Algebraic combinatorics

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Outline

1. What is algebraic combinatorics?
2. Symmetric and Schubert polynomials
3. Total positivity

What is algebraic combinatorics?

Algebraic combinatorics is concerned with interactions between combinatorics and other areas of mathematics, such as

- algebraic geometry
- representation theory
- commutative algebra
- algebraic topology

A central theme: Give proofs of nonnegativity of certain integers, by establishing combinatorial rules.

- We'll discuss two main topics in the subject, emphasizing this theme.

Symmetric polynomials I

- Let $R = \mathbb{Q}[x_1, \ldots, x_n]$ be the ring of all polynomials with rational coefficients.

- The **symmetric group** S_n acts on $f(x_1, \ldots, x_n) \in R$ by permuting the variables:

$$\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \sigma \in S_n.$$

- The polynomial $f \in R$ is **symmetric** if

$$f(x_1, \ldots, x_n) = \sigma \cdot f(x_1, \ldots, x_n), \forall \sigma \in S_n.$$

Example: $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$ is symmetric.

- Let $\Lambda \subseteq R$ be the (sub)ring of symmetric polynomials.

Symmetric polynomials II

- Λ is also a finite dimensional vector space over $\mathbb{Q}$

A **partition** λ is a decreasing sequence of nonnegative integers:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0).$$

Here $\ell = \ell(\lambda)$ is the number of **nonzero parts** of λ.

Fact: $\dim_{\mathbb{Q}} \Lambda = \#\{\lambda : \ell(\lambda) \leq n\}$

A number of explicit bases for Λ are available:

I. **Monomial symmetric functions** (fairly boring):

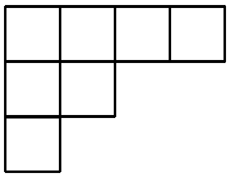
$$m_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} + \text{all symmetrizations.}$$

Example: $m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + \ldots$

Symmetric polynomials: III

II. **Schur polynomials** (interesting):

View λ as a **Young diagram**:

$$\lambda = (4, 2, 1) \leftrightarrow$$  .

A **semistandard Young tableau** of shape λ is an assignment of $\{1, 2, \ldots, n\}$ to the boxes such that

- labels weakly increase along rows
- labels strictly increase along columns

e.g., $\begin{array}{llll} 1 & 1 & 2 & 2 \\ 2 & 3 & & \\ 4 & & & \end{array}$, $\begin{array}{llll} 1 & 1 & 1 & 1 \\ 2 & 2 & & \\ 3 & & & \end{array}$, $\begin{array}{llll} 2 & 2 & 3 & 3 \\ 3 & 3 & & \\ 4 & & & \end{array}$, but not $\begin{array}{llll} 1 & 1 & 2 & 1 \\ 1 & 2 & & \\ 2 & & & \end{array}$.

Symmetric polynomials: IV

Define the **Schur polynomial**

$$s_\lambda(x_1, \ldots, x_n) := \sum_T \mathbf{x}^T,$$

as the **generating series** for all such tableaux.

Example: When $n = 3$ and $\lambda = (2, 1)$ there are eight semistandard Young tableaux:

$$\begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \quad \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \quad \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \quad \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 3 & \end{array}, \quad \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \quad \begin{array}{cc} 2 & 3 \\ 3 & \end{array}.$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

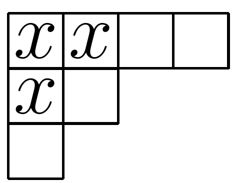
(Not obvious: $\{s_\lambda\}$ form a basis, or are symmetric!)

