problem 4

(a) Toss 12 apples at random into 100 boxes.
Find the prob that no box gets more than one apple (i.e., no multiple occupancy).
(b) Toss 100 apples into 12 boxes.
Find the prob that no box is empty.

solution 4

(a) I'm going to call the apples A1, ..., A12 and the boxes B1, ..., B100 because it doesn't change the odds to do that and all of my counting methods are for apples and boxes that are distinct.

For example, I consider outcomes to look like this:
(i) A1 into B17, A2–A12 into B13
(ii) A1–A12 into B34
etc

Use A1, ..., A12 as slots.
For the total, each slot can be filled in 100 ways (each apple can go into any of the 100 boxes). So the total is 100^{12}
For the fav, the slots can be filled in 100*99*...*89 ways
The first apple can go into any of 100 boxes but then, to not get multiple occupancy, the second apple must go into one of the 99 other boxes. And so on.

Answer is $\frac{100*99*...*89}{100^{12}}$

(b) (This is an "at least one of each" type of problem)

P(no box is empty)
= P(at least one A into B1 and at least one A into B2
and ... and at least one A into B12)

= 1 − P(B1 empty or B2 empty or ... or B12 empty)

P(B1 empty or B2 empty or ... or B12 empty)
= P(B1 empty) + ... + P(B12 empty)
− [P((B1 and B2 empty) + other 2–at–a–time terms]
+ P(B1 and B2 and B3 empty) + other 3–at–a–time terms
− the 4–at–a–time terms
+ ... + the 11–at–a–time terms

Forget the 12–at–a–time term P(B1 empty and ... and B12 empty). It's zero because the event" all boxes empty" is impossible. The balls have to go somewhere

For each prob, I'm going to use the apples as the slots
Total is 12^{100}

Do the 1–at–a–time terms.
To get the fav for P(B1 empty):
Each of the 100 apple slots can be filled in 11 ways (don't toss into B1).
So the fav here is 11^{100}

Similarly for the other one–at–a–time terms.
And there are 12 of them.

Do the 2–at–a–time terms.
To get fav for P(B1 and B2 empty):
Each of the 100 apple slots can be filled in 10 ways (don't toss into B1 or B2).
So the fav here is 10^{100}.

Same for the other 2-at-a-time terms.

And there are $\binom{12}{2}$ of them.

etc.

Answer is

$$1 - \left[\; 12 * \frac{11^{100}}{12^{100}} - \binom{12}{2} \frac{10^{100}}{12^{100}} + \binom{12}{3} \frac{9^{100}}{12^{100}} - \; \ldots + \binom{12}{11} \frac{1}{12^{100}} \; \right]$$