QUASI-LINEAR DYNAMICS IN NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. It is shown that a large subset of initial data with finite energy ($L^2$ norm) evolves nearly linearly in nonlinear Schrödinger equation with periodic boundary conditions. These new solutions are not perturbations of the known ones such as solitons, semiclassical or weakly linear solutions.

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation

\[ iq_t + \Delta q + |q|^2 q = 0, \]

where $q : \mathbb{R} \times M_x \to \mathbb{C}$, frequently appears as the leading approximation of the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium. It arises in a number of physical models in the description of nonlinear waves such as the propagation of a laser beam in a medium whose index of reflection is sensitive to the wave amplitude.

NLS has been considered with various boundary conditions: such as on $M = \mathbb{R}^d, \mathbb{T}^d$, with periodic or Dirichlet boundary conditions. One dimensional cubic NLS is integrable [17] and the explicit (or approximately explicit) solutions can be obtained as solitons, cnoidal waves, and their perturbations. There have been also many interesting results on the long time asymptotics of solutions of integrable NLS in the limit of small dispersion, see e.g. the recent monograph [10], [6, 16, 3] and references therein.

Recent results in optical communication literature (see, e.g. [2, 7, 13], and the appendix) suggest that for some initial data (highly localized pulses) the evolution is nearly linear. Based on these studies, we introduce a large class of solutions, which we call quasi-linear, for one dimensional cubic NLS with periodic boundary conditions. These solutions can be characterized by the magnitude of Fourier coefficients of the initial data. We prove that these solutions evolve nearly linearly using a normal form reduction and estimates on Fourier sums. Although we do not explicitly use integrability, we do rely

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on the integrability of the quartic normal form which is partially responsible for quasi-linear behavior. Therefore, similar results can be obtained for some nonlinear PDEs, such as \( iq_t + q_{xx} + |q|^2 q = 0 \), for which there are no integrability results. We do not study long time asymptotics but rather the finite time dynamics in the limit of spectral broadening of initial data. This broadening forces \( \|q(x,0)\|_{H^s} \) to grow to infinity, making the analysis rather nontrivial even for the finite time interval. While, we consider the focusing case, our result holds for defocusing case as well. The reader will be able to see that our proof can be immediately adapted for the defocusing case, since nowhere our arguments rely on the nonlinearity sign.

In many engineering and physics applications, nonlinearity is unavoidable while modeling and optimizing a linear behavior is much easier than a nonlinear one. Therefore, it is an important question whether a nonlinear system can be made to behave linearly. In applied mathematics and physics literature, such a behavior has been observed in e.g. \cite{1, 7, 8, 14, 15}. We believe that our result gives a systematic way to analyze this behavior in nonlinear systems when the energy is distributed over many Fourier harmonics.

2. Main Results

We consider the nonlinear Schrödinger equation with periodic boundary conditions,

\[ iq_t + q_{xx} + 2|q|^2 q = 0, \]

with initial data in \( q(0) \in L^2(-\pi, \pi) \). In \cite{4}, Bourgain proved the \( L^2 \) global well-posedness of this equation. The numerical simulations of quasi-linear regime for light wave communication systems suggest that the following statement should hold (see, e.g., \cite{7, 14})

**Observation 1.** Assume that initial data is a localized Gaussian

\[ q(x,0) = \frac{1}{\sqrt{\varepsilon}} e^{-\frac{x^2}{\varepsilon^2}} h(x), \]

where \( h(x) \) is a smooth cutoff near \( x = \pm \pi/2 \). Then the initial data evolves quasi-linearly,

\[ \|q(x,t) - e^{i(\Delta + 4P)}q(x,0)\|_2 \to 0, \]

as \( \varepsilon \to 0 \) and for \( t \leq T \), where \( T \) is a fixed positive number, and \( P = \|q(\cdot,0)\|_2^2/2\pi \).

We will prove (2) for a large class of initial data (including the ones above) characterized by the magnitude of Fourier coefficients. We will use Fourier transform in the form

\[ q(x,t) = \sum_{n \in \mathbb{Z}} u(n,t) e^{inx}, \]

\[ u(m,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(x,t) e^{-inx} dx, \]
so that the NLS equation takes the form

\[
\frac{du(m)}{dt} - m^2 u(m) + 2 \sum_{m_1 + m_2 + m_3 = m} u(m_1) u(m_2) \overline{u}(m_3) = 0.
\]

Our main result is the following theorem.

**Theorem 2.1.** Let \( P > 0 \) and \( C > 0 \) be fixed. Assume that the Fourier sequence of the initial data \( u(n, 0) = q(\cdot, 0)(n) \) satisfies

\[
\|u(\cdot, 0)\|_{L^\infty} \leq C \varepsilon^{\frac{1}{2}}, \quad \|u(\cdot, 0)\|_{L^1} \leq C \varepsilon^{-\frac{1}{2}},
\]

for sufficiently small \( \varepsilon \in (0, 1) \). Then, for each \( t > 0 \),

\[
\|q(\cdot, t) - e^{i(\Delta + 4P)}q(\cdot, 0)\|_{L^2} \lesssim \langle t \rangle \varepsilon^{1-},
\]

where \( P = \|q(\cdot, 0)\|_2^2/2\pi \), \( \langle t \rangle = \sqrt{1 + t^2} \) and the implicit constant depends only on \( C \).

**Remark 2.1.** The initial data in the observation above satisfies the hypothesis of the theorem. In fact, if \( f \) is an \( H^s \) function for some \( s > 1 \) with compact support on \((-\pi, \pi)\), then

\[
\int_{\varepsilon} f(x) = \frac{1}{\sqrt{\varepsilon}} f(x/\varepsilon)
\]

satisfies the hypothesis of the theorem.