Littlewood-Richardson numbers I

Products of symmetric polynomials are symmetric, and Schur polynomials are a vector space basis of Λ, hence:

$$s_\lambda(\mathbf{x})s_\mu(\mathbf{x}) = \sum_\nu C^\nu_{\lambda,\mu} s_\nu(\mathbf{x}).$$

In fact, the **Littlewood-Richardson number** $C^\nu_{\lambda,\mu} \in \mathbb{N}$!
One combinatorial rule to compute $C^\nu_{\lambda,\mu}$ is:

Theorem: $C^\nu_{\lambda,\mu}$ counts the number of semistandard tableau
T of shape $\nu/\lambda =$  (here $\nu/\lambda = (4,2,1)/(2,1)$) with
μ_i labels "i", for $i = 1, 2 \ldots, \ell(\mu)$ such that the "reading word" of T is a "ballot sequence".

Littlewood-Richardson numbers II

Example: Let $\lambda = \mu = (2,1)$ and $\nu = (3,2,1)$. Then $C^{\nu}_{\lambda,\mu} = 2$, as witnessed by the following two tableaux:

$$\begin{array}{|c|c|c|}\hline x & x & 1 \\ \hline x & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline x & x & 1 \\ \hline x & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}.$$

- It is <u>not</u> witnessed by $\begin{array}{|c|c|c|}\hline x & x & 1 \\ \hline x & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$ or $\begin{array}{|c|c|c|}\hline x & x & 1 \\ \hline x & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$ because they don't use exactly a μ's worth of 1's, 2's and 3's.
- It is also <u>not</u> witnessed by $\begin{array}{|c|c|c|}\hline x & x & 2 \\ \hline x & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$ because the right to left, top down reading word is 211 which sees more 2's than 1's.

Littlewood-Richardson numbers III

The Littlewood-Richardson coefficients arise in:

- Schubert calculus on Grassmannians
- Tensor product multiplicities of GL_n representations
- Induction products of representations of S_n

while their positivity controls:

- eigenvalues of sums of Hermitian matrices
- short exact sequences of finite p-groups

Basic problem(s): Extend the Littlewood-Richardson rule to generalizations of these contexts (e.g., when GL_n is replaced by other Lie groups).

A generalization: Schubert polynomials I

- Schur polynomials are cohomology classes of Schubert varieties in the Grassmannian. What's the extension to the more complicated flag manifolds?

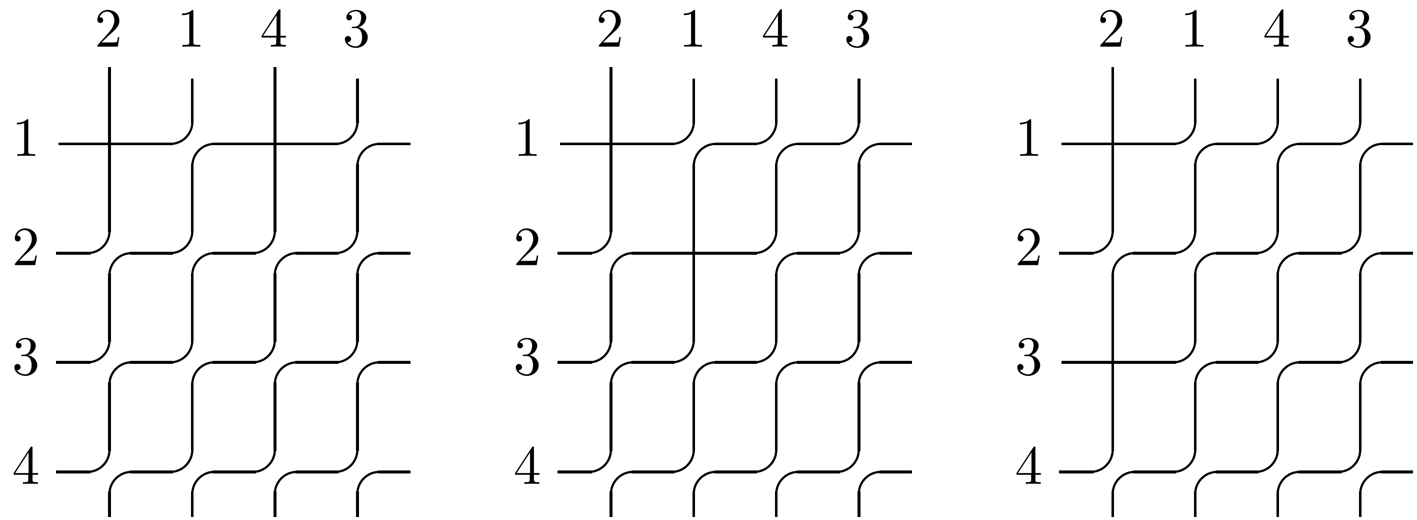
To each permutation w in the symmetric group S_n we define a **Schubert polynomial**

