I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

G. H. Hardy, commenting on Ramanujan's first letter, written on January 16, 1913

SHOW ALL WORK. INDICATE ALL REASONING.

SCORES

1.__________      7.__________
2.__________      8.__________
3.__________      9.__________
4.__________      10.__________
5.__________      11.__________
6.__________      12.__________

Total__________

Grading Scale

205-250 A
187-204 A-
179-186 B+
153-178 B
141-152 B-
135-140 C+
110-134 C

99-109 C-
87-98 D+
60-86 D
50-59 D-
0-49 E
1. (15) Let Log $z$ denote the principal branch of the logarithm.
   a. What is wrong with the following argument?
      \[
      -\frac{3\pi i}{2} = 2 \cdot \frac{3\pi i}{4} = 2 \log (e^{-\frac{3\pi i}{4}}) = \log \left(e^{-\frac{3\pi i}{4}}\right)^2 = \log e^{-\frac{3\pi i}{2}} = -\frac{i\pi}{2}.
      \]
   b. Find the image of the unit disc $|z| < 1$ under the mapping $w = \log z$.

2. (10) Evaluate (10)
      \[
      \int_C |z|dz,
      \]
   where $C$ is the semi-circle of radius 1 in the left half-plane beginning at $z = i$ and ending at $z = -i$.

3. (10) Evaluate
      \[
      \frac{1}{2\pi i} \int_{|z-1|=2} \frac{\sin z}{(z-1)^5} dz.
      \]
(Infinitesimal) analysis (i.e., calculus) is the most powerful weapon of thought yet devised by the wit of man.

W. B. Smith

4. (30) Evaluate, in two distinct ways,

\[ \int_{|z|=2} \frac{z \, dz}{z^3 + 1} \]

a. First use the residue theorem.

b. Second, integrate over a larger circle, say \(|z| = R\).

5. (10) If \( f(z) \) is analytic for \(|z| \leq \frac{3}{2}\), evaluate

\[ \int_{|z|=\frac{3}{2}} \frac{f(z) \, dz}{(z-1)(z-2) \cdots (z-n)} \]

where \( n \) is an arbitrary positive integer.
O'er Nature's laws God cast the veil of night,
Out-blazed a Newton's soul—and all was light.   

Aaron Hill

6. (30) Find the Laurent expansion about \( z_0 = 2 \) for each of the following functions. In each case, indicate where the Laurent expansion converges. Furthermore, in each case, indicate the type of singularity for \( z = 2 \), and if \( z = 2 \) is a pole, give its order.

a. \( \frac{1}{z - 2} \)

b. \( \frac{1}{(z - 2)^{1/3}} \)

c. \( \cos \left( \frac{3}{z - 2} \right) \)

d. \( \frac{\sin(z - 2)}{z - 2} \)
7. (30)
a. Prove that
\[
\sin z = \sin x \cosh y + i \cos x \sinh y.
\]
b. Using part a., show that \( w = \sin z \) is an entire function.
c. Determine where \( w = \sin z \) is conformal.
d. What is the image of a horizontal line \( y = c \neq 0 \) under the mapping \( w = \sin z \).
e. What is the image of the real axis \( y = 0 \) under the mapping \( w = \sin z \).
f. What is the image of the strip \( 0 \leq y \leq \pi \) under the mapping \( w = \sin z \).
8. (25) Let

\[ w = \frac{i - z}{i + z}. \]

a. Find the image of the real axis under the transformation \( w \).
b. If \( z \in \mathcal{H} = \{ z : \text{Im } z > 0 \} \), where does the image \( w \) lie?
c. Find the inverse transformation of \( w \).
d. Using part c., show that if \( |w| < 1 \), then \( z \in \mathcal{H} \).
e. Conclude that \( w \) maps \( \mathcal{H} \) onto the unit disc \( |w| < 1 \). (If you wish, you may also use a theorem that we proved in class.)
Battalions of figures are like battalions of men, not always as strong as is supposed.  

M. Sage

9. (20) Use the residue theorem to show that

\[ \int_{0}^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} = \frac{2\pi}{3}. \]

10. (20) How many zeros does the polynomial

\[ P(z) = z^6 - 4iz^4 - iz^2 + 5 \]

have in the upper half-plane?
Ah, why, ye Gods, should two and two make four. Alexander Pope, The Dunciad

11. (30) Use the residue theorem to prove that, for $a > 0$,

$$
\int_0^\infty \frac{\sin x \, dx}{x(x^2 + a^2)} = \frac{\pi (1 - e^{-a})}{2a^2}.
$$
Mathematical analysis is... the true rational basis of the whole system of our positive knowledge.