By continuous dependence on initial data in \( L^2 \), it suffices to prove (4) for any \( \delta > 0 \) and for any initial data in the following subset of \( L^2 \):

\[
B_{\varepsilon,C}^\delta = \{ f \in L^2 : \|f\|_{p, \delta} := \left[ \sum_{n=\infty} \|\hat{f}(n)\|_{l^p}^p \right]^{1/p} \leq C \varepsilon^{\frac{1}{2} - \frac{1}{p}}, \ p \in [1, \infty] \}.
\]

Since \( B_{\varepsilon,C}^\delta \subset H^1 \), we can introduce the Hamiltonian [11]

\[
H(u) = i \sum_n n^2 |u(n)|^2 - i \sum_{l(n)=0} u(n_1) u(n_2) \overline{u}(n_3) \overline{u}(n_4),
\]

with conjugated variables \( \{u(n), \overline{u}(n)\}_{n \in \mathbb{Z}} \), where \( l(n) = n_1 + n_2 - n_3 - n_4 \). The Hamiltonian flow is then given by

\[
\dot{u}(n) = \frac{\partial H}{\partial \overline{u}(n)}.
\]

Theorem 2.1 follows from the following by continuous dependence on initial data in \( L^2 \).

**Theorem 2.2.** Let \( P > 0 \) and \( C > 0 \) be fixed. Assume that \( \|q(0)\|_2^2 = 2\pi P \), and \( q(\cdot, 0) \in B_{\varepsilon,C}^\delta \) for some \( \delta > 0 \), and for sufficiently small \( \varepsilon \in (0, 1) \). Then, for each \( t > 0 \),

\[
\|q(\cdot, t) - e^{i(\Delta + 4P)}q(\cdot, 0)\|_2 \lesssim \langle t \rangle \varepsilon^{1-},
\]

where the implicit constant depends only on \( C \).
The proof of Theorem 2.2 is based on the normal form transformations, see, e.g., [11], [12] and [5]. In Section 3, we introduce a canonical transformation \( u = u(v) \) in the Fourier space which brings the equation into the form\(^1\), see (16) and (17) below,
\[
\dot{v}(n) = i(n^2 + 4P)v(n) + E(v)(n).
\]
We prove that the transformation \( u = u(v) \) is near-identical in the following sense.

**Proposition 2.1.** If \( u \in B^\delta_{\epsilon,C} \) or \( v \in B^\delta_{\epsilon,C} \), then
\[
\|u\|_{\ell^2} = \|v\|_{\ell^2}, \text{ and } \|u - v\|_{\ell^p, \delta} \lesssim \varepsilon^{3 - \frac{1}{p}}
\]
for \( 1 \leq p \leq \infty \), where the implicit constant depends on \( C \) and \( p \).

In particular, if \( \epsilon \) is sufficiently small, then \( u \in B^\delta_{\epsilon,C} \) implies \( v \in B^\delta_{\epsilon,2C} \) and vice versa.

Then, we estimate the error term \( E(v) \) as follows

**Proposition 2.2.** If \( v \in B^\delta_{\epsilon,C} \), then the error term \( E(v) \) in the transformed equation (6) satisfies
\[
\|E(v)\|_{\ell^p, \delta} \lesssim \varepsilon^{3 - \frac{1}{p}},
\]
for \( 1 \leq p \leq \infty \), where the implicit constant depends on \( C \) and \( p \).

Propositions 2.1 and 2.2 imply Theorem 2.2. Indeed, assume that \( q(\cdot, 0) \in B^\delta_{\epsilon,C} \) for some \( \delta > 0 \), \( C > 0 \), and for sufficiently small \( \epsilon \in (0, 1) \). Multiplying (6) with \( e^{-i(n^2 + 4P)t} \) and integrating over \( t \), we obtain
\[
v(n, t)e^{-i(n^2 + 4P)t} - v(n, 0) = \int_0^t e^{-i(n^2 + 4P)\tau} E(v)d\tau.
\]

This and Propositions 2.1 and 2.2 imply, for each \( p \in [1, \infty] \), that
\[
\|v(t) - e^{iLt}v(0)\|_{\ell^p, \delta} = \|v(t)e^{-iLt} - v(0)\|_{\ell^p, \delta} \lesssim t \varepsilon^{3 - \frac{1}{p}},
\]
where \( L(v)(n) = (n^2 + 4P)v(n) \).

Finally, Proposition 2.1 and (7) imply, for \( p \in [1, \infty] \), that
\[
\|u(t) - e^{iLt}u(0)\|_{\ell^p, \delta} \leq \|u(t) - v(t)\|_{\ell^p, \delta} + \|v(t) - e^{iLt}v(0)\|_{\ell^p, \delta} + \|e^{iLt}v(0) - e^{iLt}u(0)\|_{\ell^p, \delta}
\]
\[
\lesssim \langle t \rangle \varepsilon^{3 - \frac{1}{p}},
\]
where the implicit constant depends on \( C \). In particular, this yields the assertion of Theorem 2.2 as follows
\[
\|q(t) - e^{i(\Delta + 2P)t}q(0)\|_2 = \|u(t) - e^{iLt}u(0)\|_{\ell^2} \leq \|u(t) - e^{iLt}u(0)\|_{\ell^p, \delta} \lesssim \langle t \rangle \varepsilon^{1-}.
\]

\(^1\)Similar quasi-linear behavior can be obtained for the nonintegrable NLS \( iqt + q_{xxxx} + |q|^2q = 0 \) with the leading behavior given by \( \dot{v}(n) = i(n^4 + 4P)v(n) \).
Notation.
We will frequently use convolution with $1/|n|$, which will be denoted by
\[
\rho(n) = \frac{1}{|n|} \chi_{\mathbb{Z} \setminus \{0\}}(n)
\]
and we will also use the notation $\langle n \rangle = \sqrt{1 + |n|^2}$.
We always assume by default that the summation index avoids the terms
with vanishing denominators.
To avoid using unimportant constants, we will use $\lesssim$ sign:
$A \lesssim B$ means there is an absolute constant $K$ such that $A \leq KB$. In some
cases the constant will depend on parameters such as $p$.
$A \lesssim B(\eta-)$ means that for any $\gamma > 0$, $A \leq C_\gamma B(\eta - \gamma)$.
$A \lesssim B(\eta+)$ is defined similarly.

