**385 Differential Equations and Orthogonal Functions**
**Exam 3**

Total points: 36 (=6 $\times$6 points). Do all questions. Explain all answers. No notes, books, calculators or computers are allowed.
To answer the questions, you may use the back of the paper.

**Formulas**

In the following, $m$ and $n$ denote positive integers.

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mt \cos nt dt &= 0 \text{ if } m \neq n \\
&\quad \pi \text{ if } m = n \\
\int_{-\pi}^{\pi} \sin mt \sin nt dt &= 0 \text{ if } m \neq n \\
&\quad \pi \text{ if } m = n \\
\int_{-\pi}^{\pi} \cos mt \sin nt dt &= 0 \text{ for all } m, n \\
\int u \cos u du &= \cos u + u \sin u + C \\
\int u \sin u du &= \sin u - u \cos u + C \\
\int u^n \cos u du &= u^n \sin u - n \int u^{n-1} \sin u du \\
\int u^n \sin u du &= -u^n \cos u + n \int u^{n-1} \cos u du \\
\int \sin^2 u du &= \tfrac{1}{2}u - \tfrac{1}{4} \sin 2u + C \\
\int \cos^2 u du &= \tfrac{1}{2}u + \tfrac{1}{4} \sin 2u + C.
\end{aligned}$$

**1.** Calculate the Fourier Series of the periodic function of period 2 given by

$$f(t) = \frac{t^2}{2}$$

on $-1 < t < 1$.

**2.** At which points does the Fourier Series of the following period 4 function converge to $f$ (**you do not have to calculate the Fourier Series**)?

$$f(t) = \begin{array}{l} 2, \text{ if } 1 < t < 2 \\ 0, \text{ if } 0 < t < 1 \\ 1, \text{ if } -1 < t < 0 \\ -1, \text{ if } -2 < t < -1 \\ \frac{1}{2}, \text{ if } t = 0, \pm 2, \pm 1 \end{array}$$

**3.**

a)

Calculate the Fourier **Sine** Series of the function given by

$$f(t) = 4t$$

on $0 < t < 1$.

b)

Sketch the graph of the odd extension of $f$.

**4.**

a)

Calculate the Fourier **Cosine** Series of the function given by

$$f(t) = \begin{array}{l} t, \text{ if } 0 < 1 \leq 2 \\ 2 - t, \text{ if } 1 < t < 2 \end{array}$$

on $0 < t < 1$.

b)

Sketch the graph of the even extension of $f$.

**5.** Find a periodic solution of the endpoint problem

$$x'' + 4x = F(t)$$

$$x(0) = 0 = x(1),$$

where $F(t) = 4t$ for $0 < t < 1$.

**6.** We consider a vibrating string fixed at its endpoints 0 and 1. With special initial conditions, this gives rise to a boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(0,t) &= y(1,t) = 0 \\ y(x,0) &= f(x) \\ y_t(x,0) = 0, & \end{aligned}$$

where $f$ is the function of problem 3. Solve this boundary value problem by applying the method of separation of variables.

Hint: You may use without calculation, that the eigenvalues and eigenfunctions of the boundary value problem

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= X(1) = 0 \end{aligned}$$

are

$$\lambda_n = n^2\pi^2, X_n(x) = \sin n\pi.$$