$$\mathfrak{S}_w(x_1, \ldots, x_n) \in \mathbb{N}[x_1, \ldots, x_n].$$

A **pipe dream** is a filling of an $n \times n$ grid by crosses $+$ and elbows ╯╭.

A generalization: Schubert polynomials II

Three pipe dreams for $w = 2143 \in S_4$.



The Schubert polynomial $\mathfrak{S}_w$ is the pipe dream for w, weighted by vertical position of $+$'s.
Hence,

$$\mathfrak{S}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3.$$

A generalization: Schubert polynomials III

Schubert polynomials have numerous beautiful combinatorial properties, many of which can be explained directly, or, e.g., by **combinatorial commutative algebra.** One property is **stability**: under the natural inclusion of

$$S_n \hookrightarrow S_{n+1} : w \mapsto w \times 1,$$

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \mathfrak{S}_{w\times 1}(x_1, \ldots, x_n, x_{n+1}).$$

Fact: The set of polynomials $\{\mathfrak{S}_w\}$, $w \in S_\infty$ is a $\mathbb{Q}$-linear basis of $\mathbb{Q}[x_1, \ldots, x_n, \ldots]$.

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} D_{u,v}^w \mathfrak{S}_w, \quad \text{and } D_{u,v}^w \in \mathbb{N}.$$

Main problem: Give concise combinatorial rules for $D_{u,v}^w$.

Total positivity I

A square matrix A is **totally positive** if all of its minors are positive real numbers. (Similarly we define a **totally nonnegative** matrix.)

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ has nine 1×1 minors

$$\Delta_{\{1\},\{1\}} = |1|, \Delta_{\{1\},\{2\}} = |1|, \ldots,$$

nine 2×2 minors

$$\Delta_{\{1,2\},\{1,2\}} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \Delta_{\{2,3\},\{1,3\}} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3, \ldots$$

and the 3×3 minor $\Delta_{\{1,2,3\},\{1,2,3\}} = 1$.

Total positivity II

A systematic study of total positivity was initiated in the 1930's by Gantmacher-Krein (they showed existence of n distinct positive eigenvalues).

Relations/applications to:

- oscillations in mechanical systems
- stochastic processes and approximation theory
- Polya frequency sequences
- representation theory of S_∞
- planar resistor networks
- quantum groups
- Somos sequences

Total positivity III

Two basic questions:

Question A: How do you parametrize totally positive matrices?

Question B: How do you efficiently test for total positivity?

- The answers to both questions involve our main theme of positivity in combinatorics.

Question A: Parametrizing totally positive matrices

The matrix

$$\begin{bmatrix} d & dh & dhi \\ bd & bdh+e & bdhi+eg+ei \\ abd & abdh+ae+ce & abdhi+(a+c)e(g+i)+f \end{bmatrix}$$

for $a, b, c, d, e \in \mathbb{R}_+$ is totally positive.

