385 Differential Equations and Orthogonal Functions
Practice Problems for Exam 2

The following notes **sketch** solutions to the problems
3.2.11,12,14,15,30,
3.3.15,16,33,34,
3.5.22,24,26,39,40,
3.6.5,6,
3.8.13,14.
In an exam you would be required to give more details than are given here.

3.2.11
$W = x^3e^{2x}$ is nonzero if $x \neq 0$.

3.2.12
$W = x^{-2}(2\cos^2(\ln x)+2\sin^2(\ln x)) = 2x-2$ is nonzero for $x > 0$.

3.2.14
Imposition of the initial conditions $y(0) = 0, y'(0) = 0, y''(0) = 3$ on the general solution $y(x) = c_1e^x+c_2e^{2x}+c_3e^{3x}$ yields the three equations

$$c_1 + c_2 + c_3 = 0, c_1 + 2c_2 + 3c_3 = 0, c_1 + 4c_2 + 9c_3 = 3$$

with solution $c_1 = 3/2, c_2 = -3, c_3 = 3/2$. Hence the desired particular solution is given by $y(x) = (3e^x - 6e^{2x} + 3e^{3x})/2$.

3.2.15
Imposition of the initial conditions $y(0) = 2, y'(0) = 0, y''(0) = 0$ on the general solution $y(x) = c_1e^x + c_2xe^x + c_3x^2e^x$ yields the

three equations

$$c_1 = 0, c_1 + c_2 = 0, c_1 + 2c_2 + 2c_3 = 0$$

with solution $c_1 = 2, c_2 = -2, c_3 = 1$. Hence the desired particular solution is given by $y(x) = (2 - 2x + x^2)e^x$.

3.2.30

When the equation

$$x^2y'' - 2xy' + 2y = 0$$

is rewritten in the standard form

$$y'' + (-2/x)y' + (2/x^2)y = 0,$$

the coefficient functions $p_1(x) = (-2/x)$ and $p_2(x) = (2/x^2)$ are not continuous at $x = 0$. Thus the hypotheses of Theorem 3 are not satisfied.

3.3.15

Characteristic Equation:

$$4r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r - 2)^2(r + 2)^2 = 0;$$

Roots: 2,2,-2,-2.

$$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}.$$

3.3.16

Characteristic Equation:

$$r^4 + 18r^2 + 81 = (r^2 + 9)^2 = 0;$$

Roots: $\pm 3i, \pm 3i$.

$$y(x) = (c_1 + c_2x)\cos 3x + (c_3 + c_4x)\sin 3x.$$

3.3.33

Knowing that $y = e^{3x}$ is a solution, we divide the characteristic polynomial $r^3 + 3r^2 - 54$ by $r - 3$ and get the quadratic factor

$$r^2 + 6r + 18 = (r+3)^2 + 9$$

with roots $-3 \pm 3i$.
Hence the general solution is

$$y(x) = c_1 e^{3x} + e^{-3x}(c_2 \cos 3x + c_3 \sin 3x).$$

3.3.34

Knowing that $y = e^{2x/3}$ is a solution, we divide the characteristic polynomial $3r^3 - 2r^2 + 12r - 8$ by $3r - 2$ and get the quadratic factor

$$r^2 + 4$$

with roots $\pm 2i$.
Hence the general solution is

$$y(x) = c_1 e^{2x/3} + c_2 \cos 2x + c_3 \sin 2x.$$

3.5.22

$y_c = (c_1 + c_2 x + c_3 x^2) + c_4 e^x + c_5 e^{-x}$.
$y_p = (A + Bx + Cx^2) + De^x$ does not work as a trial solution, since we have duplications with y_c (all 4 terms are duplicated). We have to multiply the first term in the trial solution by x^3 and the second term by x to get rid of the duplications and obtain $y_p = x^3(A + Bx + Cx^2) + xDe^x$.

3.5.24
$y_c = c_1 + c_2e^{-3x} + c_3e^{4x}$.
$y_p = (A+Bx)+(C+Dx)e^{-3x}$ does not work as a trial solution, since we have duplications with y_c (all 4 terms are duplicated). We have to multiply both terms in the trial solution by x to get rid of the duplications and obtain
$y_p = x(A + Bx) + x(C + Dx)e^{-3x}$.
3.5.26
$y_c = (c_1 \cos 2x + c_2 \sin 2x)e^{3x}$.
$y_p = (A + Bx)e^{3x} \cos 2x + (C + Dx)e^{3x} \sin 2x$ does not work as a trial solution, since we have duplications with y_c (all 4 terms are duplicated). We have to multiply both terms in the trial solution by x to get rid of the duplications and obtain
$y_p = x((A + Bx)e^{3x} \cos 2x + (C + Dx)e^{3x} \sin 2x)$.
3.5.39
$y_c = c_1 + c_2x + c_3e^{-x}$.
$y_p = x^2(A + Bx) + x(Ce^{-x})$.
General solution:

$$y(x) = y_c(x) + y_p(x) = c_1 + c_2x + c_3e^{-x} - x^2/2 + x^3/6 + xe^{-x}.$$

By equating coefficients we get the system of equations

$$c_1 + c_3 = 1, c_2 - c_3 + 1 = 0, c_3 - 3 = 1$$

and the solution

$$y(x) = (-18 + 18x - 3x^2 + x^3)/6 + (4 + x)e^{-x}.$$

3.5.40
$y_c = c_1e^{-x} + c_2e^x + c_3 \cos x + c_4 \sin x$.
$y_p = A$.

