A BILINEAR FOURIER EXTENSION THEOREM AND APPLICATIONS TO THE DISTANCE SET PROBLEM

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Abstract. In this paper, we obtain a weighted version of Tao’s bilinear Fourier extension estimate for elliptic surfaces. This implies improved partial results in the direction of Falconer’s distance set conjecture in dimensions $d \geq 3$.

1. Introduction

In [11], Tao proved the following bilinear Fourier extension estimate. Let $S = \{x \in \mathbb{R}^d : x_d = x_1^2 + \ldots + x_{d-1}^2\}$ and $d\sigma$ be the surface measure on $S$. Let $\widehat{\mu}$ denote the Fourier transform of the measure $\mu$ in $\mathbb{R}^d$,

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x), \quad \xi \in \mathbb{R}^d.$$ 

**Theorem A.** Let $d \geq 2$. Let $S_1, S_2$ be compact subsets of $S$ with $d(S_1, S_2) > 1$. Then for all $q > \frac{d+2}{d}$, we have

$$\|\widehat{f_1}d\sigma \widehat{f_2}d\sigma\|_{L^q(\mathbb{R}^d)} \leq C_{q,d}\|f_1\|_{L^2(d\sigma)}\|f_2\|_{L^2(d\sigma)},$$

for all $f_j \in L^2(d\sigma)$ supported in $S_j$, $j = 1, 2$.

This theorem is proved in [11] for $d \geq 3$. For $d = 2$, it has been known for a long time and is basically the Carleson-Sjölin Theorem [2]. Previously, in [15], Wolff obtained Theorem A for the light cone in general dimensions. Tao’s proof relies on and extends the ideas in [15].

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We consider the following weighted version of the inequality (1). Fix $\alpha \in (0, d)$. Suppose $H : \mathbb{R}^d \to \mathbb{R}$ satisfies
\begin{equation}
\|H\|_\infty \leq 1, \tag{2}
\end{equation}
\begin{equation}
\int_{B(x,r)} |H(u)| \, du \leq r^\alpha, \quad \forall x \in \mathbb{R}^d, \quad \forall r > 0. \tag{3}
\end{equation}
For which $q$ and $\alpha$, the inequality
\begin{equation}
\|\hat{f}_1 \hat{f}_2\|_{L^q(Hd\xi)} \leq C_{\alpha,q,d} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)} \tag{4}
\end{equation}
holds for all $f_j \in L^2(d\sigma)$ supported in $S_j$, $j = 1, 2$?

Obviously, (2) and Theorem A imply that (4) holds for $q > \frac{d+2}{d}$. We improve this range of $q$ for $\alpha < \frac{d+2}{2}$.

**Theorem 1.** Let $d \geq 3$ and $\alpha \in (0, d)$. Assume that $H$ satisfies (2) and (3). Then, under the hypothesis of Theorem A, (4) holds for any $q > q_0(\alpha, d) := \max(1, \min(\frac{4\alpha}{d+2\alpha-2}, \frac{d+2}{d}))$.

There is no reason for this theorem to be optimal. In fact, it should be possible to improve the range of $q$ for each $\alpha \in (0, d)$. However, this theorem significantly improves the known estimates for the decay of $L^2$ spherical averages of the Fourier transform of fractal measures (see Section 3). Using this we obtain improved partial results in the direction of Falconer’s distance set conjecture in dimensions 3 and higher. Let $E$ be a compact subset of $\mathbb{R}^d$. The distance set, $\Delta(E)$, of $E$ is defined as
\[ \Delta(E) = \{|x-y| : x, y \in E\}. \]
In [5], Falconer conjectured that:

**Conjecture.** Let $d \geq 2$. Let $E$ be a compact subset of $\mathbb{R}^d$. Then,
\[ \dim(E) > \frac{d}{2} \implies |\Delta(E)| > 0. \]
Here $|\cdot|$ is the Lebesgue measure and $\dim(\cdot)$ is the Hausdorff dimension.

