446 Complex Analysis and its Applications
Test1-solutions

1.
a)
For $k \in \mathbb{Z}$ we have

$$\begin{aligned} & (1-z)^{10} && = 1024 = 2^{10} e^{2\pi k i} \\ \Leftrightarrow \quad & 1 - z && = 2e^{2\pi k i/10} \\ \Leftrightarrow \quad & z && = 1 - 2e^{2\pi k i/10}. \end{aligned}$$

Since $e^{2\pi i} = 1$, we have 10 distinct values for z that satisfy the given equation. These are evenly distributed on the circle centered at 1 with radius 2.

2.
We check the Cauchy-Riemann equations:

$$\begin{aligned} f(x+iy) &= \exp(x - iy) = \exp(x)\exp(-iy) \\ &= e^x \cos y - i e^x \sin y. \end{aligned}$$

With $u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$ we obtain $u_x(x,y) = e^x \cos y \neq -e^x \cos y = v_y(x,y)$, which is only true if $\cos y = 0$. But for such values of y we have $u_y(x,y) = -e^x \sin y \neq e^x \cos y = -v_x(x,y)$, so the Cauchy-Riemann equations are nowhere satisfied and f is nowhere analytic.

3.

a)

The image is the annulus $\{z : e^{-2} < |z| < 1\}$.

b)

The image is the whole complex plane $\mathbb{C}$.

4.

$$\begin{aligned}(1+i)^{1+i} &= \exp((1+i)\log(1+i)) \\ &= \exp((1+i)(\ln|2i| + i\text{arg }2i)) \\ &= \exp((1+i)(\ln\sqrt{2} + i(\pi/4 + 2k\pi))) \\ &= \exp(\ln\sqrt{2} - (\pi/4 + 2k\pi))\exp(i(\ln\sqrt{2} + \pi/4 + 2k\pi)) \\ &= \sqrt{2}\exp(-\pi/4 - 2k\pi)\exp(i(\ln\sqrt{2} + \pi/4)).\end{aligned}$$

5.

One way is given in the textbook on p.102. Another short way would be the following:

Any zero of cos satisfies $\cos z = \frac{e^{iz}+e^{-iz}}{2} = 0$, so $e^{iz} = -e^{-iz}$. If $z = x + iy$, then this equation can be written as

$$e^{ix}e^{y} = e^{-ix}e^{-y}.$$

The values on both sides of the equation are equal, so, in particular, their moduli are equal. Computing the moduli gives $|e^{ix}e^{y}| = e^{y}$ and $|e^{-ix}e^{-y}| = e^{-y}$, and the equation $e^{y} = e^{-y}$ is satisfied if and only if $y = 0$.

6.

$$|\sin(iy)| = \frac{e^{y} - e^{-y}}{2},$$

and if we make y large we can make the modulus of $\sin(iy)$ arbitrarily large.

7.

We calculate

$$\begin{aligned}(\sqrt{3}+i)^2 &= 3 - 1 + i2\sqrt{3} = 2 + i2\sqrt{3} \text{ and} \\ (\sqrt{3}+i)^3 &= (2 + i2\sqrt{3})(\sqrt{3}+i) \\ &= 2\sqrt{3} - 2\sqrt{3} + i(2\sqrt{3}^2 + 2) \\ &= 8i,\end{aligned}$$

so $\sqrt{3}+i$ is a cubic root of $8i$. The cubic roots of $8i$ are

$$2\exp(i(\frac{\pi}{6}+\frac{2k\pi}{3})), (k=0,1,2).$$

$\sqrt{3}+i$ is in the upper right quadrant, and $2e^{i\pi/6}$ is the only cubic root of $8i$ in this quadrant. Hence, $\sqrt{3}+i=2e^{i\pi/6}$, so $\operatorname{Arg}(\sqrt{3}+i)=\frac{\pi}{6}$ and

$$\tan\frac{\pi}{6}=\arctan(\frac{1}{\sqrt{3}}).$$