

problem #5

How many permutations of APPLETREE have an E at one of the ends (i.e., at the left end or the right end) or in the middle (i.e., the 5th spot); e.g.,

A P P L T R E E E
E A P P T R E E L
E A P P L T R E E
A P P L E E E T R
A P P L E T R E E
etc.

solution #5

$N(\text{E at left end or E at right end or E in middle})$

$$\begin{aligned} &= N(\text{E left}) + N(\text{E right}) + N(\text{E middle}) \\ &\quad - [N(\text{E left \& E right}) + \text{other two at a time terms}] \\ &\quad + N(\text{E left \& right \& middle}) \end{aligned}$$

$$N(\text{E left end}) = \frac{8!}{2! 2!} .$$

Put an E at the left end. No choice about how to do it.

WARNING It is wrong to say there are 3 ways to pick the E for the left end because the 3 E's are identical. You can't pick one of them as opposed to another of them.

Then permute the other 8 letters which include 2 E's and 2 P's. Can be done in $\frac{8!}{2! 2!}$

Similarly, the other 1-at-a-time terms are also $8!/(2! 2!)$
And there are 3 of these one-at-a-time terms.

Each 2-at-a-time term is $\frac{7!}{2!}$ (put two E's where you want them and permute the other 7 letters which include two P's)

And there are $\binom{3}{2}$ of these 2-at-a-time terms.

The 3-at-a-time-term is $\frac{6!}{2!}$ (put the three E's at the ends and in the middle and permute the other 6 letters which include two P's)

$$\text{Answer is } 3 \frac{8!}{2! 2!} - \binom{3}{2} \frac{7!}{2!} + \frac{6!}{2!} \quad [= 23040]$$