Falconer’s conjecture is open in every dimension. In [5], Falconer gave an example showing that $\frac{d}{2}$ in the conjecture is optimal and
proved that \( \dim(E) > \frac{d+1}{2} \) implies \( |\Delta(E)| > 0 \). Bourgain [1] improved this result in every dimension, and in particular proved that in \( \mathbb{R}^2 \), \( \dim(E) > \frac{13}{9} \) suffices. Later, Wolff [14] proved that in \( \mathbb{R}^2 \), \( \dim(E) > \frac{4}{3} \) suffices. This is still the best known result in \( \mathbb{R}^2 \). See [3] for a simplified proof of Wolff’s theorem. In [4], using Theorem A the author proved that \( \dim(E) > \frac{d(d+2)}{2(d+1)} \) suffices. See [14], [8], [3] and [4] for some variations and related results. In this paper, using (a variant of) Theorem 1, we prove

**Theorem 2.** Let \( d \geq 3 \). Let \( E \) be a compact subset of \( \mathbb{R}^d \) with \( \dim(E) > \frac{d}{2} + \frac{1}{3} \). Then \( |\Delta(E)| > 0 \).

**Remark.** Wolff’s result in [14] and Theorem 2 relies on a method developed by Mattila [7, 8]. As it is noted in [14], \( \frac{4}{3} \) is the best possible exponent (in \( \mathbb{R}^2 \)) one can obtain using this method. In \( \mathbb{R}^3 \), the best possible exponent is \( \frac{5}{3} \). However, it may be possible to prove Falconer’s conjecture in dimensions \( d \geq 4 \) using Mattila’s approach. In particular, it will be clear from the proof of Theorem 2 that the inequality (4) for \( \alpha = d/2 \) and for all \( q > 1 \), if true, implies Falconer’s conjecture in \( \mathbb{R}^d \), \( d \geq 4 \).

**Remark.** As in [4], the assertion of Theorem 2 can be extended to distance sets with respect to general metrics. Let \( K \) be a convex symmetric body. Assume that the boundary of \( K \) is smooth and has non-vanishing Gaussian curvature. Define \( \Delta_K(E) = \{d_K(x, y) : x, y \in E\} \), where \( d_K \) is the distance induced by \( K \). Then, the statement of Theorem 2 holds for \( \Delta_K \).

We prove Theorem 1 in Section 5. In Section 2, we describe some extensions of Theorem 1. In Section 3, we describe Mattila’s approach and in Section 4, we prove Theorem 2.
List of notations

\( \chi_A \): characteristic function of the set \( A \).

\( B(x, r) := \{ y : |x - y| < r \} \).

\( d(A, B) \): the distance between the sets \( A \) and \( B \).

\( A_R(C) := \{ x \in \mathbb{R}^d : ||x| - R| \leq C \} \).

\( C \): a constant which may vary from line to line.

\( A \lesssim B \): \( A \leq CB \).

\( A \approx B \): \( A \lesssim B \) and \( B \lesssim A \).

\( A \ll B \): \( A \leq \frac{1}{C} B \), for some large constant \( C \).

\( |A| \): length of the vector \( A \) or the measure of the set \( A \).

2. Some extensions and corollaries of Theorem 1

Following the remark on page 1381 of [11], one can easily extend Theorem 1 to more general elliptic surfaces. First let us recall the definition of elliptic surfaces from [13] and [4].

Definition 1. We say \( \phi : B(0, 1) \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R} \) is an \((M, \varepsilon_0)\)-elliptic phase if \( \phi \) satisfies

\(\text{i) } \| \phi \|_{C^\infty} < M, \)

\(\text{ii) } \phi(0) = \nabla \phi(0) = 0, \) and

\(\text{iii) For all } x \in B(0, 1), \) all eigenvalues of the Hessian \( \phi_{x_i x_j}(x) \) lie in \([1 - \varepsilon_0, 1 + \varepsilon_0]\).

We say \( S \) is an \((M, \varepsilon_0)\)-elliptic surface if \( S = \{(x, y) \in B(0, 1) \times \mathbb{R} \subset \mathbb{R}^d : y = \phi(x)\} \) for some \((M, \varepsilon_0)\)-elliptic phase \( \phi \).

We recall the following properties of elliptic phases (see, e.g., [13, 4]):

I) Let \( \phi \) be an \((M, \varepsilon_0)\)-elliptic phase and \( B(x_0, \eta) \subset B(0, 1) \). Let

\[ \tilde{\phi}(x) := \frac{1}{\eta^2} (\phi(x \eta + x_0) - \phi(x_0) - \eta x \cdot \nabla \phi(x_0)), \quad x \in B(0, 1). \]

Then \( \tilde{\phi} \) is a \((C_d M, \varepsilon_0)\)-elliptic phase.