3. Normal Form Calculations

Consider the change of variables $u(n) \to v(n)$, generated by the time $1$
flow of the Hamiltonian

\[
F = \sum_{i(m)=0} f(m_1, m_2, m_3, m_4) w(m_1)w(m_2)\bar{w}(m_3)\bar{w}(m_4),
\]

which we will specify later. Namely, solve

\[
\frac{dw}{ds} = \frac{\partial F}{\partial \bar{w}}
\]

with initial condition $w|_{s=0} = v$, thus producing a symplectic transformation
$u = u(v) := w|_{s=1}$. Let $X^s_F$ be the time $s$ map of the flow of $F$, then using
Taylor expansion [11, 12], we get

\[
H(X^s_F)(v) = H(v) + \{H, F\}(v) + \{\{H, F\}, F\}(v) + \ldots +
\]

\[
+ \left\{ \ldots \left( \left\{ \ldots \left( \left\{ \left\{ H, F \right\}, F \right\}, \ldots, F \right\} \right\} \right\} (v) +
\]

\[
+ \int_0^1 (1 - s) \left\{ \ldots \left( \left\{ \ldots \left( \left\{ H, F \right\}, F \right\}, \ldots, F \right\} \right\} (X^s_F)(v) ds,
\]

where

\[
\{A, B\} = \sum_n \left( \frac{\partial A}{\partial u(n)} \frac{\partial B}{\partial \bar{u}(n)} - \frac{\partial A}{\partial \bar{u}(n)} \frac{\partial B}{\partial u(n)} \right)
\]

is the Poisson bracket.
Recall that $H$ has a quadratic and a quartic part

\[
H = \Lambda_2 + H_4,
\]

where

\[
\Lambda_2 = i \sum m^2 |u(m)|^2.
\]
We write \( H_4 = H_4^{nr} + H_4^r \), where the superscripts “nr” and “r” denote the non-resonant and resonant terms:

\[
H_4^{nr} = i \sum_{l(m) = 0, \ q(m) \neq 0} v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4)
\]

\[
H_4^r = i \sum_{l(m) = 0, \ q(m) = 0} v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4),
\]

where \( q(m) = m_1^2 + m_2^2 - m_3^2 - m_4^2 \). Note that we can further decompose \( H_4^r \) as

\[
H_4^r = -i \sum_m |v(m)|^4 + 2i \sum_{m_1, m_2} |v(m_1)|^2|v(m_2)|^2
\]

\[
:= H_4^{r1} + H_4^{r2}.
\]

We also introduce the following notation for higher order commutators

\[
g_F^0(H) = H, \quad g_F^{k+1}(H) = \{g_F^k, F\}, \ k = 0, 1, 2, \ldots
\]

Using this and (11) in (9), we rewrite

\[
H(X_F^k) = \Lambda_2 + H_4 + g_F^1(\Lambda_2 + H_4) + \ldots + g_F^k(\Lambda_2 + H_4)
\]

\[
+ \int_0^1 (1 - s)g_F^{k+1}(\Lambda_2 + H_4)(X_F^s)ds.
\]

We will choose \( F \) so that the following cancellation property holds

\[
\{\Lambda_2, F\} = -H_4^{nr},
\]

which implies that

\[
g_F^j(\Lambda_2) = -g_F^{j-1}(H_4^{nr}).\]

Further analysis also shows that it is sufficient to take \( k = 2 \) in the Taylor expansion, and therefore, we have\(^2\)

\[
H(X_F^2) = \Lambda_2 + H_4^{r2} + H_4^{r1} + g_F^1(H_4^{nr}) + g_F^2(H_4) + \int_0^1 (1 - s)g_F^3(\Lambda_2 + H_4)(X_F^s)ds,
\]

where we had some cancellations resulting from (14). The transformed evolution equation is given by

\[
\frac{d}{dt}v(n) = \frac{\partial H(X_F^2)}{\partial \bar{v}}.
\]

Note that contribution of the “leading” terms, \( \Lambda_2 + H_4^{r2} \), is given by

\[
\frac{\partial}{\partial \bar{v}(n)} \left( i \sum m^2|v(m)|^2 + 2i \sum_{m_1, m_2} |v(m_1)|^2|v(m_2)|^2 \right) = i(n^2 + 4P)v(n).
\]

\(^2\)It turns out that \( k = 1 \) is not enough since then the term \( g_F^1(H_4) \) is present in the Hamiltonian. The direct estimate of \( g_F^1(H_4) \) produces finite order nonlinear effect (see Subsection 4.3).
Therefore, we can rewrite (15) as

\[
\frac{d}{dt} v(n) = i(n^2 + 4P)v(n) + E(v)(n),
\]

where

\[
E(v)(n) = \frac{\partial}{\partial \bar{\psi}} \left( H_4^{11} + g_{F,1}(H_4^1) + g_{F,2}(H_4) + \int_0^1 (1 - s)g_{F,3}(A_2 + H_4)(X_F^s)ds \right).
\]

3.1. Calculation of \( F \). We have

\[
\{A_2, F\} = i \sum_m m^2 \left( \bar{\psi}(m) \frac{\partial F}{\partial \bar{\psi}(m)} - v(m) \frac{\partial F}{\partial v(m)} \right)
\]

\[
= i \sum_{l(m)=0} \left( m_1^2 + m_2^2 - m_3^2 - m_4^2 \right) f(m_1, m_2, m_3, m_4) v(m_1)v(m_2)\bar{\psi}(m_3)\bar{\psi}(m_4).
\]

Therefore, to obtain (13), we let

\[
F = \sum_{l(m)=0} \frac{1}{m_1^2 + m_2^2 - m_3^2 - m_4^2} v(m_1)v(m_2)\bar{\psi}(m_3)\bar{\psi}(m_4).
\]

Alternatively, we can write

\[
F = \sum_{l(m)=0} \frac{v(m_1)v(m_2)\bar{\psi}(m_3)\bar{\psi}(m_4)}{2(m_1 - m_3)(m_2 - m_3)}.
\]

3.2. Proof of Proposition 2.1. First we state a simple corollary of Young’s inequality which will be used many times below. Recall that \( \rho(n) = 1/|n| \) for \( n \neq 0 \) and \( \rho(0) = 0 \).

