

MATH 2243 — FALL 2007 FINAL EXAM
DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA, MINNEAPOLIS

NAME: _____

ID NUMBER: _____

- (1) Do not open this exam until you are told to begin.
- (2) This exam has 14 pages including this cover and two intentionally blank pages for your use. There are 10 problems total. You have 3 hours.
- (3) No notes or books are permitted.
- (4) Only non-graphing calculators are permitted.
- (5) Please turn off all cell phones.
- (6) Place your ID card on your desk for inspection.
- (7) Good luck!

PROBLEM	POINTS	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. Solve the following differential equation:

$$\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3 - y)}.$$

You may leave your answer in implicit form.

2. Consider a population $P(t)$ satisfying the logistic equation $P'(t) = aP - bP^2$, where $\alpha = a$ is the constant birth rate per month per individual, and $\beta = bP$ is the death rate per month per individual. Assume that the initial population is 1000 individuals and there are 50 births and 20 deaths per month occurring at time $t = 0$.

- (a) Calculate the constants a and b from the data.
- (b) What is the maximum population which can be attained?
- (c) How many months does it take for the population to reach two third of the limit population?

3. Consider the differential equation $dx/dt = -x^2 + kx - 1$ containing the parameter k . Determine the number and stability or unstability of the critical points depending on the value of k . Construct the bifurcation diagram.

4. Determine whether the following statements are **true** or **false**. If you state the statement is **false** give a counterexample that demonstrates your claim. *You will be scored +2 points for a complete correct answer, 0 for no answer and -2 for an incorrect answer. Thus guessing will be penalized.*

- (a) If \mathbf{A} and \mathbf{B} are $n \times n$ matrices then $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.
- (b) Given a collection of linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in a vector space V , any vector \mathbf{x} can be expressed as $\mathbf{x} = c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k$.
- (c) If two $n \times n$ matrices \mathbf{A} and \mathbf{B} are similar, then $\det \mathbf{A} = \det \mathbf{B}$.
- (d) Suppose V is a vector space, $S \subseteq V$ is a subspace of V , $\mathbf{v} \in S$ and $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in V$. Then $\mathbf{a}, \mathbf{b} \in S$.
- (e) An $n \times n$ matrix \mathbf{A} is invertible if and only if $\det A \neq 0$.

5. Let

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

be a 3×3 matrix.

- (a) Find a collection of linearly independent vectors such that solution space corresponding to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is the set of linear combinations of those vectors.

- (b) Find all values of the coefficients k and c such that the following system has:
- (i) no solution.
 - (ii) infinitely many solutions.
 - (iii) only one solution.

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & c \\ 3 & 2 & 1 & 5 \\ 0 & 1 & k & 3 \end{array} \right]$$

6. Find the eigenvalues and a full set of linearly independent eigenvectors of the two following matrices. Determine which of them is diagonalizable and find a diagonalizing matrix S and a diagonal matrix D such that it is equal to $S^{-1}DS$.

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 3 & -3 & 5 \end{bmatrix}.$$

7. Find the general solution to the differential equation:

$$y^{(3)} - 2y'' + y' = x^2 + xe^x.$$

8. Apply the eigenvalue method to solve the following system of differential equations:

$$\begin{aligned}x_1' &= 3x_1 + x_2 + x_3 \\x_2' &= -5x_1 - 3x_2 - x_3 \\x_3' &= 5x_1 + 5x_2 + 3x_3\end{aligned}$$

9. Solve the system of linear equations

$$3x_1 + x_2 + x_3 + 6x_4 = 14$$

$$x_1 - 2x_2 + 5x_3 - 5x_4 = -7$$

$$4x_1 + x_2 + 2x_3 + 7x_4 = 17$$

10.

(a) Find the Laplace transform of $f(t) = \sin^2(t/2)$.

(b) Find the inverse Laplace transform of $F(s) = \frac{2s^2 + 1}{2s^2(s^2 + 1)}$.

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