II) Let \( S \) be a smooth compact submanifold of \( \mathbb{R}^d \) with strictly positive principal curvatures. Note that for any \( \varepsilon_0 > 0 \) and for any \( s \in S \) there is a neighborhood \( U_s \) of \( s \) and an affine bijection \( a_s \) of \( \mathbb{R}^d \) such
that \( a_s(U_s) \) is an \((M, \varepsilon_0)\)-elliptic surface, where \( M \) depends only on \( d \), \( \| \phi \|_{C^{\infty}} \) and the principal curvatures at \( s \). Moreover, by using a partition of unity, we can write \( S \) as a union of affine images of finitely many \((M, \varepsilon_0)\)-elliptic surfaces.

We have the following generalization of Theorem 1.

**Theorem 3.** Let \( d \geq 3 \) and \( \alpha \in (0, d) \). Let \( H \) be a function satisfying (2) and (3). For any \( M > 0 \), there exists \( \varepsilon_0 > 0 \) such that the following statement holds.

Let \( S_1, S_2 \) be compact subsets of diameter \( \approx 1 \) of an \((M, \varepsilon_0)\)-elliptic surface in \( \mathbb{R}^d \) with \( d(S_1, S_2) > \frac{1}{100} \). Let \( \sigma \) be the Lebesgue measure on \( S \). Then for all \( q > q_0(\alpha, d) \), we have

\[
\| \hat{f}_1 \hat{f}_2 \|_{L^q(d\sigma)} \leq C_{M,q,d} \| f_1 \|_{L^2(S_1, d\sigma_1)} \| f_2 \|_{L^2(S_2, d\sigma_2)},
\]

for all \( f_j \in L^2(d\sigma) \) supported in \( S_j \), \( j = 1, 2 \).

In the application to the distance set problem, we need the following corollary of this theorem. Recall that

**Definition 2.** A compactly supported probability measure \( \mu \) is called \( \alpha \)-dimensional if it satisfies

\[
\mu(B(x, r)) \leq C_{\mu} r^\alpha, \quad \forall r > 0, \forall x \in \mathbb{R}^d.
\]

**Corollary 1.** Let \( \mu \) be an \( \alpha \)-dimensional measure. Let \( \beta > 0 \) and \( \beta R^{-1/2} \lesssim \eta \lesssim 1 \). Let \( I_1, I_2 \) be subsets of \( A_R(\beta) = \{ x \in \mathbb{R} : |x| - R \} < \beta \} \), satisfying

\[
\text{diam}(I_j) \approx R\eta, \quad j = 1, 2, \quad d(I_1, I_2) \approx R\eta.
\]

Then for any \( q > q_0(\alpha, d) \)

\[
\| \hat{f}_1 \hat{f}_2 \|_{L^q(d \mu)} \lesssim \beta(R\eta)^{d-1} \eta^{-\frac{1}{q}} \| f_1 \|_2 \| f_2 \|_2,
\]

for any functions \( f_j \) supported in \( I_j \), \( j = 1, 2 \).
We need the following version of the uncertainty principle in the proof of the corollary. For a proof see, e.g., [4] and [16, Chapter 5]. Let \( \varphi \) be a Schwartz function satisfying

\[
\varphi(\xi) = 1, \quad \text{for } |\xi| < 2 \quad \text{and} \quad \varphi(\xi) = 0, \quad \text{for } |\xi| > 4.
\]

For each ball \( D \subset \mathbb{R}^d \) fix an affine bijection \( a_D \) of \( \mathbb{R}^d \) which maps \( D \) to \( B(0,1) \). Let \( \varphi_D := \varphi \circ a_D \).

**Lemma 2.1.** Let \( \mu \) be an \( \alpha \)-dimensional measure in \( \mathbb{R}^d \). Let \( D \) be a ball of radius \( s \) in \( \mathbb{R}^d \). Then the function \( \mu_D := |\varphi_D|^*\mu \) satisfies

i) \( \|\mu_D\|_\infty \lesssim s^{d-\alpha} \),

ii) \( \|\mu_D\|_1 \lesssim 1 \),

iii) \( \mu_D(B) := \int_B \mu_D(y) \, dy \lesssim r^\alpha \), for any ball \( B \) of radius \( r \geq 100s^{-1} \).