A. Comte

12. (20) Recall a portion of Schwarz's Lemma. Let $f(z)$ be analytic on $|z| < R$, and suppose that $|f(z)| \leq M$ on $|z| \leq R$. Assume that $f(0) = 0$. Then $|f(z)| \leq M|z|/R$ for $|z| \leq R$.

   a. Using Schwarz's Lemma, prove the following theorem. Let $f(z)$ be analytic on $|z| < R$, and suppose that $|f(z)| \leq M$ on $|z| \leq R$. Prove that $|f(z) - f(0)| \leq 2M|z|/R$ for $|z| \leq R$.

   b. State Liouville's Theorem.

   c. Use part a. to give a proof of Liouville's Theorem.
1. a. Every step is correct except

\[ 2 \log(e^{-3ni\pi}) = \log(e^{-3ni\pi})^2, \]

because \( 2 \log z = \log z^2 \) (mod \( 2\pi i \)) generally, i.e., \( i \) can be replaced by \( -i \) only in certain instances.

b. \( \log z = \log |z| + i \text{Arg} z, -\pi \leq \text{Arg} z < \pi. \)

\[ = u + iv \]

For \( |z| < 1, u \) goes from 0 to \( -\infty \) as \( |z| \to 0 \). Also, \( -\pi < v < \pi \). Thus we get

\[ \pi \]

\[ 0 \]

\[ -\pi \]

2. Let \( z = e^{i\theta}, \pi/2 < \theta < 3\pi/2 \). Thus, \( \Delta z = i e^{i\theta}d\theta \)

\[ \int_{C} \frac{z}{z^3 - 1} dz = i \int_{\pi/2}^{3\pi/2} e^{i\theta} d\theta = e^{i\pi} \bigg|_{\pi/2}^{3\pi/2} = -i - i = -2i \]

3. Use Cauchy's formula for the \( n \)-th derivative. Let \( f(z) = \sin z \) and \( n = 4 \). Thus,

\[ \frac{1}{2\pi i} \int_{C} \frac{\sin z}{(z - \frac{1}{2})^5} dz = \left( \frac{\sin z}{(z - \frac{1}{2})^4} \right) \bigg|_{z = 1} = \frac{\sin 1}{4!} \]

4. Simple poles where \( z^3 = -1 \) or \( z_k = e^{(\pi i + 2nk\pi)/3}, k = 0, 1, 2 \)

\[ \text{Res}(z_k) = \frac{z_k}{3z_k^2} \bigg|_{z = z_k} = \frac{1}{3z_k^2} \]

\[ \text{Res}(z_0) + \text{Res}(z_1) + \text{Res}(z_2) = \frac{1}{3} \left( e^{\frac{\pi i}{3}} + e^{\frac{2\pi i}{3}} + e^{\frac{5\pi i}{3}} \right) = \frac{1}{3} \left( \frac{1}{2} + i\frac{\sqrt{3}}{2} - 1 + \frac{1}{2} - \frac{i\sqrt{3}}{2} \right) = 0 \]

By the residue theorem,

\[ \int_{C} \frac{z dz}{z^3 + 1} = 2\pi i \left( \text{Res}(z_0) + \text{Res}(z_1) + \text{Res}(z_2) \right) = 0 \]

Since there are no further singularities, by Cauchy's theorem,
the integral over \(1z1 = 2\) is equal to the integral over \(1z1 = \) for any \(R > 2\). Now
\[
\left| \int_{1z1 = R} \frac{z^2 dz}{z^3 + 1} \right| \leq \int_{1z1 = R} \frac{1dz}{1z1^2 - 1} = \frac{R \cdot 2\pi R}{R^3 - 1} \rightarrow 0 \quad \text{as} \quad R \to \infty.
\]

5. \(z = 1\) is the only pole on the interior of \(1z1 = 3/2\).
\[
\text{Res}_{z = 1} = \lim_{z \to 1} \frac{(z-1)f(z)}{(z-1)(z-2)(z-3) \cdots (z-n)} = \frac{f(1)}{(-1)(-2) \cdots (1-n)}
\]
\[
\int_{|z|=13/2} \frac{f(z)}{(z-1)(z-2) \cdots (z-n)} = 2\pi i \left( \frac{f(1)}{(-1)(-2) \cdots (1-n)} \right) = \frac{\left(\frac{-1}{2}\right)^{n-1} f(1)}{(n-1)!}
\]

6. a. \(\frac{1}{z-2}\) is a Laurent exp. about \(z = 2\). It converges for all \(z \neq 2\). 2 is a pole of order 1.