Lemma 3.1. For any \( p > 1 \), for any choices of \( \pm \) signs

\[
\| \sum_j w(\pm j) \rho(\pm j) \|_{\ell_p^n} \leq \| w \|_{\ell_p^n}.
\]

With some abuse of notation, we will denote each sum of the above form by \( w \ast \rho \).

Proof. Recall that by Young’s inequality, \( \| w \ast \rho \|_{\ell_p^n} \leq \| w \|_{\ell_p^n} \rho \|_{\ell_q^n} \), where \( 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \). The lemma follows since \( \rho \in \ell^q \) for any \( q > 1 \).

Proof of Proposition 2.1. First note that the equality of the \( \ell^2 \) norms follows from Hamiltonian formalism. Indeed, it is straightforward to verify that \( \{F, Q\} = 0 \) (where \( Q(u) = \| u \|_2^2 \)), which implies \( \ell^2 \) norm conservation. To prove the second statement, we should estimate the time 1 map of the Hamiltonian flow of \( F \):

\[
\frac{dw(n)}{ds} = \frac{\partial F}{\partial \bar{\psi}(n)} = \sum_{m_1 + m_2 - m_3 - n = 0} \frac{w(m_1)w(m_2)\bar{\psi}(m_3)}{(m_1 - n)(m_2 - n)}.
\]
Multiplying with $e^{\delta |n|}$, we estimate (assuming that $w \in B^\delta_{\epsilon', C}$)

\[
|e^{\delta |n|} \frac{d}{ds} w(n)| \leq \\
\leq \sum_{m_1 + m_2 - m_3 = n=0} e^{-\delta (|m_1| + |m_2| - |m_3| - |n|)} |w(m_1)e^{\delta |m_1|}w(m_2)e^{\delta |m_2|}w(m_3)e^{\delta |m_3|}| \\
\leq \sum_{m_1 + m_2 - m_3 = n=0} \frac{1}{|m_1 - n||m_2 - n|} |w(m_1)e^{\delta |m_1|}w(m_2)e^{\delta |m_2|}w(m_3)e^{\delta |m_3|}| \\
\leq \|w\|_{\ell^\infty, \delta} \sum_{m_1, m_2} \frac{|w(m_1)|e^{\delta |m_1|}|w(m_2)|e^{\delta |m_2|}}{|m_1 - n||m_2 - n|} \leq \|w\|_{\ell^\infty, \delta} \| |w| e^{\delta |\cdot|} \ast \rho \|_{\ell^2}^2 \\
\leq \|w\|_{\ell^\infty, \delta} \|w\|_{\ell^{\infty, \delta}} \rho \|_{\ell^q}^2,
\]

where $\frac{1}{q} + \frac{1}{\delta} = 1$. Thus, for any $1 \leq q < \infty$, we obtain

\[
\| \frac{d}{ds} w \|_{\ell^\infty, \delta} \leq \|w\|_{\ell^\infty, \delta} \| |w| e^{\delta |\cdot|} \ast \rho \|_{\ell^2}^2 \leq \|w\|_{\ell^\infty, \delta} \|w\|_{\ell^{\infty, \delta}} \| \rho \|_{\ell^q}^2.
\]

Similarly, using Lemma 3.1, we obtain

\[
\| \frac{d}{ds} w \|_{\ell^1, \delta} \leq \|w\|_{\ell^\infty, \delta} \|w| e^{\delta |\cdot|} \ast \rho \|_{\ell^2}^2 \leq \|w\|_{\ell^\infty, \delta} \|w\|_{\ell^{2-\delta}}^2.
\]

The last two inequalities imply that if $w(0) \in B^\delta_{\epsilon', C}$ (or $w(1) \in B^\delta_{\epsilon', C}$) then

\[
\|w(s) - w(0)\|_{\ell^\infty, \delta} \leq e^{\frac{3}{\delta}}, \quad \|w(s) - w(0)\|_{\ell^1, \delta} \leq e^{\frac{1}{\delta}}.
\]

This and interpolation estimates completes the proof of Proposition 2.1. □

3.3. Cancellation property of $H^{r_2}_t$. We claim that $\{H^{r_2}_t, F\} = 0$. Indeed, by (8), $F$ has the phase invariant property

\[
F(v) = F(ve^{\psi}),
\]

but the evolution induced by $H^{r_2}_t$ is just uniform phase rotation,

\[
v(n, t) = e^{i\Delta t} v(n, 0).
\]

Thus,

\[
\{H^{r_2}_t, F\} := \frac{d}{dt} F(X^{t=0}_{H^{r_2}_t}) = 0.
\]

Therefore, we can rewrite the error term (17) as

\[
E(v) = \frac{\partial}{\partial \theta} \left( H^{\Delta_1} + g_\Delta(H^{\Delta_1}) + g_\Delta^2(H_1) + \int_0^1 (1-s)g_\Delta^3(A_2 + H_1)(X^{t}_{H^{r_2}_t})ds \right).
\]
4. Proof of Proposition 2.2

Assuming that \( v \in B_{\epsilon,C}^\delta \), we should prove that the \( \ell^p \) norm of each of the summands in (20) is \( \lesssim \epsilon^{3/2-1/p} \) for \( p = 1 \) and \( p = \infty \). We will do this in the case \( \delta = 0 \) to simplify the exposition. The proof for the case \( \delta > 0 \) is similar by using the simple multiplication by \( \epsilon^{|\delta|} \) argument we used in the proof of Proposition 2.1.

