385 Differential Equations and Orthogonal Functions
Exam 3-solutions

1.

$$\begin{aligned} L &= 1 \\ a_0 &= \frac{1}{1}\int_{-1}^{1} \frac{t^2}{2} dt = \frac{1}{3} \\ a_n &= \frac{1}{1}\int_{-1}^{1} \frac{t^2}{2} \cos n\pi t dt \\ &= \frac{2\cos n\pi}{n^2\pi^2} = \frac{2(-1)^n}{n^2\pi^2} \\ b_n & \ 0 \text{ , since f is even} \\ f(t) &\approx \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2\pi^2} \cos n\pi t. \end{aligned}$$

2.
The Fourier Series converges to f at points where f is continuous and at points where the average of the right-sided and the left-sided limit

$$\frac{f(t+) + f(t-)}{2}$$

is equal to $f(t)$ (cf. Theorem 1 in 9.2). These are all points except the odd integers.

3.

$$\begin{aligned} L &= 1 \\ b_n &= \frac{2}{1}\int_{0}^{1} 4t \sin n\pi t dt \\ &= \frac{-8\cos n\pi}{n\pi} \\ &= \frac{8(-1)^{n+1}}{n\pi} \\ f(t) &\approx \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin n\pi t. \end{aligned}$$

The graph looks like figure 9.3.4 in the textbook.

4.

$$
\begin{aligned}
L &= 2 \\
a_0 &= \tfrac{2}{2}\Big(\int_0^1 t\,dt + \int_1^2 (2-t)\,dt\Big) = \tfrac{1}{3} \\
a_n &= \tfrac{2}{2}\Big(\int_0^1 t\cos\tfrac{n\pi t}{2}\,dt + \int_1^2 (2-t)\cos\tfrac{n\pi t}{2}\,dt\Big) \\
&= \tfrac{8\cos n\pi/2}{n^2\pi^2} - \tfrac{4}{n^2\pi^2} - \tfrac{4\cos n\pi}{n^2\pi^2} \\
f(t) &\approx \tfrac{1}{6} + \sum_{n=1}^{\infty}\Big(\tfrac{8\cos n\pi/2}{n^2\pi^2} - \tfrac{4}{n^2\pi^2} - \tfrac{4\cos n\pi}{n^2\pi^2}\Big)\cos\tfrac{n\pi t}{2}.
\end{aligned}
$$

The graph looks like figure 9.3.3 in the textbook.

5.
This is Example 2 in Section 9.3.

6.
This is problem A in 9.6 (p. 623 ff). The coefficients A_n in (22) are then the coefficients calculated in problem 3.