**Proof of Corollary 1.** Note that \( f_1 * f_2 \) is contained in a ball \( D \) of radius \( \approx R\eta \). Therefore

\[
\|\hat{f}_1 \hat{f}_2\|_{L^q(d\mu)} = \|(\hat{f}_1 \hat{f}_2) * \varphi_D^\gamma\|_{L^q(d\mu)}
\]

\[
\lesssim \|\hat{f}_1 \hat{f}_2\|_{L^q(|\varphi_D^\gamma|*d\mu)} \|\varphi_D^\gamma\|_1^{1/q}
\]

\[
\lesssim \|\hat{f}_1 \hat{f}_2\|_{L^q(\mu_D)}.
\]

Let \( e \) be the unit vector in the direction of the center of mass of \( I_1 \cup I_2 \). Let \( \{e_1 = e, e_2, ..., e_d\} \) be an orthogonal basis for \( \mathbb{R}^d \). Let \( T : \mathbb{R}^d \to \mathbb{R}^d \) be the linear map which satisfies

\[
T(e_1) = \frac{1}{R^2}e_1, \quad T(e_j) = \frac{1}{R\eta}e_j, \quad j = 2, 3, ..., d,
\]

In view of I) and II) above, \( C_j = TI_j \) is contained in \( \approx \frac{\beta}{R^2} \)-neighborhood of an affine image of a surface \( S_j \), \( j = 1, 2 \), where the surfaces \( S_1, S_2 \) satisfy the hypothesis of Theorem 3 (with \( M \) independent of \( R, \eta, I_1, I_2 \)).

Let \( g_j(x) = f_j(T^{-1}x), \quad j = 1, 2 \). Note that \( g_j \) is supported in \( C_j \), \( j = 1, 2 \). We have

\[
\hat{g}_j(\xi) = \frac{1}{\det(T)} \hat{g}_j(T^{-1}(\xi)) = (R\eta)^d \eta \hat{g}_j(T^{-1}(\xi)), \quad j = 1, 2.
\]
Therefore,
\begin{equation}
\|\hat{f}_1 \hat{f}_2\|_{L^q(\mu)} = \left( R\eta \right)^{2d-\frac{d}{q}} \left[ \int |\hat{g}_1(T^{-1}x)\hat{g}_2(T^{-1}x)|^{q} \mu_D(x)dx \right]^{1/q}
\end{equation}

\begin{equation}
= \left( R\eta \right)^{2d-\frac{d}{q}} \eta^{2-\frac{1}{q}} \left[ \int |\hat{g}_1(x)\hat{g}_2(x)|^{q} \mu_D(Tx)dx \right]^{1/q}
\end{equation}

\begin{equation}
= \left( R\eta \right)^{2d-\frac{d}{q}} \eta^{2-\frac{1}{q}} \left( R\eta \right)^{\frac{d-\alpha}{q}} \|\hat{g}_1\|_{L^q(H_{dx})},
\end{equation}

where \( H(x) = (R\eta)^{a-d} \mu_D(Tx) \). Using Lemma 2.1, it is easy to see that \( H \) satisfies the conditions (2) and (3) (possibly with a constant other than 1 which can be scaled out). Since \( g_j \) is supported in \( C_j \), using Theorem 3 we obtain
\begin{equation}
\|\hat{g}_1 \hat{g}_2\|_{L^q(\mu)} \lesssim \frac{\beta}{R\eta^2} \|g_1\|_2 \|g_2\|_2.
\end{equation}

We also have
\begin{equation}
\|g_j\|_2 = \left( R\eta \right)^{-\frac{d}{2}} \eta^{-\frac{1}{2}} \|f_j\|_2, \quad j = 1, 2.
\end{equation}

Using (8), (9), (10) and (11), we have
\begin{equation}
\|\hat{f}_1 \hat{f}_2\|_{L^q(\mu)} \lesssim \left( R\eta \right)^{2d-\frac{d}{q}} \eta^{2-\frac{1}{q}} \left( R\eta \right)^{\frac{d-\alpha}{q}} \left( R\eta \right)^{d-\frac{1}{q}} \|f_1\|_2 \|f_2\|_2
\end{equation}

\begin{equation}
= \beta \left( R\eta \right)^{d-\frac{\alpha}{q}} \eta^{-\frac{1}{q}} \|f_1\|_2 \|f_2\|_2.
\end{equation}

\[\square\]

3. Application to the distance set problem

In [7] (also see ([16, 8]), Mattila developed a method to attack the distance set problem. Mattila’s approach was used in [7, 1, 14, 6, 4]. We refer the reader to [14] and [4] for the following version of Mattila’s theorem.