4.1. Estimate of \( \partial_{\sigma(k)} H_4^{11} \). Recall that

\[
H_4^{11} = \sum_m |v(m)|^4,
\]

and hence

\[
\frac{\partial H_4^{11}}{\partial \tilde{v}(k)} = 2i |v(k)|^2 v(k).
\]

We estimate the contribution of this term as

\[
\left\| \frac{\partial H_4^{11}}{\partial \tilde{v}(\cdot)} \right\|_{\ell^\infty} \lesssim \|v^3\|_{\ell^\infty} \lesssim \epsilon^{3/2},
\]

and

\[
\left\| \frac{\partial H_4^{11}}{\partial \tilde{v}(\cdot)} \right\|_{\ell^1} \lesssim \|v^3\|_{\ell^1} = \|v\|_{\ell^3}^3 \lesssim \epsilon^{3/2-1}.
\]

4.2. Estimate of \( \partial_{\sigma(k)} g_F^1(H_4^{11}) \). Using the definition

\[
g_F^1(H_4^{11}) = \sum_n \frac{\partial F}{\partial v(n)} \frac{\partial H_4^{11}}{\partial \tilde{v}(n)} - \frac{\partial F}{\partial \tilde{v}(n)} \frac{\partial H_4^{11}}{\partial v(n)}
\]

\[
= 2i \sum_{m_2 + m_3 - m_4 = 0} \frac{v(m_2) \tilde{v}(m_3) \tilde{v}(m_4)}{(m_3 - m_2)(m_4 - m_2)} v^2(n) \tilde{v}(n) - c.c.,
\]

where we have used the Hamiltonian structure\(^3\)

\[
\frac{\partial H}{\partial \tilde{v}(n)} = -\frac{\partial H}{\partial v(n)}, \quad \frac{\partial F}{\partial \tilde{v}(n)} = -\frac{\partial F}{\partial v(n)}.
\]

We calculate

\[
\left| \frac{\partial}{\partial \tilde{v}(k)} g_F^1(H_4^{11}) \right| \leq 4 \sum_{m_2 + m_3 - m_4 = k} \frac{|v(m_2)||v(m_4)||v(n)|^3}{|k - m_2||m_4 - m_2|} + 6 \sum_{-m_2 + m_3 + m_4 = k} \frac{|v(m_2)||v(m_3)||v(m_4)||v(n)|^2}{|m_3 - m_2||m_4 - m_2|} + 2 \sum_{-n + m_3 + m_4 = k} \frac{|v(m_3)||v(m_4)||v(n)|^3}{|m_3 - k||m_4 - k|}.
\]

\(^{3}\)These identities can be derived from the following ones: \( \Re(H) = 0, \partial_v H(v, \tilde{v}) + \partial_{\tilde{v}} H(v, \tilde{v}) = 0 \) and \( \partial_{\tilde{v}} H(v, \tilde{v}) = \frac{\partial H}{\partial \tilde{v}(v, \tilde{v})} \).
We should estimate the \( \ell^1 \) and \( \ell^\infty \) norms of each summand in the variable \( k \). Since the estimates are similar, we present it only for the first term:

\[
\left\| \sum_{n_1+m_2-m_4=k} \frac{|v(m_2)||v(m_4)||v(n)|^3}{|k-m_2||m_4-m_2|} \right\|_{\ell^p} \lesssim \left\| v \right\|_{\ell^\infty}^3 \left\| \sum_{n_1+m_2-m_4=k} \frac{|v(m_2)||v(m_4)|}{|k-m_2||m_4-m_2|} \right\|_{\ell^p}
\]

\[
= \left\| v \right\|_{\ell^\infty}^3 \left( \left\| |v| \right\| \ast |v| \right) * \rho \left\| \ell^p \right. \\
\lesssim \left\| v \right\|_{\ell^\infty}^3 \left( \left\| |v| \right\| \ast \rho \right) \left\| \ell^p \right.
\]

\[
\lesssim \left\| v \right\|_{\ell^\infty}^3 \left( \left\| |v| \right\| \ast \rho \right) \left\| \ell^p \right.
\]

by Lemma 3.1. The same estimate holds for the remaining three terms. This is stronger than the required bound.

The bounds for \( \frac{\partial}{\partial \vartheta(k)} g^j_k(H_4) \) will be obtained inductively. We start with the case \( j = 1 \).

4.3. **Estimate of** \( \frac{\partial}{\partial \vartheta(k)} g^1_k(H_4) \). The commutator is given by

\[
g^1_k(H_4) = \sum_j \left( \frac{\partial H_4}{\partial \vartheta(j)} \frac{\partial F}{\partial \vartheta(j)} - \frac{\partial H_4}{\partial \vartheta(j)} \frac{\partial F}{\partial \vartheta(j)} \right).
\]

Let \( f(j) \) denote \( \frac{\partial F}{\partial \vartheta(j)} \). Note that \( g^1_k(H_4) \) is a sum of terms of the form

\[
H_4(v_1, v_2, v_3, v_4) = \sum_{n_1-n_2+n_3-n_4=0} v_1(n_1)v_2(n_2)v_3(n_3)v_4(n_4)
\]

where one of \( v_i \)'s is \( \tilde{f} \) or \( \tilde{f} \) and the others are \( v \) or \( \tilde{v} \).

