

problem 8

How many 9-letter words contain exactly 4 different letters. For example:

A B B A C C B A D

Q Z Z Z Z Z Z A B

P Q X Y P Q X Y P

solution 8

Step 1 Pick 4 letters. Can be done in $\binom{26}{4}$ ways

Step 2 Let's say that the letters from step 1 are A, B, C, D

Now I need to count words using only A, B, C, D *and* including *at least one of each* (so that my word doesn't end up with *fewer* than 4 letters) (I don't want to count words like AAAAAAAAA, ABBBBBBBBB etc)

$$N(\text{at least one of each}) = \text{total} - N(\text{no A or no B or no C or no D})$$

$$\text{Total} = 4^9 \quad (\text{each spot in the word can be filled with A or B or C or D})$$

$$\begin{aligned} N(\text{no A or no B or no C or no D}) &= N(\text{no A}) + N(\text{no B}) + N(\text{no C}) + N(\text{no D}) \\ &\quad - [N(\text{no A \& no B}) + \text{other 2-at-a-time terms}] \\ &\quad + [N(\text{no A \& no B \& no C}) + \text{other 3-at-a-time terms}] \end{aligned}$$

(Forget about subtracting no A & no B & no C & no D. Can't happen)

Each 1-at-a-time term is 3^9 . And there are 4 of them

Each 2-at-a-time term is 2^9 . And there are $\binom{4}{2}$ of them

Each 3-at-a-time term is 1. And there are 4 of them

Final answer is

$$\binom{26}{4} * [4^9 - [4 * 3^9 - \binom{4}{2} * 2^9 + 4]]$$