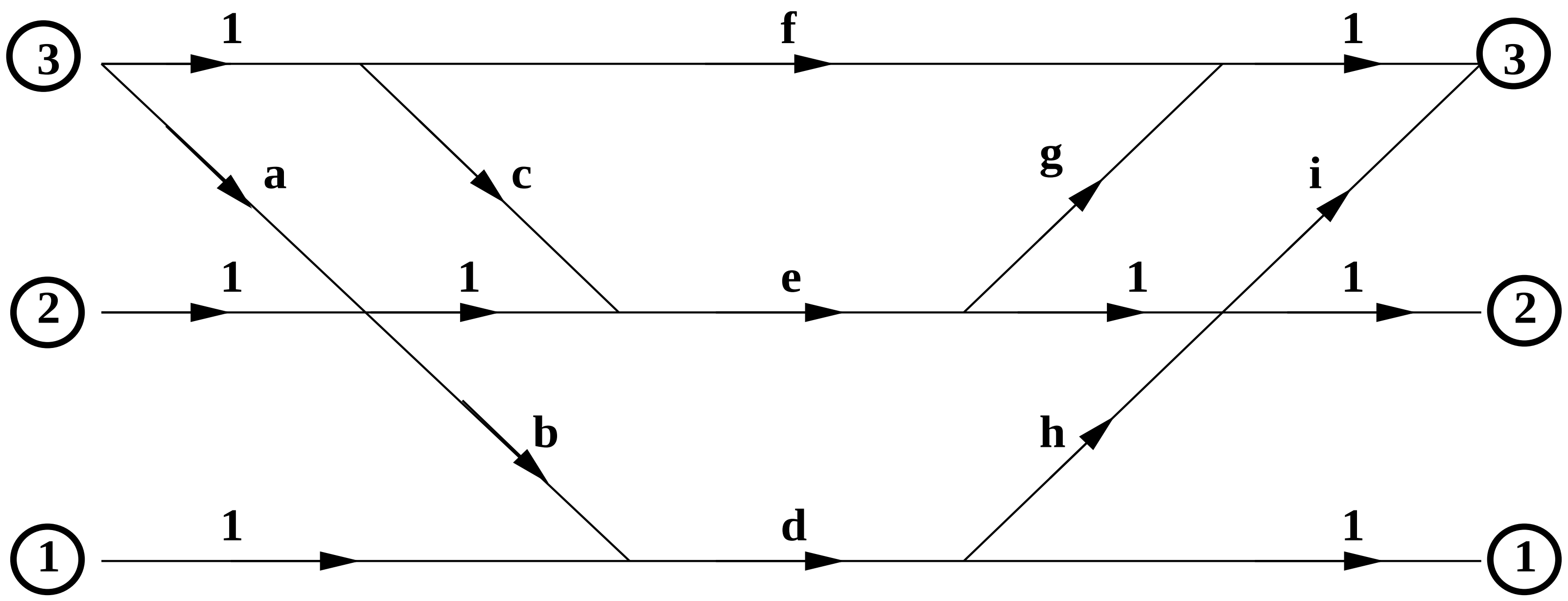
Fact: Every 3×3 totally positive is of this form

Our goals for Question A:

- Give a combinatorial construction of this matrix (and the $n \times n$ generalization).
- Give a combinatorial explanation of the positivity of minors.

Planar networks I

Consider a **planar network** Γ with **edge weighting** ω:



- The **sources** are the extreme left nodes and the **sinks** are the extreme right nodes.
- The **weight** of a directed path in Γ is the product of the weights of its edges.

Planar networks II

- The **weight matrix** $A(\Gamma,\omega)$ is the $n \times n$ matrix with

$$A_{ij} = \text{ sum of weights of paths from source } i \text{ to sink } j.$$

Theorem: $A(\Gamma,\omega)$ is totally positive.

