

446 Complex Analysis and its Applications
Assignment 1-selected solutions

7.10

If $S = 1 + z + z^2 + \dots + z^n$, then

$$S - zS = (1 + z + z^2 + \dots + z^n) - (z + z^2 + \dots + z^{n+1}) = 1 - z^{n+1}.$$

Hence, $S = \frac{1-z^{n+1}}{1-z}$, provided $z \neq 1$. That is,

$$1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Setting $z = e^{i\theta}$ ($0 < \theta \leq 2\pi$) in this identity, we obtain

$$1 + e^{i\theta} + e^{i2\theta} + \dots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - z}.$$

Now the real part of the left-hand side here is evidently

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta;$$

and, to find the real part of the right-hand side, we write that side in the form

$$\frac{1 - \exp(i(n+1)\theta)}{1 - \exp(i\theta)} \frac{\exp(-i\frac{\theta}{2})}{\exp(-i\frac{\theta}{2})} = \frac{\exp(-i\frac{\theta}{2}) - \exp(i\frac{(2n+1)\theta}{2})}{\exp(-i\frac{\theta}{2}) - \exp(i\frac{\theta}{2})},$$

which becomes

$$\frac{\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} - \cos \frac{(2n+1)\theta}{2} - i \sin \frac{(2n+1)\theta}{2}}{-2i \sin \frac{\theta}{2}} \frac{i}{i},$$

or

$$\frac{(\sin \frac{\theta}{2} + \sin \frac{(2n+1)\theta}{2}) + i(\cos \frac{\theta}{2} - \cos \frac{(2n+1)\theta}{2})}{2 \sin \frac{\theta}{2}}.$$

The real part of this is

$$\frac{1}{2} + \frac{\sin \frac{(2n+1)\theta}{2}}{2 \sin \frac{\theta}{2}},$$

and we arrive at Lagrange's identity.

10.5.

a)

$$\begin{aligned}
 (\pm\sqrt{A}\exp(i\frac{\alpha}{2}))^2 &= (\pm 1)^2(\sqrt{A})^2(\exp(i\frac{\alpha}{2}))^2 \\
 &= A\exp(i\frac{\alpha}{2} + i\frac{\alpha}{2}) \\
 &= \sqrt{a^2 + 1}\exp(i\alpha) \\
 &= |a + i|\operatorname{Arg}(a + i) \\
 &= a + i.
 \end{aligned}$$

b)

$\pi/2 > \alpha/2 > 0$, since $a + 1$ is in the upper half plane. This means that $\cos(\frac{\alpha}{2}) = \sqrt{\frac{1+\cos\alpha}{2}} > 0$, and $\sin(\frac{\alpha}{2}) = \sqrt{\frac{1-\cos\alpha}{2}} > 0$. Hence,

$$\begin{aligned}
 \sqrt{A}\exp(i\frac{\alpha}{2})^2 &= \sqrt{A}(\cos(\frac{\alpha}{2}) + i\sin(\frac{\alpha}{2})) \\
 &= \sqrt{A}(\sqrt{\frac{1+\cos\alpha}{2}} + i\sqrt{\frac{1-\cos\alpha}{2}}) \\
 &= \frac{1}{\sqrt{2}}(\sqrt{A + A\cos\alpha} + i\sqrt{A - A\cos\alpha}) \\
 &= \frac{1}{\sqrt{2}}(\sqrt{A + a} + i\sqrt{A - a}).
 \end{aligned}$$