b. \(\frac{1}{z} = \frac{1}{z(z-2)} = \frac{1}{2z(1+z-2)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n} \left(\frac{z-2}{2}\right)^{n}
\]
\[
\frac{1}{(z-2)^{13/2}} = \sum_{n=0}^{\infty} (-1)^{n} \left(\frac{z-2}{2}\right)^{n-13} \quad \text{is a pole of order 13. The series converges for} \quad 0 < \left|z-2\right| < 2.
\]

c. \(\cos \left(\frac{3}{z-2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} \left(\frac{3}{z-2}\right)^{2n} \quad \text{2 is an essential isolated singularity. The series converges for} \quad 0 < \left|z-2\right| < 20.
\]
d. \(\sin \left(\frac{z-2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} \left(\frac{z-2}{2}\right)^{2n+1} \quad \text{2 is a removable singularity. The convergence is} \quad 0 \leq \left|z-2\right| < 20.
\]

7a. \(\sin (x + iy) = \frac{1}{2i} \left( e^{i(x+iy)} - e^{-i(x+iy)} \right) = \frac{1}{2i} \left( e^{ix-y} - e^{-ix+y} \right) = \frac{1}{2i} \left( e^{-y} \cos x + i \sin x - e^{y} \cos x - i \sin x \right) = \frac{1}{2i} \left( e^{-y} - e^{y} \right) \cos x + \frac{1}{2} \left( e^{-y} + e^{y} \right) \sin x = \left( \sin y \right) \cos x + \left( \cos y \right) \sin x
\)
b. Use the Cauchy-Riemann equations: \( u = \sin x \cosh y, \)
\( v = \cos x \sinh y. \)
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \sin x \cosh y, & \frac{\partial v}{\partial y} &= \cos x \cosh y; & \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= \cos x \sinh y, & \frac{\partial v}{\partial x} &= -\sin x \sinh y; & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{align*}
\]
Thus, the Cauchy-Riemann eqns. hold, if \( f \) is continuous.

Hence, \( f(z) \) is analytic everywhere.

c. \( f'(z) = \cosh z \neq 0 \) if \( z \neq (2n+1)\frac{\pi}{2}, \ n \in \mathbb{Z}. \)

d. If \( y = 0, \) \( u = \sin x \cosh 0, \) \( v = \cos x \sinh 0 \)
\( I = \sin^2 x + \cos^2 x = (\frac{u}{\cosh c})^2 + (\frac{v}{\sinh c})^2 \) is an ellipse.

e. If \( y = 0, \) \( u = \sin x, \) \( v = 0. \) Thus, the image is \([-1, 1]\)
on the real axis.

f. By d.e., we get all points on the closed region bounded by the ellipse along with \( y = \pi. \)

8. a. \( |W(z)| = \left| \frac{z - x}{\bar{z} + x} \right|. \) We get the circle \( |W| = 1. \)

b. If \( z \in \mathbb{C}, \) \( \left| \frac{z - \bar{z}}{\bar{z} + z} \right| < 1 \) (distance from \( z \) to \( i \) less than
the distance from \( z \) to \( -i. \)

c. \( W(\bar{z} + z) = \bar{z} - z, \) or \( z(1 + W) = i(\bar{z} + w) \) or \( z = \frac{i(1 - w)}{w + 1} \)

d. \( z = \frac{i(1 - w)(1 + \bar{w})}{|w + 1|^2} = \frac{i(1 - |w|^2)}{|w + 1|^2} = \frac{i(|w|^2) - 2w}{|w + 1|^2} \)
Since \( |w| < 1, \) we see \( \lim z \to 0, \) i.e. \( z \in \mathbb{C}. \)

e. Thus, every pt. in \( |w| < 1 \) is the image of a pt. in \( \mathbb{H}, \) thus, \( W \) maps \( \mathbb{H} \) onto \( |w| < 1. \)
9. Let \( z = e^{i\theta} \). So \( dz = i e^{i\theta} d\theta \), or \( d\theta = \frac{dz}{iz} \). Also,
\[
\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}, \text{ thus,}
\]
\[
\int_{0}^{2\pi} \frac{dz}{5 + 4\sin \theta} = \frac{1}{i} \int_{0}^{2\pi} \frac{dz}{z^2 - 2iz(z - \frac{1}{z})} = \frac{1}{2i} \int_{0}^{2\pi} \frac{dz}{z^2 + 5z + 2i}.
\]
\[
-2iz^2 + 5z + 2i = 0 \implies z = -\frac{5 \pm \sqrt{25 - 16}}{-4i} = -\frac{5 \pm 3}{-4i}.
\]
The pole corresponding to the + sign lies inside \( |z| = 1 \), the other lies outside \( |z| = 1 \). Thus, \( z = -i/2 \) lies inside.
\[
R_{-i/2} = \frac{4}{-4iz + 5} \bigg|_{z = -i/2} = \frac{4}{-2 + 5} = \frac{1}{3}
\]