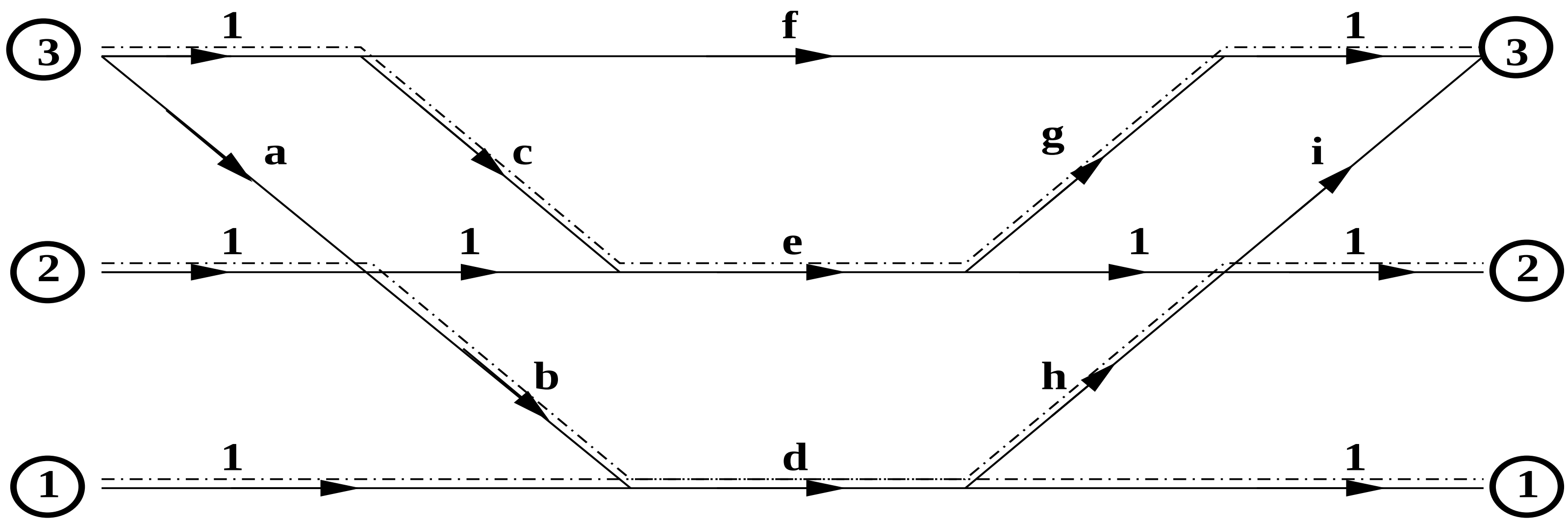
Idea of proof: Consider the combinatorial interpretation of the determinant of A:

$$\det(A) = \sum_{w \in S_n} \sum_{\pi} \operatorname{sgn}(w) \omega(\pi),$$

where

- the inner sum is over all **families** of paths $\pi = (\pi_1, \ldots, \pi_n)$ from the sources to the sinks
- $\omega(\pi) = \omega(\pi_1) \cdots \omega(\pi_n)$

Planar network III



- path $1 \to 2$ has weight: dh
- path $2 \to 1$ has weight: bd
- path $3 \to 3$ has weight: ceg
- $w = 213$ and $\mathrm{sgn}(w) = -1$

Hence family contributes $-(dh)(bd)(ceg)$ to $\det(A)$.

Planar network IV

The signed expression for $\det(A)$ doesn't manifest the claimed positivity!
However, we can cancel all negative contributions.

