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AN IDENTITY FOR THE DEDEKIND ETA-FUNCTION INVOLVING TWO INDEPENDENT COMPLEX VARIABLES

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Abstract

The authors prove a new identity for the Dedekind eta-function that involves third powers of the eta-function, with each of the two cubes being a function of a different complex variable.

1. *Introduction*

Recall that the Dedekind eta-function $\eta(\tau)$ is defined for $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H} = \{\tau : Im\ \tau > 0\}$ by

$$\eta(\tau) = q^{1/24}(q;q)_\infty,$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

The purpose of this paper is to prove the following striking identity for the eta-function, of which we know no other examples of a similar type.

Theorem 1.1. *For $w, z \in \mathcal{H}$,*

$$27\eta^3(3w)\eta^3(3z) = \eta^3\Big(\frac{w}{3}\Big)\eta^3\Big(\frac{z}{3}\Big) + i\eta^3\Big(\frac{w+1}{3}\Big)\eta^3\Big(\frac{z+1}{3}\Big) - \eta^3\Big(\frac{w+2}{3}\Big)\eta^3\Big(\frac{z+2}{3}\Big). \quad (1.1)$$

We describe now the genesis of (1.1). In preparing his doctoral thesis [**2**], the second author searched for modular equations involving

$$u_1(\tau) := \frac{\eta(\tau/m)}{\eta(\tau)} \qquad \text{and} \qquad v_1(\tau) := u_1(n\tau), \quad (1.2)$$

(and various modular transforms thereof). His goal was to generalize the modular equations of 'irrational kind' for the Weber functions

$$\mathfrak{f}(\tau) := e^{-\pi i/24}\frac{\eta\left((\tau+1)/2\right)}{\eta(\tau)}, \qquad \mathfrak{f}_1(\tau) := \frac{\eta\left(\tau/2\right)}{\eta(\tau)}, \qquad \mathfrak{f}_2(\tau) := \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)},$$

discussed in Section 75 of Weber's book [**3**]; that is, the case $m = 2$ in (1.2). For example, if $n = 3$, letting

$$u(\tau) := \mathfrak{f}(\tau), \qquad u_1(\tau) := \mathfrak{f}_1(\tau), \qquad u_2(\tau) := \mathfrak{f}_2(\tau),$$

and

$$v(\tau) = \mathfrak{f}(3\tau), \qquad v_1(\tau) := \mathfrak{f}_1(3\tau), \qquad v_2(\tau) := \mathfrak{f}_2(3\tau),$$

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one can prove the identity

$$u^2v^2 = u_1^2v_1^2 + u_2^2v_2^2, \qquad \tau \in \mathcal{H}.$$

Generally, Weber's modular equations depend on n, and increase in complexity as n increases.

In attempting to generalize these modular equations, the second author began with an appropriately normalized set of transforms (under modular substitutions) of $u_3(\tau) := \eta(\tau/3)/\eta(\tau)$. However, he eventually realized that the modular equations obtained for these 'generalized Weber functions' did not appear to vary as n increased. Moreover, the single identity that he found was completely general in that the second parameter $n\tau$ was not related to τ in any way; that is, the equation held for two completely independent complex variables. Simplification then gave the identity for the eta-function given in Theorem 1.1 above. The identity was then verified in many cases to tens of thousands of decimal places.

2. *Proof of Theorem* 1.1

Let $q = e^{2\pi i w}$, $Q = e^{2\pi i z}$, and $\rho = e^{2\pi i/3}$. Then (1.1) is equivalent to the identity

$$\begin{aligned}27q^{3/8}Q^{3/8}(q^3;q^3)_\infty^3(Q^3;Q^3)_\infty^3 &= q^{1/24}Q^{1/24}(q^{1/3};q^{1/3})_\infty^3(Q^{1/3};Q^{1/3})_\infty^3\\ &\quad+i\rho^{1/4}q^{1/24}Q^{1/24}(\rho q^{1/3};\rho q^{1/3})_\infty^3(\rho Q^{1/3};\rho Q^{1/3})_\infty^3\\ &\quad-i\rho^{-1/4}q^{1/24}Q^{1/24}(\rho^{-1}q^{1/3};\rho^{-1}q^{1/3})_\infty^3(\rho^{-1}Q^{1/3};\rho^{-1}Q^{1/3})_\infty^3\end{aligned}$$

or

$$\begin{aligned}27q^{1/3}Q^{1/3}(q^3;q^3)_\infty^3(Q^3;Q^3)_\infty^3 &= (q^{1/3};q^{1/3})_\infty^3(Q^{1/3};Q^{1/3})_\infty^3\\ &\quad+i\rho^{1/4}(\rho q^{1/3};\rho q^{1/3})_\infty^3(\rho Q^{1/3};\rho Q^{1/3})_\infty^3\\ &\quad-i\rho^{-1/4}(\rho^{-1}q^{1/3};\rho^{-1}q^{1/3})_\infty^3(\rho^{-1}Q^{1/3};\rho^{-1}Q^{1/3})_\infty^3. \qquad (2.1)\end{aligned}$$