Hence,
\[
\int_{0}^{2\pi} \frac{1}{5 + 4\sin \theta} = \frac{1}{i} 2\pi i \cdot \frac{1}{3} = \frac{2\pi}{3}
\]

10. Let \( z = x \) real. \( P(x) = x^6 - 4ix^4 + x^2 + 5 = x^6 + 5 - i(4x^4 + x^2) \)

Note that \( x^6 + 5 = 0 \) and \( 4x^4 + x^2 = 0 \) have no real zeros.

Consider \( \tan^{-1} \frac{-4ix^4 + x^2}{x^6 + 5} \). Since the imaginary part of \( P(x) \)
is always negative as \( x \) moves from \(-\infty \) to \( \infty \), since \( \arg = \frac{d\theta}{dx} \) \( x = \infty \)
\[\Delta P(x) \bigg|_{-\infty, \infty} = 0. \text{ Now,}
\]
\[
P(R e^{i\theta}) = Re^{i\theta} - 4iR^4 e^{4i\theta} - iRe^{i\theta} + 5
\]
\[
= R e^{i\theta} (1 - 4iR^{-2} e^{-2i\theta} - iR^{-4} e^{-4i\theta} + 5R^{-6} e^{-6i\theta}),
\]
we see that \( \Delta P(z) \bigg|_{z \in C_R} = 6\pi + o(1) \) as \( R \to \infty \).

Thus, \( \Delta P(z) \bigg|_{C} \to 6\pi \) as \( R \to \infty \). By the argument principle,
\( P(z) \) has \( \frac{1}{2\pi} \cdot 6\pi = 3 \) zeros in \( \mathbb{C} \).

11. Since the integrand is even,
\[
\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + a^2)} \ dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + a^2)} \ dx.
\]
\[
\int_{C} \frac{e^{iz}}{z(z^2+a^2)} \, dz = 2\pi i \frac{2i}{a^2} = -\frac{2\pi}{a^2} \quad \text{(by residue theorem)}
\]

On \( C_0 \), let \( z = \varepsilon e^{i\theta}, 0 \leq \varepsilon \leq \pi \). Thus, as \( \varepsilon \to 0 \),
\[
\int_{C_0} \frac{e^{iz}}{z(z^2+a^2)} \, dz = i\varepsilon \int_{C_0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta} (\varepsilon^2 e^{2i\theta} + a^2)} \, d\theta \to i\int_{0}^{\pi} \frac{d\theta}{\varepsilon a^2} = -\frac{\pi}{a^2}
\]

Thus, letting \( \varepsilon \to 0 \) and \( R \to \infty \), we have
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+a^2)} \, dx + \int_{0}^{\frac{\pi}{2}} \frac{e^{ix}}{x(x^2+a^2)} \, dx = \int_{-\infty}^{\infty} \frac{e^{-ix}}{x(x^2+a^2)} \, dx + \int_{0}^{\frac{\pi}{2}} \frac{e^{ix}}{x(x^2+a^2)} \, dx = 2\int_{0}^{\infty} \frac{\sin x}{x(x^2+a^2)} \, dx
\]
\[
= -\frac{\pi}{a^2} i + \frac{\pi}{a^2}
\]

or
\[
\int_{0}^{\infty} \frac{\sin x}{x(x^2+a^2)} \, dx = \frac{\pi (1-e^{-a})}{2a^2}
\]

12a. Apply Schwarz's Lemma to \( g(z) = f(z) - f(0) \). Note that \( g(0) = 0 \). By the maximum modulus theorem,
\[
|g(z)| \leq |f(z)| + |f(0)| \leq M + M = 2M.
\]
Thus, by Schwarz's Lemma, \( |g(z)| \leq 2M |z| / R \).

b. Every bounded entire function is a constant.

c. Let \( z_0 \in C \). Suppose \( f(z) \) is a bounded entire function, say \( |f(z)| \leq M, z \in C \). By part a,
\[
|f(z_0) - f(0)| \leq 2M |z_0| / R \to 0 \quad \text{as} \ R \to \infty.
\]
Thus, \( f(z_0) - f(0) = 0 \), i.e., for all \( z_0, f(z_0) = f(0) \), i.e., \( f(z) \) is a constant.