MATH 448, COMPLEX ANALYSIS
PROBLEM SET NO. 3
DUE MARCH 18, 2015

1. Let \( f(z) \) be analytic on \(|z| \leq 1\). Using polar coordinates, show that
\[
\frac{1}{\pi} \int_{|z|\leq 1} f(x+iy) \, dx \, dy = f(0).
\]

2. The \( n \)th Bell number \( B(n) \), \( n \geq 0 \), is defined by
\[
e^z - 1 = \sum_{n=0}^{\infty} \frac{B(n)}{n!} z^n, \quad |z| < \infty. \tag{0.1}
\]
The Bell numbers appear prominently in combinatorics. For example, \( B(n) \) is the number of ways of subdividing a set of \( n \) elements into subsets. For example, \( B(3) = 5 \). If \( a, b, \) and \( c \) are the three elements in our set of 3 elements, the subsets are:
\[
\{a, b, c\}; \quad \{a, b\}, \{c\}; \quad \{a, c\}, \{b\}; \quad \{b, c\}, \{a\}; \quad \{a\}, \{b\}, \{c\}.
\]
Use (0.1) to show that \( B(0) = 1, B(1) = 1, B(2) = 2, \) and \( B(3) = 5 \).

3. Recall that \( f(z) \) has period \( p \) if \( f(z+p) = f(z) \) for every complex number \( z \). Let \( f(z) \) be an entire function with periods 1 and \( i \). Prove that \( f(z) \) is a constant.

4. Let \( f(z) \) be analytic on the domain \(|z| < R\), except at the point \( z = a \), where it has a simple pole with residue \( \kappa \). Prove that
\[
f(z) = \sum_{n=0}^{\infty} \left( \frac{f^{(n)}(0)}{n!} + \frac{\kappa}{a^{n+1}} \right) z^n + \kappa \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}.
\]

5. The Bernoulli numbers \( B_n \), \( 0 \leq n < \infty \), are defined by
\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi. \tag{0.2}
\]
Using (0.2), prove that
\[
(a) \ \cot z = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n} z^{2n}}{(2n)!} z^{2n-1}, \quad |z| < \pi,
\]
\[
(b) \ \log \left( \frac{z}{e^z - 1} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{n n!} z^n, \quad |z| < 2\pi.
\]

6. Find the Laurent expansion of each of the following functions in neighborhoods of the given points:
\[\begin{align*}
(a) \ &z^2 \sin \left( \frac{1}{z-1} \right), \quad z_0 = 1, \\
(b) \ &\frac{1}{z^2 - 1}.
\end{align*}\]
(b) \( \cos \left( \frac{z^2 - 4z}{(z - 2)^2} \right), \quad z_0 = 2 \),

c) \( \log \left( \frac{z - a}{z - b} \right), \quad z_0 = \infty \).

The next two problems are for extra credit.

7. Let \( f(z) \) be analytic on a simply connected domain \( D \). Let \( C \) be a simple closed contour in \( D \). Let \( z_k \in I(C), 1 \leq k \leq n, \) with \( z_i \neq z_j, \) if \( i \neq j \). Let \( g_n(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \). Prove that

\[
P(z) = \frac{1}{2\pi i} \int_C \frac{f(w) g_n(w) - g_n(z)}{g_n(w)} dw
\]

is a polynomial of degree \( n - 1 \) that coincides with \( f(z) \) at the points \( z_1, z_2, \ldots, z_n \).

(The polynomial \( P(z) \) is called Lagrange’s interpolation polynomial.)

8. Let \( C \) be a simple closed contour, and suppose that \( f(z) \) is analytic on \( C \) and \( E(C) \). Suppose that \( \lim_{z \to \infty} f(z) = A \). Prove Cauchy’s integral formula for an infinite domain, i.e.,

\[
\frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw = \begin{cases} A - f(z), & \text{if } z \in E(C), \\ A, & \text{if } z \in I(C). \end{cases}
\]

Hint: Suppose first that \( z \in E(C) \). Consider a circle \( C_z \) centered at \( z \) such that \( C_z \) and \( I(C_z) \) are contained in \( E(C) \). Now consider a large circle \( C_R \) centered at the origin and such that both \( C \) and \( C_z \) are contained on \( I(C_R) \). By drawing lines connecting curves, make an appropriate application of Cauchy’s theorem and Cauchy’s integral formula.