446 Complex Analysis and its Applications
Practice Exam 2-solutions

1.

There are infinitely many parametric representations, an example is

$$z(t) = \begin{array}{ll} t & ,0 \leq t \leq 1 \\ 1 + i(t-1) & ,1 \leq t \leq 2 \\ 1 + i - (t-2) & ,2 \leq t \leq 3 \\ i + (-1-i)(t-3) & ,3 \leq t \leq 4 \\ -1 + (t-4) & ,4 \leq t \leq 5. \end{array}$$

2.

a)

$$\begin{aligned} \int\limits_C f dz &= \int\limits_{\pi}^{2\pi} f(z(\phi))z'(\phi)d\phi \\ &= \int\limits_{\pi}^{2\pi} (z(\phi)-1)z'(\phi)d\phi \\ &= \int\limits_{\pi}^{2\pi} (1+e^{i\phi}-1)ie^{i\phi}d\phi \\ &= \int\limits_{\pi}^{2\pi} ie^{2i\phi}d\phi \\ &= (1-1)/2 = 0. \end{aligned}$$

b) $z(t) = t, 0 \leq t \leq 2$ and

$$\begin{aligned} \int\limits_C f dz &= \int\limits_{0}^{2} f(z(t))z'(t)dt \\ &= \int\limits_{0}^{2} (z(t)-1)z'(t)dt \\ &= \int\limits_{0}^{2} (t-1)1dt \\ &= 2-2-0 = 0. \end{aligned}$$

3.
$C : z(t) = e^{-it}, 0 \leq t \leq 2\pi$ and

$$\begin{aligned}
\int\limits_C f dz &= \int\limits_0^{2\pi} (e^{-it})^m (\overline{e^{-it}})^n (-i) e^{-it} dt \\
&= -i \int\limits_0^{2\pi} \exp(-imt + int - it) dt \\
&= -i \int\limits_0^{2\pi} \exp(it(n-1-m)) dt \\
&= -i \tfrac{1}{n-1-m} \tfrac{1}{i} \exp(it(n-1-m))|_0^{2\pi} \\
&= 0
\end{aligned}$$

if $m \neq n - 1$ and

$$\begin{aligned}
\int\limits_C f dz &= \int\limits_0^{2\pi} (e^{-it})^m (\overline{e^{-it}})^n (-i) e^{-it} dt \\
&= -i \int\limits_0^{2\pi} \exp(-imt + int - it) dt \\
&= -i \int\limits_0^{2\pi} \exp(it(n-1-m)) dt \\
&= -i \int\limits_0^{2\pi} 1 dt \\
&= -2\pi i
\end{aligned}$$

if $m = n - 1$.

4.
The contour $C_1 + C$ is a simply closed contour, and f is analytic on and inside $C + C_1$. By the Theorem of Cauchy and Goursat,

$$\int\limits_C f dz + \int\limits_{C_1} f dz = \int\limits_{C+C_1} f dz = 0.$$

Hence,

$$\int\limits_{C_1} f dz = -\int\limits_C f dz = 4 - 2\pi i.$$

5.
48.1.a):
The Cauchy Integral Formula with

$$f(z) = e^{-z}, z_0 = \pi i/2, m = 1$$

gives

$$\int\limits_C \frac{e^{-z}}{z - \pi i/2} dz = 2\pi i f(\pi i/2) = 2\pi$$

(note that f is analytic on and inside C, as required for the Cauchy Integral Formula).
For the second question, the integrand f is analytic on both contours and in the region between these. The principle of the transformation of paths (Section 46, Corollary 2) yields that the value of the integral is again 2π.

6.
a)

$$f(z) = e^z, f^{(n)}(z) = e^z, f^{(n)}(1) = e$$

for all $n \geq 0$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = e\sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n.$$

b)
Using

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

and setting $w = z - 1$, we obtain

$$f(z) = ee^{z-1} = e\sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n.$$

The radius of convergence is the distance from te point we expand about to the next singular point. f has no singular points, so the radius is ∞ (as for all entire functions).

