Let \( w = \frac{az+b}{cz+d} \). Since \((1+i)(1-i)=2\), let us simplify by replacing \( a \) by \((1-i)a\) and \( b \) by \((1-i)b\). Thus, we want
\[
\text{Re}(1+i)w = \text{Re}(1+i)(1-i)\frac{az+b}{cz+d} = \text{Re}2\frac{az+b}{cz+d} = 0 \quad \text{when} \quad z = e^{i\theta}.
\]
Now
\[
\frac{e^{i\theta}+1}{-e^{i\theta}+1} = \frac{(e^{i\theta}+1)(e^{-i\theta}+1)}{|1-e^{i\theta}|^2} = \frac{e^{i\theta}-e^{-i\theta}}{|1-e^{i\theta}|^2} = \frac{2i\sin\theta}{|1-e^{i\theta}|^2}
\]
which is purely imaginary. Hence, we can take
\[
w = \frac{z+1}{-z+1}
\]
or any real multiple thereof.

We are moving the circle \( |z|=1 \) one unit to the right. Thus, \( w = z+1 \).

We are rotating the real axis by \( 90^\circ \) in the counterclockwise direction and then adding \( \frac{1}{2} \) to the points. Hence, \( w = iz + \frac{1}{2} \).

We want to shift the center \( z_0 \) to the origin. Thus, we want \( w - 0 = z - z_0 \). But we also want to dilate (or magnify) the radius from \( z \) to \( z_1 \). Thus, we want
\[
w = \frac{z-z_0}{z_1}
\]
Recall that \( w = \beta \frac{z-x}{z-x} \), \( |\beta| = 1 \), \( \Im w > 0 \) takes \( z \) onto \( 1 \).

We want a circle of radius \( 4 \). Thus, take \( \beta = 4 \).

Let \( w = 4 \frac{z-x}{z-x} \). We do not want the real axis to map to \( 1 \).

Instead, we want \( \text{Re} z = \frac{1}{2} \) to map to \( 1 \). Thus, let \( z = iu + \frac{1}{2} \) (rotate real axis to the imaginary axis and add \( \frac{1}{2} \)). Also let \( x = i/2 \). Thus, as \( w = \frac{z-1/2}{i/2} \)
\[ W = 4 \frac{2 - \frac{1}{2}}{\frac{2}{2} + \frac{1}{2}} = 4 \frac{-\frac{1}{2}}{-\frac{1}{2} + 1} = \frac{4 \cdot 2}{2 - 1} \]

Lastly, we want to move the center to \( 4i \). So,

\[ W = \frac{4 \cdot 2}{2 - 1} + 4i \]

is the desired transformation.

The text gives the transformation \( W = \frac{4z + 4}{z} \), which also has the desired mapping property.

We showed in class that the general LFT mapping \( |z| < 1 \) onto \( |W| < 1 \) has the form

\[ W = \beta \frac{z - \alpha}{z^2 - 1}, \quad \beta \neq 0, \beta > 0, \text{ or } W = \beta z \quad \text{if} \quad \alpha = 0, \]

Thus it is identical to what is requested with \( \lambda = -\beta \) and \( \delta = \alpha \).

Since \( f(z) \) is real and nonzero on \( C \), either \( f(z) > 0 \) or \( f(z) < 0 \)

on \( C \), since \( f(z) \) is continuous on \( C \). In other words, either \( \arg f(z) = 0 \) or \( \arg f(z) = \pi \) on \( C \). Thus \( \Delta f(z)|_C = 0 \). By the principle of the argument, since \( f(z) \) is analytic, \( f(z) \) has no zeros on \( \mathbb{I}(C) \). Let \( c \) be an arbitrary nonreal complex number. As \( z \) traverses \( C \), \( \arg f(z) - c \) always stays in the upper half plane if \( \text{Im} \ c < 0 \) or always stays in the lower half plane if \( \text{Im} \ c > 0 \). Thus \( \Delta f(z) - c \) \( \leq \pi \). But since \( C \) is closed and \( |\Delta f(z) - c| = 2n\pi \) for some integer \( n \), we must have \( n = 0 \). Thus, by the principle of the argument, \( f(z) - c \) has no zeros on \( \mathbb{I}(C) \), i.e., \( f(z) \) never assumes the value \( c \) on \( \mathbb{I}(C) \), for \( c \) is arbitrary with \( \text{Im} \ c \neq 0 \). Thus \( f(z) \) is real on \( \mathbb{I}(C) \). By an easy application of the Cauchy-Kovalevskaya equations \( f(z) \) is constant on \( \mathbb{C} \setminus \mathbb{I}(C) \). By the identity theorem, \( f(z) \) is constant on \( D \).
Let \( g(z) = f(z)/z^2 \). As \( f(0) = f'(0) = 0 \), \( g(z) \) is analytic on \( |z| < 1 \). Consider \( |z| \leq r < 1 \).