To estimate \( \left\| \frac{\partial}{\partial \vartheta(k)} g^1_k(H_4) \right\|_{\ell^p} \), we use duality:

\[
\left( 21 \right) \left\| \frac{\partial}{\partial \vartheta(k)} g^1_k(H_4) \right\|_{\ell^p} \leq \sup_{\left\| h \right\|_{\ell^p} = 1} \sum_k \left\| \frac{\partial}{\partial \vartheta(k)} g^1_k(H_4) \right\|_{\ell^p} \left\| h(k) \right\|.
\]

Note that the sum in the right hand side of (21) is bounded by the sum of the following two terms

\[
H_4(\left| f(\left| v \right|, \left| v \right|, \left| v \right|, \left| v \right|) \right|, \left| h \right|, \left| v \right|, \left| v \right|), \quad H_4(\left| f(\left| h \right|, \left| v \right|, \left| v \right|) \right|, \left| v \right|, \left| v \right|, \left| v \right|)
\]

and similar terms obtained by permuting the arguments. The following lemma will be used repeatedly to estimate these terms and the ones appearing in the higher order commutators.

**Lemma 4.1.** I) For any \( q \in [1, \infty] \) and any permutation \( (i_1, i_2, i_3, i_4) \) of \( (1, 2, 3, 4) \), we have

\[
\left| H_4(v_1, v_2, v_3, v_4) \right| \leq \left\| v_{i_1} \right\|_{\ell^q} \left\| v_{i_2} \right\|_{\ell^q} \left\| v_{i_3} \right\|_{\ell^1} \left\| v_{i_4} \right\|_{\ell^1}.
\]

II) For any \( q \in [1, \infty] \) and any permutation \( (i_1, i_2, i_3) \) of \( (1, 2, 3) \), we have

\[
\left\| f(v_1, v_2, v_3) \right\|_{\ell^q} \leq \left\| v_{i_1} \right\|_{\ell^q} \left\| v_{i_2} \right\|_{\ell^q} \left\| v_{i_3} \right\|_{\ell^\infty} \leq \left\| v_{i_3} \right\|_{\ell^\infty}.
\]
Proof. I) Note that for any permutation we can write
\[
H_4(v_1, v_2, v_3, v_4) = \sum_j v_{i_1}(j) v_{i_2} * v_{i_3} * v_{i_4}(j).
\]
The statement follows from Hölder’s and Young’s inequalities.
II) This is easy to verify following the proof of Proposition 2.1 with \( \delta = 0. \)

Using Lemma 4.1, we obtain
\[
H_4(\|f(|v|, |v|), |h|, |v|, |v|) \lesssim \|h\|_{\ell^p} \|v\|_{\ell^p} \|v\|_1 \|f(|v|, |v|, |v|)\|_{\ell^1}^2
\lesssim \|h\|_{\ell^p} \|v\|_{\ell^p} \|v\|_2 \|v\|_{\ell^2}^{2 - \epsilon}
\lesssim \|h\|_{\ell^p} \varepsilon^{\frac{1}{2} - \frac{1}{p}}.
\]
Similarly, we have
\[
H_4(\|f(|h|, |v|), |v|, |v|, |v|) \lesssim \|f(|h|, |v|, |v|)\|_{\ell^p} \|v\|_{\ell^p} \|v\|_2 \|v\|_{\ell^2}^2
\lesssim \|h\|_{\ell^p} \|v\|_{\ell^p} \|v\|_2 \|v\|_{\ell^2} \lesssim \|h\|_{\ell^p} \varepsilon^{\frac{1}{2} - \frac{1}{p}}.
\]
Similar bounds follow for the terms obtained by permuting the arguments. Therefore, we have
\[
\left\| \frac{\partial}{\partial \beta(\ell^p) g_{\beta}(H_4)} \right\|_{\ell^p} \lesssim \varepsilon^{\frac{1}{2} - \frac{1}{p}}.
\]
Note that this gives an error of order 1 when \( p = 2. \) This explains why we consider higher order commutators (see footnote 2).

This proof motivates the following generalization:

**Lemma 4.2.** Consider \( H_4(|v|, |v|, |v|, |v|). \) Repeatedly \( (N \times \text{times, say}) \) replace one of the \( v \)'s with \( f(|v|, |v|, |v|). \) Finally, replace one of the \( v \)'s with \( h. \) We denote any such function by \( H_{4,N}(f, h, v). \) Then, for any \( p \in [1, \infty], \) we have
\[
|H_{4,N}(f, h, v)| \lesssim \|h\|_{\ell^p} \|v\|_{\ell^p} \|v\|_2 \|v\|_{\ell^2} \|v\|_{\ell^2}^{2N - \frac{1}{2} - \frac{1}{2}} \lesssim \|h\|_{\ell^p} \varepsilon^{N - \frac{1}{2} - \frac{1}{2}}.
\]

**Proof.** First by using part II of Lemma 4.1 repeatedly, we see that any composition of \( f \)'s satisfy
\[
\|f(\ast, \ast, \ast)\|_{\ell^2} \lesssim \|v\|_{\ell^p} \|v\|_{\ell^p}^{2n - \frac{1}{2} - \frac{1}{2}},
\]
where \( n \) is the number of \( f \)'s appearing in the composition.