7.
See Example 3 in Section 56.

8.

$$\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

so

$$\begin{aligned} z^2 \sin(1/z^2) &= z^2 \textstyle\sum_{n=0}^{\infty}(-1)^n \frac{(z^{-2})^{2n+1}}{(2n+1)!} \\ &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-4n-2+2} \\ &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-4n}. \end{aligned}$$

9.

$$\frac{1}{(z-2)^2} = \frac{d}{dz}\frac{1}{2-z}$$

and

$$\begin{aligned}\tfrac{1}{2-z} &= \tfrac{1}{z-3}\tfrac{-1}{1-\frac{-1}{z-3}}\\ &= \tfrac{1}{z-3}\sum_{n=0}^{\infty} -(\tfrac{-1}{z-3})^n\\ &= -\sum_{n=0}^{\infty}(-1)^n(z-3)^{-n-1},\end{aligned}$$

so

$$\begin{aligned}\tfrac{1}{(z-2)^2} &= \tfrac{d}{dz}(-\sum_{n=0}^{\infty}(-1)^n(z-3)^{-n-1})\\ &= -\sum_{n=0}^{\infty}(-1)^n(-n-1)(z-3)^{-n-2}.\end{aligned}$$

$$\begin{aligned}\tfrac{1}{z+1} &= \tfrac{1}{z-3}\tfrac{1}{1-\frac{-4}{z-3}}\\ &= \tfrac{1}{z-3}\sum_{n=0}^{\infty}(-4)^n(z-3)^{-n}\\ &= \sum_{n=0}^{\infty}(-4)^n(z-3)^{-n-1}.\end{aligned}$$

We obtain

$$\begin{aligned}\tfrac{1}{(z-2)^2}+\tfrac{1}{z+1} &= -\sum_{n=0}^{\infty}(-1)^n(-n-1)(z-3)^{-n-2}+\sum_{n=0}^{\infty}(-4)^n(z-3)^{-n-1}\\ &= \sum_{n=2}^{\infty}(-1)^{n+1}(-n+1)(z-3)^{-n}+\sum_{n=1}^{\infty}(-4)^{n+1}(z-3)^{-n}\\ &= \tfrac{16}{z-3}+\sum_{n=2}^{\infty}(-1)^{n+1}(-n+1+4^{n+1})(z-3)^{-n}.\end{aligned}$$

Note that using the geometric series is valid in all cases, since $|1/(z-3)| < 1$ $|4/(z-3)| < 1$and in the annulus under consideration.

10.

For the Laurent Series, see Example 4 in Section 62.
To use the Theorem in Section 66, we write

$$\phi(z) = 1/z, m = 4, z_0 = 2$$

and obtain

$$\text{Res}_{z=2}f(z) = \frac{\phi^{(3)}(2)}{3!} = -\frac{6/2^4}{3!} = -1/16$$

and calculate the integral as in the textbook.

11.
See Example 5 in Section 62.

12.
The singular points are $1, -3i, 3i$. Only 1 is enclosed by C, so we only need to consider the residue at this point. One way to calculate it is setting

$$p(z) = \frac{3z^3+2}{z^2+9}, q(z) = z-1,$$

so

$$p(1) \neq 0, q(1) = 0, q'(1) \neq 0$$

and Theorem 2 in Section 69 gives

$$\text{Res}_{z=1} f(z) = \frac{p(1)}{q'(1)} = 1/2,$$

so

$$\int\limits_C f dz = 2\pi i 1/2 = \pi i.$$

13.
The singular points are the points $\pi/2 + n\pi, n \in \mathbb{Z}$.
To apply Theorem 1 in Section 69, we set

$$p(z) = \sin z, q(z) = (\cos z)^2.$$

Then for each $n \in \mathbb{Z}$,

$$p(\pi/2+n\pi) \neq 0, q(\pi/2+n\pi) = 0 = q'(\pi/2+n\pi), q''(\pi/2+n\pi) \neq 0,$$

so f has poles of order 2 at these points.

14.
See Example 3 in Section 69.

15.
The singular points are at $-2i, 2i$. Again we use the theorem in Section 66: For $z_0 = -2i$:

$$\phi(z) = \frac{ze^z}{(z-2i)^3}, m = 3, z_0 = -2i,$$

so $\phi(-2i) \neq 0$ and

$$\mathrm{Res}_{z=-2i} f(z) = \frac{\phi''(-2i)}{2!} = \ldots,$$

and for $z_0 = 2i$ we set

$$\phi(z) = \frac{ze^z}{(z+2i)^3}, m = 3, z_0 = 2i,$$

and calculate the residue the same way.