446 Orthogonal Functions and its Applications
Assignment 8-solutions

99.1

We have $u(s,t) = s^3 - 3st^2$, so $u_t(s,t) = -6st$ and $u_s(s,t) = 3s^2 - 3t^2$. To calculate v using the formula on page 353, we have to integrate first along the horizontal path from (0,0) to $(x, 0)$ and then along the vertical path from (x, o) to (x, y). The firts integral of $-u_t$ is 0:

$$\int_{(0,0)}^{(x,0)} -6stds = -3st|_{s=0,t=0}^{s=x,t=0} = 0.$$

Hence, we only need to consider the second integrand, and

$$v(x,y) = \int_{(0,0)}^{(x,y)} u_s dt = \int_{(0,0)}^{(x,y)} s^3 - 3st^2 = 3s^2t - t^3|_{(0,0)}^{(x,y)} = 3x^2y - y^3.$$

The analytic function is
$f(x + iy) = u(x,y) + iv(x,y) = x^3 - 3xy^2 + i(3x^2y - y^3)$,
so $f(z) = z^3$.

99.3

Using the theorem in Section 98:
We have $h(x,y) = u^2 - v^2$ and $f(x+iy) = e^{x+iy} = e^x \cos y + ie^x \sin y$, so
$u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Then

$$H(x,y) = h(u(x,y), v(x,y)) = (e^x \cos y)^2 - (e^x \sin y)^2 = e^{2x} - \cos 2y$$

is normal due to this theorem. To verify this directly:

$$H_{xx}(x,y) + H_{yy}(x,y) = 4e^{2x} \cos 2y - 4e^{2x} \cos 2y = 0.$$

99.5

$$f(z) = z^2 = x^2 - y^2 + i2xy,$$

so $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$ and

$$H(x,y) = h(u(x,y), v(x,y)) = e^{-x^2+y^2} \cos 2xy.$$

The normal derivative along the positive y-axis is

$$H_x(0,y) = -2xe^{-x^2+y^2}\cos 2xy - e^{-x^2+y^2}2y\sin 2xy|_{x=0} = 0,$$

and along the positive x-axis we similarly obtain

$$H_y(x,0) = 0.$$

99.6
Here,

$$H(x,y) = h(u(x,y),v(x,y)) = 4xy + e^{-x^2+y^2}\cos 2xy.$$

Then

$$H_x(x,0) = 0$$

and

$$H_x(0,y) = 0.$$