

446 Complex Analysis and its Applications
Exam 2-solutions

1.

The Cauchy Integral Formula with

$$g(z) = \frac{e^z}{(z+1)^3}, z_0 = \pi i/2, m = 1$$

gives

$$\int_C \frac{e^z}{z^2(z+1)^3} dz = \int_C \frac{g(z)}{z^2} dz = 2\pi g'(0) = -4\pi i$$

(note that g is analytic on and inside C , as required for the Cauchy Integral Formula).

Another way to calculate the integral is to calculate the residue at 0 and use the Residue Theorem (as in b) below).

b)

Residue at $z_0 = 0$:

Write

$$\phi(z) = \frac{e^z}{(z+1)^3}, m = 2, z_0 = 0,$$

then $\phi(z_0) = \phi(0) \neq 0$ and

$$\text{Res}_{z=0} \frac{e^z}{z^2(z+1)^3} = \text{Res}_{z=0} \frac{\phi(z)}{z^2} = \frac{\phi'(0)}{1!} = -2$$

(using the theorem in Section 66).

Residue at $z_0 = -1$:

Write

$$\phi(z) = \frac{e^z}{z^2}, m = 3, z_0 = -1,$$

then $\phi(z_0) = \phi(-1) \neq 0$ and

$$\text{Res}_{z=-1} \frac{e^z}{z^2(z+1)^3} = \text{Res}_{z=-1} \frac{\phi(z)}{(z+1)^3} = \frac{\phi''(-1)}{2!} = \frac{11}{2e}.$$

The value of the integral is

$$\int_C \frac{e^z}{z^2(z+1)^3} dz = 2\pi i (\text{Res}_{z=0} \frac{e^z}{z^2(z+1)^3} + \text{Res}_{z=-1} \frac{e^z}{z^2(z+1)^3}) = 2\pi i (-2 + \frac{11}{2e}).$$

2.

a)

The maximal disks/annuli are $|z| < 2$, $2 < |z| < 4$ and $|z| > 4$.

b)

In $|z| < 2$:

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-4} \\ &= -\frac{1}{2} \frac{1}{1-\frac{z}{2}} + \frac{1}{4} \frac{1}{1-\frac{z}{4}} \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n + \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{4^{n+1}} - \frac{1}{2^{n+1}}\right) z^n. \end{aligned}$$

Note that using the geometric series is valid in both cases, since $|z/2| < 1$ and $|z/4| < 1$ in the disk under consideration.

In $2 < |z| < 4$:

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-4} \\ &= \frac{1}{z} \frac{1}{1-\frac{2}{z}} + \frac{1}{4} \frac{1}{1-\frac{z}{4}} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{4^{n+1}} z^n + \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{4^{n+1}} z^n + \sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n}. \end{aligned}$$

Note that using the geometric series is valid in both cases, since $|2/z| < 1$ and $|z/4| < 1$ in the annulus under consideration.

In $|z| > 4$:

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-4} \\ &= \frac{1}{z} \frac{1}{1-\frac{2}{z}} - \frac{1}{z} \frac{1}{1-\frac{4}{z}} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{4}{z}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{4^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1} - 4^{n-1}}{z^n}. \end{aligned}$$

Note that using the geometric series is valid in both cases, since $|2/z| < 1$ and $|4/z| < 1$ in the annulus under consideration.

3.

Only the singular point at $3i$ is enclosed by C , so we can use the Cauchy Integral Formula with

$$g(z) = \frac{1}{(z + 3i)}, z_0 = 3i, m = 0$$

and obtain

$$\int_C \frac{1}{(z + 3i)(z - 3i)} dz = \int_C \frac{g(z)}{z - 3i} dz = 2\pi g(3i) = \pi/3.$$

(note that g is analytic on and inside C , as required for the Cauchy Integral Formula).

Another way to calculate the integral is to calculate the residue at $3i$.

4.

Using

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

and setting $w = 1/z^2$ we obtain

$$e^{1/z^2} = \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!}.$$

We have infinitely many nonzero terms with negative coefficients of z , so 0 is an essential singular point.

b)

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

so

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!},$$

and since there are no nonzero terms with negative coefficients of z , 0 is a removable singular point.

Another way to see this is to check that the limit of $\sin z/z$ at 0 exists; the rule of L'Hospital gives

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = \lim_{z \rightarrow 0} \frac{\cos z}{1} = 1,$$

and this also tells us that 0 is a removable singular point.

c)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

so

$$\begin{aligned} \frac{e^z - 1}{z^2} &= \frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1}{z^2} \\ &= \frac{1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} - 1}{z^2} \\ &= \frac{\sum_{n=1}^{\infty} \frac{z^n}{n!}}{z^2} \\ &= \sum_{n=1}^{\infty} \frac{z^{n-2}}{n!} \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}. \end{aligned}$$

There is only one nonzero term with a negative exponent of z , namely the term z^{-1} , and 0 is a pole of order 1.

5.

a)

There are infinitely many parametric representations, an example is

$$\begin{aligned} z(t) = & \begin{aligned} & 1 - i + 2it & , 0 \leq t \leq 1 \\ & 1 + i - 2(t - 1) & , 1 \leq t \leq 2 \\ & -1 + i - 2i(t - 2) & , 2 \leq t \leq 3 \\ & -1 - i + 2(t - 3) & , 3 \leq t \leq 4. \end{aligned} \end{aligned}$$

b)

We cannot apply the principle of the transformation of paths (Cor. 2 in Section 46) directly, since C and C_1 intersect. The easiest way is to take some auxiliary contour that does not intersect C or C_1 , say, C_2 given by $z(t) = 2e^{it}$, $0 \leq t \leq 2\pi$. Then this corollary

says that

$$\int_C f(z)dz = \int_{C_2} f(z)dz$$

and

$$\int_{C_1} f(z)dz = - \int_{C_2} f(z)dz$$

(the "-" due to the negative orientation of C_1). Hence,

$$\int_{C_1} f(z)dz = - \int_C f(z)dz = -\omega.$$

6.

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{i^n - (-i)^n}{2in!} z^n \end{aligned}$$

would be a possible representation.

$$\begin{aligned} \frac{1}{(2-z)^2} &= \frac{d}{dz} \left(\frac{1}{2-z} \right) \\ &= \frac{d}{dz} \left(\frac{1}{2} \frac{1}{1-\frac{z}{2}} \right) \\ &= \frac{d}{dz} \left(\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} z^n \right) \\ &= \frac{d}{dz} \left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \right) \\ &= \sum_{n=0}^{\infty} \frac{d}{dz} \left(\frac{1}{2^{n+1}} z^n \right) \\ &= \sum_{n=0}^{\infty} \frac{n}{2^{n+1}} z^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} z^n. \end{aligned}$$

To calculate the Taylor Series of f_3 , we note that $f_3(z) = f_1(z)f_2(z)$.

The product of series' is given by the formula

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) z^n,$$

so

$$f_3(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{i^k - (-i)^k}{2ik!} \frac{n-k+1}{2^{n-k+2}} \right) z^n.$$