446 Complex Analysis and its Applications
extra problems-solutions

1.
We have to look for z points satisfying

$$(1+z)^4 = 16i.$$

These are the points satisfying

$$1+z = 2e^{i(\frac{pi}{8}+2\pi n/4)}, n \in \mathbb{N}\ ,$$

so

$$z_n = 2e^{i(\frac{pi}{8}+2\pi n/4)} - 1.$$

A closer look reveals that $z_m = z_{m+4}$ for all $m \in \mathbb{N}$, so we have only four distinct solutions

$$z_n = 2e^{i(\frac{pi}{8}+2\pi n/4)} - 1, n \in \{1,2,3,4\}.$$

2.
See Appendix 2, Figure 8, p.443.

3.

$$2+2i = \sqrt{8}e^{\pi/4+2\pi n},$$
$$i = e^{-\pi/2+2\pi n}, n \in \mathbb{N}\ .$$

4.
The only accumulation point is 0, the set is neither open nor closed.

5.
It is not continuous in 0, so it is not analytic. f is continuous in 1, but not analytic (polar Cauchy-Riemann equations fail). The real part is not harmonic.

6.
See textbook p. 93.

7.

This is a special case of a problem dealt with on p.93 in the textbook.
A way to see show this is to note the definition

$$\text{Log } z = \ln|z| + i\text{Arg}z,$$

and from a picture one can easily see that

$$\lim_{x+iy\to -1 y>0} = \pi$$

and

$$\lim_{x+iy\to -1 y<0} = -\pi,$$

so the function Arg is not continuous in a point contained in the negative real axis. Hence the imaginary part of Log is not continuous, so Log itself is not continuous.

8.
See textbook p.101.

9.
See textbook p.101.

10.
To be entire, a function has to be single-valued. Entire functions are cos, z^3, exp. Those functions are also single-valued, additionally tan and Log.
11.

$i^i = \exp(i\log i) = \exp(i((ln|i|+iargi))) = \exp(i((0+i(\pi/2+i2\pi n)))) = \exp(-\pi/2+2\pi n).$