**Theorem 4.** Fix \( \alpha \in \left[ \frac{d}{2}, \frac{d+1}{2} \right] \) and \( q_0 \in [1, 2] \) such that \( \alpha (1 + \frac{1}{q_0}) \geq d \). Assume that for all \( q > q_0 \), for all \( \alpha \)-dimensional measures \( \mu \), for all \( R > 1 \) and for all \( f \) supported in \( A_R(1) \), we have
\begin{equation}
\left| \int f^\vee(u) \, d\mu(u) \right| \leq C_{q,\mu} R^{\frac{d-\alpha}{2} - \frac{\alpha}{q_0}} \|f\|_2,
\end{equation}

where \( A_R(1) \) is the annulus centered at the origin with inner radius \( R \) and outer radius \( R+1 \).
where \( f^\vee \) is the inverse Fourier transform of \( f \). Then Falconer’s conjecture holds for \( \alpha \), i.e.

\[
\dim(E) > \alpha \Rightarrow |\Delta(E)| > 0.
\]

In light of Theorem 4, Theorem 2 is a corollary of the following

**Theorem 5.** Let \( \alpha \in (0, d) \) and \( q > q_0(\alpha, d) \). For all \( \alpha \)-dimensional measures \( \mu \), for all \( R > 1 \) and for all \( f \) supported in \( A_R(1) \), (12) holds.

Like Theorem 2, Theorem 5 was first proved in [14] for \( d = 2 \).

**Remark.** By duality and the uncertainty principle (see [4]), the inequality (12) implies that for every \( \beta < \frac{\alpha}{2q} \)

\[
\| \hat{\mu}(R \cdot) \|_{L^2(S^{d-1})} \lesssim R^{-\beta}.
\]

In fact, one can easily keep track of the constant \( C_{q,\mu} \) in (12) and obtain the statement

\[
\| \hat{\mu}(R \cdot) \|_{L^2(S^{d-1})} \leq C_{\alpha,\beta} R^{-\beta} \sqrt{I_\alpha(\mu)},
\]

for any \( \beta < \frac{\alpha}{2q} \) (see [14, 3]). Here \( I_\alpha(\mu) \) is the \( \alpha \)-dimensional energy of the measure \( \mu \),

\[
I_\alpha(\mu) := \int \int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = C_{\alpha,d} \int \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{d-\alpha}} d\xi.
\]

Combining the result of Theorem 5 with the previously known partial results [7, 9, 14, 10, 8, 3, 4], we see that the inequality (13) holds for every

\[
\beta < \begin{cases} 
\frac{\alpha}{2}, & \alpha \in (0, \frac{d-1}{2}], \\
\frac{d-1}{4}, & \alpha \in [\frac{d-1}{2}, \frac{d}{2}], \\
\frac{d+2\alpha-2}{8}, & \alpha \in [\frac{d}{2}, \frac{d+2}{2}], \\
\frac{\alpha-1}{2}, & \alpha \in [\frac{d+2}{2}, d). 
\end{cases}
\]

The range of \( \beta \) is optimal for each \( \alpha \in (0, 2) \) for \( d = 2 \) (see, e.g., [9, 14, 3]). In higher dimensions, the range is optimal for \( \alpha \leq \frac{d-1}{2} \) (see [9]). However, there is no reason to believe that the range is optimal for \( \alpha > \frac{d-1}{2} \) and \( d \geq 3 \).
4. Proof of Theorem 5

The proof is same as the proof given in [4] except a minor change in the inequality (22) below. Fix $\alpha \in (0,d)$. Let $f$ be supported in $A_R(1)$ with $L^2$ norm 1. Below, we prove that for each $q > q_0(\alpha, d)$

\begin{equation}
\|f^\vee\|_{L^2(d\mu)} \lesssim R^{d\frac{1}{2} - \frac{\alpha}{2q}}. \tag{15}
\end{equation}