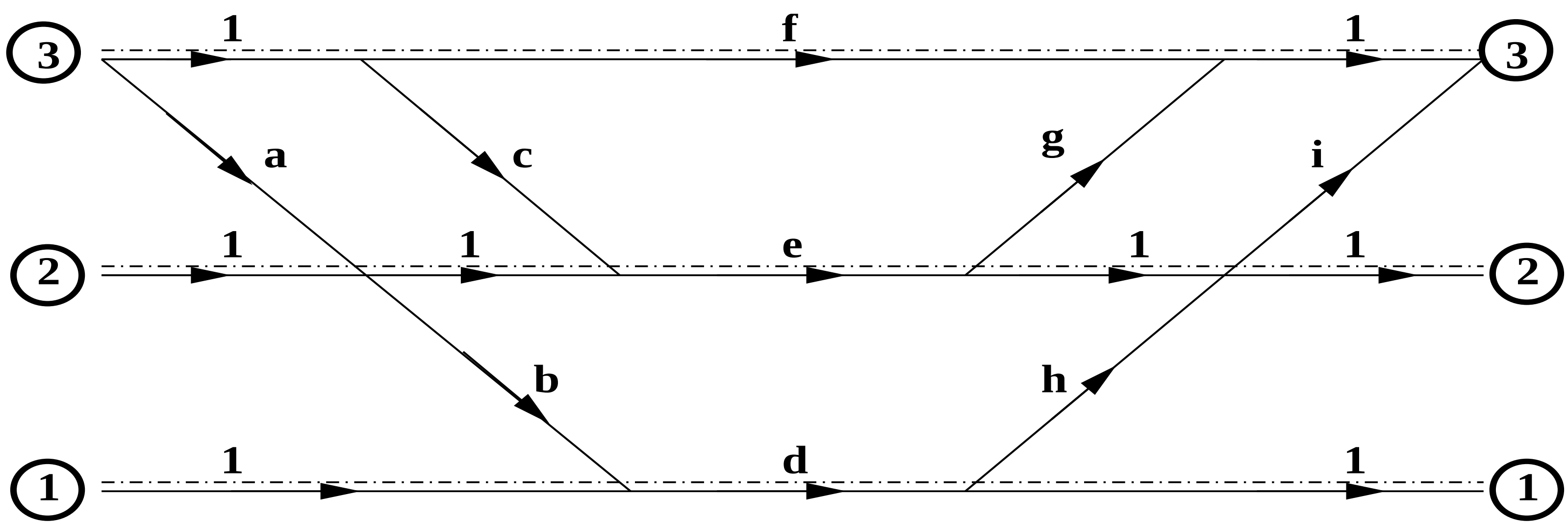
- The paths intersect, so we can "path switch" at an intersection to change the sgn
- This gives a positive contribution to $\det(A)$ which cancels with our negative contribution.

Thus our theorem is immediate from:

Lindström's Lemma: $\det(A) = \sum_{\pi} \omega(\pi)$ where $\pi = (\pi_1, \ldots, \pi_n)$ are nonintersecting families of paths (necessarily sending source i to sink i).

Planar network V

In our example, there's only one nonintersecting path family:



and so $\det(A) = fed (> 0)$.

