MATH 104 Final Exam Review Problems– Fall 2004

1. Let (a_n) be a sequence wih the property that there exists a real number r, $0 < r < 1$, and an integer N_0 such that for all $n \geq N_0$, $\mid a_n - a_{n-1} \mid \leq r \mid a_{n-1} - a_{n-2} \mid$. Then prove (a_n) is a Cauchy sequence.

2. Let (a_n) be a sequence and let R be a real number, $0 < R < 1$ such that $\lim_{n\to\infty}[a_n]^{\frac{1}{n}} = R$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

3. Let $(a_n$ be a bounded sequence. Prove there exists a subsequence of (a_n) that converges to $\lim\,\inf_{n\to\infty} a_n$.

4. Give counterexamples to show that the following "definitions" of $\lim_{x\to c} f(x) = L$ are *not* correct.

(i) For every $\delta > 0$ there exists an $\epsilon > 0$ such that $0 <\mid x - c \mid< \delta$ implies that $\mid f(x) - L \mid< \epsilon$.

(ii) For every $\epsilon > 0$, there exists a $\delta > 0$ such that $\mid f(x) - L \mid< \epsilon$ implies that $0 <\mid x - c \mid< \delta$.

5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $r \in \mathbb{Q}$, $f(r) = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

6. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\mid f(x) \mid \leq \mid x \mid$ for all $x \in \mathbb{R}$, then f is continuous at $x = 0$.

7. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If for each $c \in [a, b]$ there exists a $\delta > 0$ such that f is increasing on $(c - \delta, c + \delta) \cap [a, b]$, then f is an increasing function on $[a, b]$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If f is an even function $(f(-x) = f(x))$, then f' is an odd function $(f(-x) = -f(x))$.

9. Let $f : [0, 1] \to [0, 1]$ be a differentiable function such that $\mid f'(x) \mid\neq 1$. Show that there exists exactly one point c such that $f(c) = c$.

10. Suppose f is an integrable function on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Show $\int_a^b f \geq 0$.

11. Suppose that f is an integrable function on $[a, b]$. Then

$$\lim_{h\to 0} \int_a^{a+h} f = 0.$$

12. Let $f : [-a, a] \to \mathbb{R}$ be defined by $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)^n}$. Prove f is continuous on $[-a, a]$.

13. Study theorems/proofs given in class. Some will appear on the exam.

14. Suppose f is a bounded real function on $[a, b]$, and f^2 is Riemann integrable. Does it follow f is Riemann integrable?

15. Let f be defined for all real x and suppose that

$$| f(x) - f(y) | \leq (x - y)^2$$

for all real x and y. Prove f is constant.

16. Suppose f is defined and differentiable for every $x > 0$ and moreover $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove $g(x) \to 0$ as $x \to +\infty$.

17. Let f be a continuous function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

18. Suppose f is differentiable on $[a, b]$, $f(a) = 0$ and there is a real number A such that $| f'(x) | \leq A | f(x) |$ on $[a, b]$. Prove $f(x) = 0$ for all $x \in [a, b]$. *Hint*: Fix $x_0 \in [a, b]$, let

$$M_0 = sup \mid f(x) \mid, M_1 = sup \mid f'(x) \mid$$

for $a \leq x \leq x_0$. For any such x,

$$| f(x) | \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$.

19. Suppose X, Y, Z are metric spaces, and Y is compact. Let $f : X \to Y$, let g be a continuous injection of Y into Z and put $h(x) = g(f(x))$ for $x \in X$. Prove f is uniformly continuous if h is.

20. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \sqrt{a_n}/n$.

21. Suppose that X is a nonempty complete metric space, and $\{G_n\}$ is a sequene of dense open subsets of X. Prove Baire's theorem: $\bigcap_1^\infty G_n$ is not empty.