(12) can be obtained from (15) using Cauchy-Schwarz inequality. As in [4], we use the bilinear approach. It suffices to prove (15) for functions $f$ supported in a subset of $A_R(1)$ of diameter $\ll R$. Consider a dyadic decomposition of $A_R(1)$ into spherical caps, $I$, with dimensions $2 \times 2^n \times \ldots \times 2^n$ for $R^{\frac{1}{2}} \ll 2^n \ll R$.

We say $I$ has sidelength $2^n$ and write $\ell(I) = 2^n$. The unique cap of sidelength $2^{n+1}$ which contains $I$ is called the parent of $I$. Let $I$ and $J$ be caps with the same sidelength. We say $I$ and $J$ are related, $I \sim J$, if they are not adjacent but their parents are.

Let $f_I := f \chi_I$. As in [4], we have

\begin{equation}
\|f^\vee\|^2_{L^2(d\mu)} \leq \sum_{R^{\frac{1}{2}} \ll 2^n \ll R} \sum_{\ell(I) = 2^n, I \sim J} \|f^\vee_I f^\vee_J\|_{L^1(d\mu)} + \sum_{I \in I_E} \|f^\vee_I\|^2_{L^2(d\mu)}
= : S_1 + S_2. \tag{16}
\end{equation}

Here $I_E$ is a set of dyadic caps with sidelengths $\approx R^{\frac{1}{2}}$ satisfying the finite overlapping property:

\begin{equation}
\| \sum_{I \in I_E} \chi_I \|_\infty \lesssim 1. \tag{17}
\end{equation}

First, we obtain a bound for $S_2$. Since each $I \in I_E$ is contained in a ball $D$ of radius $CR^{\frac{1}{4}}$, we have $f^\vee_I = f^\vee_I * \varphi_D^\vee, (\varphi_D$ is defined in the Section 2). Using this and Cauchy-Schwarz inequality, we have

\begin{equation}
|f^\vee_I| \leq (|f^\vee_I|^2 * |\varphi_D^\vee|)^{\frac{1}{2}} \|\varphi_D^\vee\|_1^{\frac{1}{2}} \lesssim (|f^\vee_I|^2 * |\varphi_D^\vee|)^{\frac{1}{2}}. \tag{18}
\end{equation}
Using this, Fubini’s theorem and Lemma 2.1, we obtain
\begin{equation}
\|f_I^\vee\|_{L^2(d\mu)}^2 \leq \int |f_I^\vee(x)|^2 (\mu * |\varphi_I^\vee|)(x) dx \lesssim \|f_I^\vee\|^2_2 R^{\frac{d-\alpha}{2}} = \|f_I\|^2_2 R^{\frac{d-\alpha}{2}}.
\end{equation}

Using (19) and (17), we obtain
\begin{equation}
S_2 = \sum_{I \in \mathcal{I}} \|f_I^\vee\|_{L^2(d\mu)}^2 \lesssim R^{\frac{d-\alpha}{2}} \sum_{I \in \mathcal{I}} \|f_I\|^2_2 \lesssim R^{\frac{d-\alpha}{2}} \|f\|^2_2 = R^{\frac{d-\alpha}{2}}.
\end{equation}
This term is harmless since \( \frac{d-\alpha}{2} \leq d - 1 - \frac{\alpha}{q_0(\alpha,d)} \), for \( \alpha \in (0,d) \).

In the remaining part of the paper we prove that for \( q > q_0(\alpha,d) \),
\begin{equation}
S_1 \lesssim R^{d-1-\frac{\alpha}{q}}. \quad \text{By a standard } L^2\text{-orthogonality argument (see e.g. [13, 15, 3, 4]), it suffices to prove that for each } q > q_0(\alpha,d), \text{ for each } n \text{ and } I \sim J \text{ with } |I| = |J| = 2^n
\end{equation}
\begin{equation}
\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \lesssim C_{\alpha,q,d} R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2.
\end{equation}