Now, note that \( H_4 \) has four arguments. Let \( N_j \) be the number of \( f \)'s appearing in the \( j \)th argument. Only one of the arguments contains \( h, \) say the first one. Using part I of Lemma 4.1 estimate
\[
|H_4(v_1, v_2, v_3, v_4)| \lesssim \|v_1\|_{\ell^p} \|v_2\|_{\ell^p} \|v_3\|_1 \|v_4\|_1.
\]
Using (22), we estimate
\begin{equation}
\|v_2\|_{L^1} \leq \|v\|_{L^1} \lesssim \|v\|_{L^1} \|v\|_{L^\infty}^{2(N_2 + N_3 + N_4)}
\end{equation}

Next, note that $v_1$ is either $|h|$ (in which case we stop) or $|f(v_1, v_2, v_1)|$. In the latter case, without loss of generality, $v_{1,1}$ contains $|h|$. We estimate, using (22) and a simple induction,
\begin{equation}
\|v_1\|_{L^p} \lesssim \|v_{1,1}\|_{L^p} \|v_{1,2}\|_{L^\infty} \|v_{1,3}\|_{L^\infty} \lesssim \|h\|_{L^p} \|v\|_{L^\infty}^{2N_1}.
\end{equation}

Combining (23) and (24) yields the claim of the lemma. □

4.4. **Estimate of $\frac{\partial}{\partial \overline{v}(k)} g^N_F(H_4)$ for $N \geq 2$.** As above we will estimate $\frac{\partial}{\partial \overline{v}(k)} g^N_F(H_4)$ by duality:
\begin{equation}
\left\| \frac{\partial}{\partial \overline{v}(k)} g^N_F(H_4) \right\|_{L^p} \leq \sup_{\|h\|_{L^p} = 1} \sum_k \left| \frac{\partial}{\partial \overline{v}(k)} g^N_F(H_4) \right| \|h(k)\|.
\end{equation}

By following the calculation in the previous section, this sum can be bounded by a finite sum of functions $H_4, N(f, h, v)$. Therefore, Lemma 4.2 implies that
\begin{equation}
\left\| \frac{\partial}{\partial \overline{v}(k)} g^N_F(H_4) \right\|_{L^p} \lesssim \varepsilon^{N-\frac{1}{2} - \frac{1}{p}} \lesssim \varepsilon^{\frac{3}{2} - \frac{1}{p}}.
\end{equation}

4.5. **Remainder Estimates.** To estimate the remainder term,
\begin{equation}
\frac{\partial}{\partial \overline{v}(k)} \int_0^1 (1 - s)g^3_F(A_2 + H_4)(X^s_F(v)) ds,
\end{equation}

it is sufficient to find a bound on
\begin{equation}
\sup_{s \in [0,1]} \left\| \frac{\partial}{\partial \overline{v}(k)} g^3_F(A_2 + H_4)(X^s_F(v)) \right\|_{L^p}.
\end{equation}

Furthermore, since $g_F(A_2) = -H_4^{\text{nr}}$, we only need to find bounds on
\begin{equation}
\sup_{s \in [0,1]} \left\| \frac{\partial}{\partial \overline{v}(k)} g^3_F(H_4^{\text{nr}})(X^s_F(v)) \right\|_{L^p} \quad \text{and} \quad \sup_{s \in [0,1]} \left\| \frac{\partial}{\partial \overline{v}(k)} g^3_F(H_4)(X^s_F(v)) \right\|_{L^p}.
\end{equation}

Since we will have to estimate the composite function derivative, we first study the bound on the derivatives of $X^s_F(v)$, more precisely, let
\begin{equation}
w(m) = [X^s_F(v)](m),
\end{equation}

which is the solution of
\begin{equation}
\frac{dw(m)}{ds} = \sum_{m_1, m_2} \frac{w(m_1) w(m_2) w(m_1 + m_2 - m)}{(m_1 - m)(m_2 - m)}.
\end{equation}
Differentiating this equation with respect to initial condition \( w(n)|_{s=0} = v(n) \) and using the notation \( D_n \), we obtain

\[
\frac{d}{ds} D_n w(m) = \sum_{m_1, m_2} D_n w(m_1) w(m_2) \bar{w}(m_1 + m_2 - m) \frac{1}{(m_1 - m)(m_2 - m)} \\
+ \sum_{m_1, m_2} w(m_1) D_n w(m_2) \bar{w}(m_1 + m_2 - m) \frac{1}{(m_1 - m)(m_2 - m)} \\
+ \sum_{m_1, m_2} w(m_1) w(m_2) D_n \bar{w}(m_1 + m_2 - m) \frac{1}{(m_1 - m)(m_2 - m)}.
\]

We have a similar formula for \( \frac{d}{ds} D_n \bar{w}(m) \). Note that at \( s = 0 \), we have

\[
\left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} = \left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} = 1.
\]

We will prove that both of these norms remain bounded for \( s \in [0, 1] \). Taking the \( \ell^\infty_m \ell^1_n \) of (25), we obtain

\[
\left\| \frac{d}{ds} D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \leq \sup_m \sum_{m_1, m_2} \frac{|D_n w(m_1)||w(m_2)||w(m_1 + m_2 - m)|}{|m_1 - m||m_2 - m|} + \ldots
\]

\[
\leq \left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \sup_m \sum_{m_1, m_2} \frac{|w(m_2)||w(m_1 + m_2 - m)|}{|m_1 - m||m_2 - m|}
\]

\[
\leq \left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \left\| ((w * \rho) * \rho \right\|_{\ell^\infty} \leq \left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \varepsilon^{1-}.
\]

The remaining two terms give similar estimates which imply that

\[
\sup_{0 \leq s \leq 1} \left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \lesssim 1.
\]

Similarly, we obtain

\[
\sup_{0 \leq s \leq 1} \left\| D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \lesssim 1.
\]

**Remark 4.1.** For \( \delta > 0 \), a similar argument implies

\[
\left\| e^{\delta|m-n|} D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \lesssim 1, \quad \left\| e^{\delta|m-n|} D_n w(m) \right\|_{\ell^\infty_m \ell^1_n} \lesssim 1.
\]

and similarly for \( D_n \bar{w}(m) \). The rest of the argument follows as in other sections.