\[
|g(z)| = |f(z)/z^2| \leq M/r^2.
\]

By the maximum modulus theorem, on \( |z| = r \),

\[
|g(z)| \leq M/r^2.
\]

or

\[
|f(z)| \leq M|z|^2/r^2. \quad (\ast)
\]

But \((\ast)\) is valid for every \( r < 1 \). Thus, \((\ast)\) holds for \( r = 1 \), i.e.,

\[
|f(z)| \leq M|z|^2 \quad \text{for} \quad |z| < 1.
\]

b. Clearly, \( \Phi(z) \) has a double zero at \( z = 0 \).

As \( \text{Re}(f(z)) < 1/4 \) on \( |z| < 1 \), \( 1 - 2f(z) \neq 0 \) on \( |z| < 1 \). Hence, \( \Phi(z) \) is analytic on \( |z| < 1 \) and satisfies the conditions of part a. We need to determine \( M \).

To do this, let \( f(z) = u + iv \), with \( u, v \) real. Then

\[
|\Phi(z)|^2 = \frac{u+iv}{1-2u-2iv} \frac{u-iv}{1-2u+2iv} = \frac{u^2+v^2}{(1-2u)^2+4v^2}
\]

\[
= \frac{\frac{u^2+v^2}{4[(\frac{1}{2}-u)^2+v^2]}}{4[\frac{1}{4}+v^2]} = \frac{1}{4}.
\]

Thus, \( |\Phi(z)| < 1/2 \) for \( |z| < 1 \). By part a,

\[
|\Phi(z)| \leq \frac{1}{2} |z|^2, \quad |z| < 1.
\]

Now \( \Phi - 2\Phi' = f \) or \( f = \frac{\Phi}{1+2\Phi} \). So,

\[
|f(z)| = \frac{|\Phi(z)|}{|1+2\Phi(z)|} \leq \frac{|\Phi(z)|}{1-|2\Phi(z)|} \leq \frac{\frac{1}{2} |z|^2}{1-|z|^2} = \frac{|z|^2}{2(1-|z|^2)}
\]

for \( |z| < 1 \).
5. Let $p > 0$. Let $z_0$ be fixed but arbitrary with $|z_0| > p$. Let $C$ be the circle $|z - z_0| = r_0$, where $r_0$ is chosen so that

$$\exists z : |z| \leq p \cap \exists z : |z - z_0| \leq r_0 \implies \emptyset.$$

Let $\epsilon > 0$ be given, but choose $\epsilon$ also such that

$$\epsilon < \inf_{|z| = p} |e^{\frac{1}{2}z}|. \quad (1)$$

As $f_n(z) \to e^{\frac{1}{2}z}$ uniformly for $|z| > p$, $\exists N$ such that

$$|f_n(z) - e^{\frac{1}{2}z}| < \epsilon \quad \text{for} \quad |z| > p. \quad (2)$$

Now apply Rouché’s theorem to $f_n(z)$ and $e^{\frac{1}{2}z}$ on $|z - z_0| = r_0$. Note that, in particular, (2) holds for $|z - z_0| = r_0$, and so from (1) and (2), on $|z - z_0| = r_0$,

$$|f_n(z) - e^{\frac{1}{2}z}| < |e^{\frac{1}{2}z}|.$$

By Rouché’s theorem, $f_n(z)$ and $e^{\frac{1}{2}z}$ have the same no. of zeros on $|z - z_0| < r_0$. However, $e^{\frac{1}{2}z}$ has no zeros on $|z - z_0| < r_0$. Thus, $f_n(z)$ has no zeros on $|z - z_0| < r_0$. Since $z_0$ is arbitrary and (1) and (2) hold for any $z$, $|z| > p$, the zeros of $f_n(z)$ must lie on $|z| = p$.

Note on (1): Let $W = \frac{1}{2}z$. Then $e^W$ is bounded from above and from below on the compact set $|W| \leq \frac{p}{4}$. Thus, we can choose $\epsilon > 0$ so that (1) holds uniformly on $|z| > p$. 