Let \( e \) be the unit vector which is in the direction of the center of mass of \( I \cup J \). Consider a tiling of \( \mathbb{R}^d \) with rectangles \( P \) of dimensions \( 100 \times 100 \times ... \times 100 \), the long axis being in the direction \( e \). For each \( P \), let \( a_P \) be an affine bijection from \( \mathbb{R}^d \) to \( \mathbb{R}^d \) which maps \( P \) to the unit cube. Let \( \phi \) be a Schwartz function satisfying
\begin{equation}
\phi(x) \geq \chi_{B(0,1)}(x), \quad x \in \mathbb{R}, \quad \text{supp}(\hat{\phi}) \subset B(0,1).
\end{equation}

Let \( \phi_P := \phi \circ a_P \) and \( f_{I,P} := \hat{f_I^\vee} \phi_P \). Using (21) and the fact that the rectangles \( P \) tile \( \mathbb{R}^d \), we obtain
\begin{equation}
\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \lesssim \sum_P \int |f_I^\vee(x)f_J^\vee(x)| \phi_P(x) d\mu(x)
\lesssim \sum_P \|f_I^\vee f_J^\vee\|_{L^2(\mu)} \|\phi_P\|_{L^{q'}(\mu)}^{1/q'},
\end{equation}
where \( q' = \frac{q}{q-1} \).

To estimate \( \|f_I^\vee f_J^\vee\|_{L^2(\mu)} \), we use Corollary 1 of Theorem 1. Let \( I_P \) be the support of \( f_{I,P} \). Note that \( I_P \) is contained in \( I + \text{supp}(\hat{\phi}_P) \subset I + P_{\text{dual}} \), where \( P_{\text{dual}} \) is the dual of \( P \) centered at the origin. We have
Lemma 4.1. \( I + P_{\text{dual}} \) is contained in a spherical cap of dimensions \( 10 \times \frac{11}{10} 2^n \times \ldots \times \frac{11}{10} 2^n \) in \( A_R(10) \) which contains \( I \).

See [4] for the elementary proof. Using Lemma 4.1 for \( I \) and \( J \), we see that \( I_P \) and \( J_P \) have diameter \( \approx 2^n \); they are contained in \( A_R(10) \) and \( d(I_P, J_P) \approx 2^n \). Therefore, Corollary 1 implies that

\[
\| f_I^\vee f_J^\vee \|_{L^q(\mu)} \lesssim 2^{n(d-1-\frac{q}{2})} \left( \frac{2^n}{R} \right)^{-\frac{1}{q}} \| f_I, f_J \|_2 \| f_I, f_J \|_2
\]

(23)

Now, we estimate \( \| \phi_P \|_{L^1(\mu)} \). Using the Schwartz decay of \( \phi_P \), we have

\[
\| \phi_P \|_{L^1(\mu)} \leq \sum_{j=1}^{\infty} 2^{-Mj} \int \chi_{2^j P}(x) \, d\mu(x).
\]

Note that \( 2^j P \) can be covered by \( \approx \frac{R}{2^n} \) balls of radius \( \approx \frac{2^n}{R} \). Since \( \mu \) is \( \alpha \)-dimensional, we get

(24)

\[
\| \phi_P \|_{L^1(\mu)} \lesssim \sum_{j=1}^{\infty} 2^{-Mj} 2^{n\alpha-n} R^{1-\alpha} \lesssim 2^{n\alpha-n} R^{1-\alpha}.
\]

Using (22), (23), (24) and then Cauchy-Schwarz inequality, we get

\[
\| f_I^\vee f_J^\vee \|_{L^1(\mu)} \lesssim R^{1-\frac{n}{q}} 2^{n(\alpha(1-\frac{1}{2})+\frac{d-2}{q})} \sum_P \| f_I, f_J \|_2 \| f_I, f_J \|_2
\]

\[
\lesssim R^{1-\frac{n}{q}} 2^{n(\alpha(1-\frac{1}{2})+\frac{d-2}{q})} \left[ \sum_P \| f_I, f_J \|_2^2 \right]^\frac{1}{2} \left[ \sum_P \| f_I, f_J \|_2^2 \right]^\frac{1}{2}
\]

(25)

