COMBINATORIAL MODELS IN SCHUBERT GEOMETRY

My principal mathematical interests so far have been the construction of combinatorial models to study some central problems in flag varieties and their Schubert subvarieties. Combinatorial methods, such as Schützenberger's celebrated theory of jeu de taquin have had great influence on areas ranging from algebraic geometry and representation theory to combinatorics and random matrix theory. My own research has produced new models which expand and enrich such applications and their variegated interconnections.

Schubert calculus. The **flag variety** G/P has **Schubert varieties** X_w indexed by elements of a (quotient) Weyl group W (where G is a complex, connected, reductive Lie group and P its parabolic subgroup). Let

$$C_{u,v,w} = \#\{\text{points of } G/P \text{ in } g_1 \cdot X_u \cap g_2 \cdot X_v \cap g_3 \cdot X_w\} \text{ (for generic } g_1, g_2, g_2 \in G)$$

be the **Schubert intersection number**. The main open problem asks for an explicit, manifestly positive combinatorial rule for $C_{u,v,w}$. For Grassmannians this is solved by the *Littlewood-Richardson rule*, a product of Schützenberger's theory. In work with H. Thomas, I gave the first *root-system uniform rule* in this subject, solving the problem for all *minuscule* G/P. We also have a K-theory generalization in terms of our new theory of *jeu de taquin for increasing tableaux*, the first cohomological extension of Schützenberger's theory. This theory also seems widely applicable, and I've connected it to topics as different as degeneracy loci of vector bundles, and longest increasing subsequences of random words.

In addition, an old question about the Littlewood-Richardson rule is to give a formulation that manifests its S_3-symmetry $u \leftrightarrow v \leftrightarrow w$. H. Thomas and I solved this problem, by inventing three-dimensional objects called *cartons*.

In connection to equivariant symplectic geometry, we have extensions to model equivariant K-theory of Grassmannians. Among other things, I am working on further developing this theory, and extending it root-system uniformly to minuscule G/P.

Singularities of Schubert varieties. Understanding the singularities of Schubert varieties is a difficult and heavily studied problem because of the importance of these varieties to representation theory. In work with A. Woo, we discover a general combinatorial model for this problem in terms of new permutation combinatorics called *interval pattern avoidance*. This model governs all reasonable singularity measures, including the famous *Kazhdan-Lusztig polynomials*. It led to conjectures, solved by algebraic geometers and graph theorists alike, and opens the door to deep questions.

Our theory utilizes *Kazhdan-Lusztig ideals*. A. Woo and I solved the problem of finding a Gröbner basis for these ideals. This sheds light on A_n Schubert polynomials, and answers a question of A. Buch and R. Rimanyi. I am pursuing extensions to other Lie types.

I am presently working on the problem of computing multiplicities of Schubert varieties. In the key *vexillary case*, L. Li (a postdoc at U. Illinois) and I are working on a geometric rule using Gröbner degeneration. This work extends my earlier work with A. Knutson and E. Miller that furthermore developed a Gröbner basis theory of *geometric vertex decompositions*, an analogy with vertex decompositions of simplicial complexes.

Summary and future plans. This is an exciting time to be working as a combinatorialist in Schubert geometry, with several different mathematical areas making contributions. I plan to recruit and train graduate students and postdoctoral scholars in Urbana. Together, we can develop new foundational models and explore their ramifications.