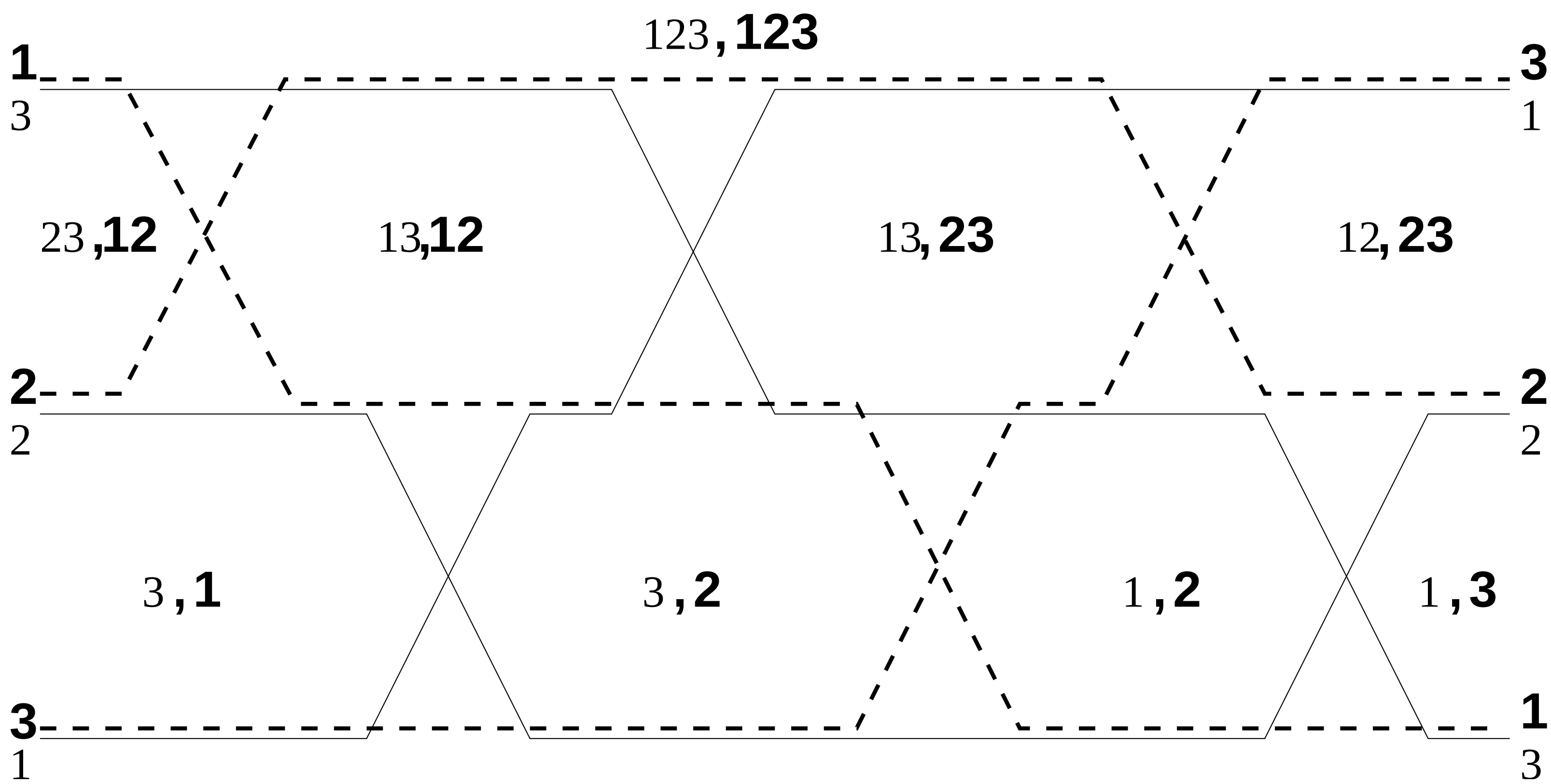
For smaller minors of A, the same argument works.

Question B: efficient tests for total positivity

Theorem: (Gasca-Peña) A square matrix is totally positive if and only if its **initial minors** $\Delta_{I,J}$ (where $1 \in I \cup J$) are positive. Thus one has an n^2 check (and this is tight).

Goal: Set up a combinatorial framework where this test is a special case.

Double wiring diagrams



- Any two wires of the same color cross exactly once
- There are n^2 chambers
- Each chamber indexes a **chamber minor**, determined by which wires of each color are below it

Chamber minors and positivity

Theorem: (Fomin-Zelevinsky) A square matrix A is totally positive if and only if its chamber minors are positive.
Proof follows from
Theorem: (Fomin-Zelevinsky) Any minor of A is a subtraction free rational expression in the chamber minors of a given double wiring diagram.

Example: $\Delta_{2,1} = \frac{\Delta_{3,2}\Delta_{23,12}+\Delta_{3,1}\Delta_{13,23}}{\Delta_{13,12}}$

Main conjecture: Every minor is a Laurent polynomial with positive coefficients, in chamber minors of an arbitrary double wiring diagram.

- Some minors are not chamber minors, e.g., $\Delta_{1,3}$.
- What is a combinatorial rule for this expansion?

Summary and Overview

This talk discussed the theme of positivity in algebraic combinatorics, through two major topics, with relations to other branches of mathematics:

- Symmetric polynomials
- Total positivity of matrices

In each case, we describe:

- a known positivity theorem
- a motivating positivity conjecture/problem

Some references

- S. Fomin and A. Zelevinsky, *Total positivity: tests and parametrizations*, The Mathematical Intelligencer **22**(2000), 23–33.
- W. Fulton, *Young tableau*, Cambridge University Press 1997.
- R. Stanley, *Enumerative Combinatorics II*, Cambridge University Press, 1999.