Using the Schwartz decay of \( \phi \), the fact that the rectangles \( P \) tile \( \mathbb{R}^d \) and Plancherel formula, we get

\[
\| f_I^\vee f_J^\vee \|_{L^1(\mu)} \lesssim R^{1-\frac{n}{q}} 2^{n(\alpha(1-\frac{1}{2})+\frac{d-2}{q})} \| f_I \|_2 \| f_J \|_2.
\]

The exponent of \( 2^n \) in (25) is non-negative and \( 2^n \lesssim R \). Therefore

\[
\| f_I^\vee f_J^\vee \|_{L^1(\mu)} \lesssim R^{1-\frac{n}{q}} R^{(1-\frac{1}{2})+\frac{d-2}{q}} \| f_I \|_2 \| f_J \|_2
\]

(26)

\[
= R^{d-1-\frac{n}{q}} \| f_I \|_2 \| f_J \|_2.
\]

This finishes the proof of Theorem 5.
5. Proof of Theorem 1

The proof of Theorem A is quite technical and lengthy. The proof of Theorem 1 is a very simple modification of this proof. Therefore, we won’t repeat the whole argument but just describe the modification we make. We also note that the “epsilon-removal” lemma in [12, Lemma 2.4] which was used in [11] to reduce the proof to a localized restriction estimate remains valid in our case. In fact, Theorem 2 can be proved using only the localized version of (4),

$$\|f_1 d\sigma f_2 d\sigma\|_{L^q(B(0,M),Hd\xi)} \leq C_{\epsilon,\alpha,q,d} M^\epsilon \|f_1\|_{L^2(\sigma)} \|f_2\|_{L^2(\sigma)},$$

since we don’t prove an endpoint result. We warn the reader that in [11] the letter \(n\) is used for dimension. We use the letter \(d\) for dimension and \(d = n + 1\).

The only restriction on \(q\) in the argument in [11] comes from the interpolation of two estimates, Lemma 5.2 of [11] and the inequality (23) on page 1372 in that paper. More explicitly, using the notation of [11], let

$$F := \sum_{T_1 \in \mathcal{T}_1 : T_1 \not\sim B} \sum_{T_2 \in B} \phi_{T_1} \phi_{T_2}.$$  

The argument in [11] requires an estimate of the form (here \(R, \delta\) and \(\epsilon\) are some parameters in the proof and \(B\) is an \(R^{1-\delta}\)-ball)

$$\|F\|_{L^q(B)} \lesssim R^\epsilon R^{C\delta} (\#T_1)^{1/2} (\#T_2)^{1/2}, \quad \forall R > 1.$$  

For \(q = \frac{n+3}{n+1} = \frac{d+2}{d}\), this estimate can be obtained by interpolating the assertion of Lemma 5.2,

$$\|F\|_{L^1(B)} \lesssim R^\epsilon R (\#T_1)^{1/2} (\#T_2)^{1/2},$$

and the inequality (23) on page 1372,

$$\|F\|_{L^2(B)} \lesssim R^\epsilon R^{-(d-2)/4} (\#T_1)^{1/2} (\#T_2)^{1/2}.$$  

In our case, the whole argument works with the minor modification of replacing the Lebesgue measure with \(H(x)dx\) in the natural places. In
addition, in the step above, we modify the interpolation by
\[
\|F\|_{L^q(B,Hdx)} \leq \|F\|_{L^2(B,dx)} \left[ \int_B |H(x)|^{2/(2-q)}dx \right]^{\frac{1}{2} - \frac{1}{2q}} \\
&\leq \|F\|_{L^2(B,dx)} \left[ \int_B |H(x)|^{\frac{1}{q}}dx \right]^{\frac{1}{2} - \frac{1}{2q}} \\
& \lesssim \|F\|_{L^2(B,dx)} R^{\frac{\alpha}{q} - \frac{\alpha}{2}} \\
& \lesssim R^\epsilon R^{-\frac{d-2}{4}} R^{\frac{\alpha}{2} - \frac{\alpha}{4}} (\#T_1)^{1/2} (\#T_2)^{1/2}.
\]
We used (2) in the second inequality above and (3) in the third one. Therefore (27) holds for \(-\frac{d-2}{4} + \frac{\alpha}{q} - \frac{\alpha}{2} < 0\) and \(q \geq 1\). Thus, the statement holds if
\[
q > \max \left( 1, \min \left( d + 2, \frac{4\alpha}{d} \right) \right).
\]

REFERENCES


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