General solution:

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^x + c_3 \cos x + c_4 \sin x - 5.$$

By equating coefficients we get the system of equations

$$c_1 + c_2 + c_3 - 5 = 0, -c_1 + c_2 + c_4 = 0, c_1 + c_2 - c_3 = 0, -c_1 + c_2 - c_4 = 0$$

and the solution

$$y(x) = (5e^{-x} + 5e^x + 10 \cos x - 20)/4.$$

3.6.5

Complementary solution:

$$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{k/m}$. Trial solution:

$$x_p(t) = A \cos \omega t + B \sin \omega t.$$

Substitution and equating coefficients gives

$$B = 0, A = F_0/(k - m\omega^2),$$

so

$$x(t) = x_c(t) + x_p(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + (F_0/(k - m\omega^2)) \cos \omega t.$$

Imposition of the initial conditions gives

$$x(t) = (x_0 - (F_0/(k - m\omega^2))) \cos \omega_0 t + (F_0/(k - m\omega^2)) \cos \omega t.$$

3.6.6

Complementary solution:

$$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{k/m}$. If we write this DE in the form

$$x'' + \omega_0^2 x = (F_0/m)\cos\omega_0 t,$$

then this is Equation (13) in 3.6 in the textbook. There the particular solution is calculated to be

$$x_p(t) = (F_0/2m\omega_0)t\sin\omega_0 t$$

(Equation 14). Therefore, the general solution is

$$x(t) = x_c(t) + x_p(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + (F_0/2m\omega_0)t\sin\omega_0 t.$$

Imposition of the initial conditions gives

$$x(t) = \frac{2mv_0 + F_0 t}{2m\omega_0}\sin\omega_0 t.$$

3.8.13

a)

With $\lambda = 1$, the general solution of $y'' + 2y' + y = 0$ is

$$y(x) = Ae^{-x} + Bxe^{-x}.$$

But then $y(0) = A = 0$ and $y(1) = e^{-1}(A + b) = 0$. Hence $A = 0 = B$ and λ is not an eigenvalue.

b)

If $\lambda < 1$, then the equation $y'' + 2y' + y = 0$ has the general solution

$$y(x) = Ae^{rx} + Be^{sx},$$

where r and s are the two complex roots $(-2 \pm \sqrt{4 - 4\lambda})/2$ of the characteristic equation.

The conditions $y(0) = y(1) = 0$ yield

$$A + B = 0, Ae^r + Be^s = 0.$$

If $A, B \neq 0$, then it follows that $e^r = e^s$. But $r \neq s$, so this is impossible and there is no eigenvalue $\lambda < 1$.

c)

If $\lambda > 1$ let $\lambda - 1 = \alpha^2$, so $\lambda = 1 + \alpha^2$. Then the characteristic equation

$$r^2 + 2r + \lambda = (r+1)^2 + \alpha^2 = 0$$

has roots $-1 \pm \alpha i$, so

$$y(x) = e^{-x}(A \cos \alpha x + B \sin \alpha x).$$

Now $y(0) = A = 0$, so $y(x) = Ae^{-x} \sin \alpha x$. Next, $y(1) = Ae^{-1} \sin \alpha = 0$, so $\alpha = n\pi$ with an integer n. Thus the $n - th$ positive eigenvalue is $\lambda_n = n^2\pi^2 + 1$. Because $\lambda = \alpha^2 + 1$, the eigenfunction associated with λ_n is

$$y_n(x) = e^{-x} \sin n\pi x.$$

3.8.14

If $\lambda = 1 + \alpha^2$ then we first impose the initial condition $y(0) = 0$ on the solution

$$y(x) = e^{-x}(A \cos \alpha x + B \sin \alpha x)$$

found in problem 13, and find that $A + 0$. Hence

$$\begin{aligned} y(x) &= Be^{-x} \sin \alpha x \\ y'(x) &= B(-e^{-x} \sin \alpha x + e^{-x} \alpha \cos \alpha x), \end{aligned}$$

so the condition $y'(1) = 0$ yields $-\sin \alpha + \alpha \cos \alpha = 0$, that is, $\tan \alpha = \alpha$.