4.5.1. *Estimation of \( \partial_n v \bar{g}_0^2 \left( H_1 \right)(X^2_n(v)) \).* We estimate this expression rather than the one containing \( H_1^{ar} \) (as we should have) because it simplifies the
notation and still implies the estimate for the required expression. Note that
\[
\left\| \frac{\partial}{\partial \vartheta(k)} g^2_k(H_4)(X^k_F(v)) \right\|_{\ell^4_k} \leq \left\| \sum_j \frac{\partial g^2_k(H_4)}{\partial \vartheta(j)} \frac{\partial \vartheta(j)}{\partial \vartheta(k)} \right\|_{\ell^4_k} + \left\| \sum_j \frac{\partial g^2_k(H_4)}{\partial \vartheta(j)} \frac{\partial \vartheta(j)}{\partial \vartheta(k)} \right\|_{\ell^4_k} \leq \left\| \frac{\partial g^2_k(H_4)}{\partial \vartheta(j)} \right\|_{\ell^4_k} \| D_k \|_{\ell^\infty_k \ell^4_k} + \left\| \frac{\partial g^2_k(H_4)}{\partial \vartheta(j)} \right\|_{\ell^4_k} \| D_k \|_{\ell^\infty_k \ell^4_k} \leq \varepsilon^{\frac{1}{2}}.
\]
by (26) and the estimates we obtained in Subsection 4.4. Similarly, using (27), we obtain
\[
\left\| \frac{\partial}{\partial \vartheta(k)} g^3_k(H_4)(X^k_F(v)) \right\|_{\ell^\infty_k} \leq \varepsilon^{\frac{3}{2}}.
\]

4.5.2. Estimation of $\frac{\partial}{\partial \vartheta(k)} g^3_k(H_4)(X^k_F(v))$. As above, we only need to prove that
\[
\left\| \frac{\partial g^3_k(H_4)}{\partial \vartheta(k)} \right\|_{\ell^p} \leq \varepsilon^{\frac{3}{2}} - \frac{1}{p},
\]
for $p = 1$ and $p = \infty$. This was done in Subsection 4.4.

Appendix A. Nonlinear fiber optics application

One of the most important applications of NLS concerns light-wave communication systems, where optical pulses in a retarded time frame evolve according to the one dimensional NLS
\[
(28) \quad iA_{\varphi} + Sd(z) A_{\tau \tau} + g(z) |A|^2 A = 0.
\]
Here $z$ is the rescaled distance, $\tau$ is the rescaled retarded time, $A$ is the amplitude of the optical wave envelope, $d(z)$ is the group velocity dispersion, which is usually piecewise constant, and $S$ is the dispersion strength parameter. Finally, $g(z) > 0$ is the nonlinear coefficient which accounts for the losses and amplifications. For the derivation of NLS from Maxwell’s equations, one can consult many references, e.g. [9]. It is a standard assumption that $d(z)$ and $g(z)$ are periodic.

In general, in light-wave communication, the information is transmitted with localized pulses (with Gaussian or exponential tails) in allocated time slots. The presence of pulse corresponds to “1” and the absence of pulse corresponds to “0” in binary format. Naturally, it is preferable that the incoming waveform would appear undistorted at the end of the transmission line. It can be achieved by optimizing an individual pulse, so it would propagate without distortion, and sending such pulses together, keeping them sufficiently far apart (i.e. taking time slots sufficiently large), so they would not interact. Such regime is usually called “soliton regime” in the optical communication literature, where the word “soliton” does not usually mean that the equation is integrable. The pulses could be, for example, dispersion managed solitons, which are approximately periodic localized solutions of
the above equation. In other words, the main feature of the soliton regime is that the pulses do not interact (or rather pulse to pulse interaction is weak compared to the pulse self-interaction) during the propagation through the transmission line.

An alternative regime (often called the quasi-linear regime) has been found when the pulses strongly overlap during the transmission, see e.g. the survey paper [7]. Surprisingly, it was observed that up to a linear transformation of the transmitted waveform, the pulses appeared undistorted. Note that even though the pulses spread over many time slots, the average optical energy ($L^2$ norm square) per bit does not change and therefore nonlinear effects remain strong. It is usually implicitly assumed in the engineering literature that “nonlinearity gets averaged out” due to the high frequency of the initial data.

In this article, we rigorously explain the quasi-linear phenomenon for a model problem when $d(z)$ and $g(z)$ are constant and all bits are occupied by 1’s, in the limit of vanishing pulse width. This case (of all identical 1’s) leads to the formulation with periodic boundary conditions. Although, this is a special case, we hope that our proof can be extended to the more general case: pseudo-random sequence of 1’s and 0’s. Note that constant $d(z)$ and $g(z)$ assumption is not restrictive since if the evolution is quasi-linear on each interval where $d(z), g(z)$ are constant, then the evolution is quasi-linear on their union.

There has been previous work on quasi-linear regime. In [15], the limit of the short pulse width for dispersion managed NLS on the real line is considered. An effective evolution equation was derived which turned out to be integrable and weakly nonlinear. The equation was later improved in [1]. On the real line the energy disperses to infinity and therefore nonlinearity becomes small. This leaves an open question: what will happen if the energy does not disperse to infinity or in other words, there is an infinite bit stream. The problem considered in this paper models this situation: nonlinearity remains strong which is due to the periodic boundary conditions.

Finally, we note that on $\mathbb{R}$, the dispersion strength $S$ and pulse width $\varepsilon$ can be combined into a single effective parameter $S/\varepsilon^2$ by scaling $\tau$. This implies that the limits $S \to \infty$ and $\varepsilon \to 0$ are equivalent. This is not the case in our model since the characteristic $\tau$-scale, bit size, is already present. Therefore, the two parameter problem in $S, \varepsilon$ should be considered. However, the limit $S \to \infty$ is insufficient to achieve quasi-linear evolution and must be supplemented with $\varepsilon \to 0$. On the other hand, the limit $\varepsilon \to 0$ does produce quasi-linear evolution with $S$ being fixed but arbitrary. This motivated us to consider only this case. We put $S = 1$ in order not to obscure the exposition.

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