligned}$$

1.5.21

$$\begin{aligned}
&\rho &&= \exp(\int(-3\tfrac{dx}{x})) = x^{-3}\\
&\tfrac{d}{dx}(yx^{-3}) &&= \cos x\\
\Rightarrow\ &yx^{-3} &&= \sin x + C\\
&y(x) &&= x^3\sin x + Cx^3\\
&y(2\pi) &&= 0\\
&C &&= 0\\
\Rightarrow\ &y(x) &&= x^3\sin x.
\end{aligned}$$

1.5.22

$$\begin{aligned}
&\rho &&= \exp(\textstyle\int -2x dx) = \exp(-x^2)\\
&\tfrac{d}{dx}(y\exp(-x^2)) &&= 3x^2\\
\Rightarrow\ &y\exp(-x^2) &&= x^3 + C\\
&y(x) &&= (x^3 + C)\exp(x^2)\\
&y(0) &&= 5\\
&C &&= 5\\
\Rightarrow\ &y(x) &&= (x^3 + 5)\exp(x^2).
\end{aligned}$$

1.5.23

$$\begin{aligned}
&\rho &&= \exp(\textstyle\int 2 - \frac{3}{x} dx) = x^{-3}\exp(2x)\\
&\tfrac{d}{dx}(yx^{-3}\exp(2x)) &&= 4e^{2x}\\
\Rightarrow\ &yx^{-3}\exp(2x) &&= 4e^{2x}\\
&y(x) &&= 2x^3 + Cx^3\exp(-2x)
\end{aligned}$$

1.5.24

$$\begin{aligned}
&\rho &&= \exp(\textstyle\int \frac{3x}{x^2+4} dx) = (x^2+4)^{3/2}\\
&\tfrac{d}{dx}(y(x^2+4)^{3/2}) &&= x(x^2+4)^{1/2}\\
\Rightarrow\ &y(x^2+4)^{3/2} &&= \tfrac{1}{3}(x^2+4)^{1/2} + C\\
&y(x) &&= \tfrac{1}{3} + C(x^2+4)^{-3/2}\\
&C &&= 16/3\\
&y(x) &&= \tfrac{1}{3} + \tfrac{16}{3}(x^2+4)^{-3/2}
\end{aligned}$$

1.6.12

Homogeneous DE;

$$\begin{aligned}
&xvv' &&= \sqrt{v^2+4}\\
&\textstyle\int \frac{vdv}{\sqrt{v^2+4}} &&= \textstyle\int \frac{dx}{x}\\
\Rightarrow\ &\sqrt{v^2+4} &&= \ln x + C\\
&v^2 + 4 &&= (\ln x + C)^2\\
&4x^2 + y^2 &&= x^2(\ln x + C)^2.
\end{aligned}$$

1.6.13

Homogeneous DE;

$$\begin{aligned} & xv' && = \sqrt{v^2+1} \\ & \int \tfrac{dv}{\sqrt{v^2+1}} && = \int \tfrac{dx}{x} \\ \Rightarrow\ & \ln(v+\sqrt{v^2+1}) && = \ln x + C \\ & v+\sqrt{v^2+1} && = Cx \\ & y+\sqrt{x^2+y^2} && = Cx^2. \end{aligned}$$

1.6.17

Substitution $v = 4x + y$ leads to

$$\begin{aligned} & v' && = v^2+4 \\ & x && = \int \tfrac{dv}{v^2+4} = \tfrac{1}{2}\tan^{-1}\tfrac{v}{2} + \tfrac{C}{2} \\ \Rightarrow\ & v && = 2\tan(2x-C) \\ & y && = 2\tan(2x-C) - 4x. \end{aligned}$$

1.6.20

Bernoulli; Substitution $v = y^3$ leads to

$$\begin{aligned} v' + 6xv &= 18x \\ \rho &= e^{3x^2} \\ y^3 &= 3 + Ce^{-3x^2}. \end{aligned}$$

1.6.31

$$M(x,y) = 2x + 3y, N(x,y) = 3x + 2y.$$

To verify exactness, just check

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To calculate the solution:

$$\begin{aligned} F(x,y) &= \int 2x+3ydy = x^2+3xy+g(y) \\ F_y &= 3x+g'(y) = 3x+2y = N \\ g'(y) &= 2y \\ g(y) &= y^2 \\ x^2+3xy+y^2 &= C. \end{aligned}$$

1.6.32

$$M(x,y) = 4x-y, N(x,y) = 6y-x.$$

To verify exactness, just check

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To calculate the solution:

$$\begin{aligned} F(x,y) &= \int 4x-ydy = 2x^2-xy+g(y) \\ F_y &= -x+g'(y) = 6y-x = N \\ g'(y) &= 6y \\ g(y) &= 3y^2 \\ 2x^2-xy+3y^2 &= C. \end{aligned}$$

2.1.12
Solution of the equation

$$P' = kP^2$$

by separation of variables gives

$$P(t) = \frac{1}{C-kt}.$$

Now $P(0) = 12$ implies $C + \frac{1}{12}$, so $P(t) = \frac{12}{1-12kt}$. Then $P(10) = 24$ implies that $k = \frac{1}{240}$, so finally $P(t) = \frac{240}{20-t}$. Hence $P = 48$

when $t = 15$, that is, in the year 2003. And obviously $P \to \infty$ as $t \to 20$. **2.3.4**
Equation:

$$v' = -kv^2, v(0) = v_0, x' = v, x(0) = x_0.$$

Solution:

$$\begin{aligned} -\int \frac{dv}{v^2} &= \int k dt \\ \frac{1}{v} &= kt + C \\ C &= \frac{1}{v_0} \\ x'(t) &= v(t) = \frac{v_0}{1+v_0 kt} \\ x(t) &= \frac{1}{k}\ln(1 + v_0 kt) + x_0. \end{aligned}$$

3.1.33

$$r^2 - 3r + 2 = 0; r_{1,2} = 1, 2; y(x) = c_1 e^x + c_2 e^{2x}.$$