Math446
Exam3-Solutions

1.
We have to find the number of zeros of $z^4 + \frac{5}{4}z^3 + 5z^2 - iz - 2i$ inside the circle $n = 1/2$. We use the same ideas as in the example in Section 80.
We write

$$f(z) = -2i, g(z) = z^4 + \frac{5}{4}z^3 + 5z^2 - iz.$$

Then, for $|z| = 1/2$,

$$|f(z)| = 2 > \frac{63}{32} = \frac{1}{2^4} + \frac{5}{4}\frac{1}{2^3} + 5\frac{1}{2^2} + \frac{1}{2^4} \geq |g(z)|,$$

so $z^4 + \frac{5}{4}z^3 + 5z^2 - iz - 2i$ and f have the same number of zeros in $|z| < 1/2$ (counted with multiplicities), namely 0.

2.
This problems follows exactly the same ideas as the example in Section 72. The integrand

$$f(z) = \frac{1}{z^4 + 1}$$

has four singular points, two of which are interior to the simply closed arc given in the textbook on p.255, these points are at $z_1 = e^{i\pi/4}$ and $z_2 = e^{-i\pi/4}$. Using the formula for residues in Section 69, we calculate the residues to be $\frac{1}{4}e^{-i3\pi/4}$ and $\frac{1}{4}e^{-i\pi/4}$. The Residue Theorem gives

$$\int_{-R}^{R} f(z)dz = 2\pi i(\frac{1}{4}e^{-i3\pi/4} + \frac{1}{4}e^{-i\pi/4}) - \int_{C_R} f(z)dz.$$

We estimate the integral on the right-hand side using the standard estimation

$$|\int_{C_R} f(z)dz| \leq L(C_R) \max_{z\in C_R} |f(z)|.$$

The maximum is estimated via the triangle inequation, so for $z \in C_R$

$$|z^4| = R^2,$$

$$|z^4+1| = |z^4-(-1)| \geq |z^4| - |-1| = R^4 - 1,$$

so

$$\max_{z\in C_R} |f(z)| \leq \frac{1}{R^4-1}$$

and

$$|\int_{C_R} f(z)dz| \leq \pi R \frac{1}{R^4-1},$$

which tends to 0 as R tends to ∞.
Since f is even, the value of the integral is

$$\begin{aligned}
\int\limits_0^\infty f(x)dx &= \tfrac{1}{2}\text{ P.V. } \int\limits_{-\infty}^\infty f(z)dz \\
&= \tfrac{1}{2}(2\pi i(\tfrac{1}{4}e^{-i3\pi/4} + \tfrac{1}{4}e^{-i\pi/4}) - \lim_{R\to\infty} \int_{C_R} \tfrac{z^2dz}{(z^2+9)(z^2+4)^2}) \\
&= \tfrac{1}{2}2\pi i(\tfrac{1}{4}e^{-i3\pi/4} + \tfrac{1}{4}e^{-i\pi/4}) \\
&= \tfrac{\pi}{2\sqrt{2}}.
\end{aligned}$$

3.
Let $T(z) = \frac{az+b}{cz+d}$. The equations

$$\frac{a(1)+b}{c(1)+d} = 2,$$

$$\frac{a(2)+b}{c(2)+d} = 3$$

and

$$\frac{a(3)+b}{c(3)} = i$$

have (among others) the solution $a = -6+i, b = 18-5i, c = i-4, d = 10-3i$, so

$$T(z) = \frac{(-6+i)z + 18 - 5i}{(i-4)z + 10 - 3i}$$

is the desired transformation.

4.
The map $f_1(z) = -z$ maps the lower half plane onto the upper half plane. The map $f_2(z) = \frac{z-i}{z+i}$ is one of the functions mapping the upper half plane into the unit circle (we also could take any other of the functions doing so). $f_3(z) = 2z+2+i$ takes the unit disk onto the circle of radius 2 centered at $2+i$. The desired map is

$$f_3(f_2(f_1(z))) = \frac{(4+i)z+1}{z-i}.$$

5.
We split f into

$$f_1(z) = z+1,$$

$$f_2(z) = \frac{1}{z},$$
$$f_3(z) = 1 + z.$$

Then $f(z) = f_3(f_2(f_1(z)))$. f_1 just translates the circle centered at 1 with radius 1 to the circle centered at 2 with radius 1, it has the representation $|z-2| = 1$ and satisfies the circle equation $x^2 + y^2 - 4x + 3 = 0$.

The map $f_2(z) = \frac{1}{z}$ maps this circle into the circle given by

$$3(u^2 + v^2) - 4u + 1 = 0,$$

completing the squares transforms this into

$$(u - \frac{2}{3})^2 + v^2 = (\frac{1}{3})^2,$$

which is the circle centered at $\frac{2}{3}$ with radius $\frac{1}{3}$.

The transformation f_3 is just a polynomial. The center of the image is $f_3(\frac{2}{3}) = \frac{5}{3}$. The radius is $\frac{1}{3}$, so the result is the circle around $\frac{5}{3}$ with radius $\frac{1}{3}$.

6.

We use the same ideas as in the example in Section 74, so $f(z) = \frac{e^{iz}}{z^2+1}$. The singular points enclosed are at i and $-i$, the residue at i is $\frac{e^{-a}}{2i}$. As in 2. we show that the integral along C_R tends to 0 as R tends to ∞. Taking the real part of $\frac{1}{2}2\pi i\frac{e^{-a}}{2i} = \frac{\pi e^{-a}}{2}$, we obtain the claim.

7.

The inverse of the sine function maps the interior of this semiellipse to the rectangle with vertices $-\pi/2, \pi/2, -\pi/2 + i, \pi/2 + i$. The transformation $f_1(z) = 2iz$ then maps this rectangle to the rectangle with vertices $\pi i, \pi i, -2 - \pi i, -2 + \pi i$.