To prove (2.1), we use Jacobi's identity [**1**, p. 285]

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \qquad (2.2)$$

Observe that

$$\rho^{n(n+1)/2} = \begin{cases}1, & \text{if } n \equiv 0, 2 \pmod 3,\\ \rho, & \text{if } n \equiv 1 \pmod 3.\end{cases}$$

Hence, by (2.2),

$$\begin{aligned}(\rho q^{1/3};\rho q^{1/3})_\infty^3 &= \sum_{n=0}^{\infty}(-1)^n(2n+1)\rho^{n(n+1)/2}q^{n(n+1)/6}\\ &= \sum_{\substack{n=0\\ n\equiv 0,2 \ (\mathrm{mod}\ 3)}}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6} + \rho\sum_{\substack{n=0\\ n\equiv 1 \ (\mathrm{mod}\ 3)}}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6}\\ &= (q^{1/3};q^{1/3})_\infty^3 + (\rho-1)\sum_{n=0}^{\infty}(-1)^{3n+1}(6n+3)q^{(3n+1)(3n+2)/6}\\ &= (q^{1/3};q^{1/3})_\infty^3 - 3(\rho-1)q^{1/3}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{3n(n+1)/2}\\ &= (q^{1/3};q^{1/3})_\infty^3 - 3(\rho-1)q^{1/3}(q^3;q^3)_\infty^3, \qquad (2.3)\end{aligned}$$

where we used (2.2) twice again. For brevity, set

$$A := (q^{1/3};q^{1/3})_\infty^3, \qquad B := (Q^{1/3};Q^{1/3})_\infty^3, \qquad C := q^{1/3}(q^3;q^3)_\infty^3, \qquad D := Q^{1/3}(Q^3;Q^3)_\infty^3.$$

Using the notation above, (2.3), its analogue with ρ replaced by ρ^{-1}, and their analogues, with q replaced by Q, in (2.1), we find that it suffices to prove that

$$\begin{aligned}27CD = AB &+ i\rho^{1/4}\left(A-3(\rho-1)C\right)\left(B-3(\rho-1)D\right)\\ &- i\rho^{-1/4}\left(A-3(\rho^{-1}-1)C\right)\left(B-3(\rho^{-1}-1)D\right).\end{aligned}\tag{2.4}$$

Observe that $\rho^{1/4}=(\sqrt{3}+i)/2$. Thus, the coefficient of AB on the right-hand side of (2.4) is equal to

$$1+i\rho^{1/4}-i\rho^{-1/4}=1+i\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)-i\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=0.\tag{2.5}$$

Next, the coefficients of AD and BC on the right-hand side of (2.4) are each equal to

$$\begin{aligned}&-3i\rho^{1/4}(\rho-1)+3i\rho^{-1/4}(\rho^{-1}-1)\\ &=-3i\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\left(\frac{-3+i\sqrt{3}}{2}\right)+3i\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\left(\frac{-3-i\sqrt{3}}{2}\right)\\ &=-\frac{3i}{4}(-\sqrt{3})+\frac{3i}{4}(-\sqrt{3})\\ &=0.\end{aligned}\tag{2.6}$$

The coefficient of CD on the right-hand side of (2.4) is equal to

$$\begin{aligned}&9i\rho^{1/4}(\rho-1)^2-9i\rho^{-1/4}(\rho^{-1}-1)^2\\ &=9i\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\left(\frac{-3+i\sqrt{3}}{2}\right)^2-9i\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\left(\frac{-3-i\sqrt{3}}{2}\right)^2\\ &=9i\left(\frac{6\sqrt{3}-6i}{4}-\frac{6\sqrt{3}+6i}{4}\right)\\ &=9i(-3i)\\ &=27.\end{aligned}\tag{2.7}$$

Hence, using the calculations (2.5)–(2.7) in (2.4), we see that (2.4) indeed has been shown, and so this completes the proof.

References

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