



Ramanujan's Eisenstein series and new hypergeometric-like series for $1/\pi^2$

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Abstract

Using hypergeometric identities and certain representations for Eisenstein series, we uniformly derive several new series representations for $1/\pi^2$.

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1. Introduction

In his famous paper [37,38, pp. 36–38], Ramanujan recorded 17 hypergeometric-like series representations for $1/\pi$. Proofs of the first three series representations were briefly sketched by Ramanujan [38, p. 36]. These three series belong to the classical theory of elliptic functions, while the latter fourteen series depend on Ramanujan's alternative theories of elliptic functions. In 1928, Chowla [25,26, pp. 87–91] gave the first published proof of a general series representation for $1/\pi$ and used it to derive the first of Ramanujan's series for $1/\pi$ [37, Eq. (28)]. It was not until 1987 that proofs of all 17 formulas were found by Borwein and Borwein [15]. These authors subsequently discovered many further series for $1/\pi$ [16–20], where [20] is coauthored with Bailey. Chudnovsky and Chudnovsky [27] independently proved several of Ramanujan's series

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representations for $1/\pi$ and established new ones as well. Further particular series representations for $1/\pi$ as well as some general formulas were subsequently derived by Berndt and Chan [11], Berndt et al. [12], Chan et al. [21], Chan and Liaw [22], Chan et al. [23], and Chan and Verrill [24]. In a recently communicated paper, the authors [3] employed Ramanujan's ideas expressed in Section 13 of his fundamental paper [37,38, p. 36] and used them in conjunction with 12 identities for Eisenstein series recorded without proofs by Ramanujan in Section 10 of [37,38, pp. 33–34] and with further identities of this type to prove 13 of Ramanujan's 17 identities from [37] as well as to establish many new series representations for $1/\pi$. In recent years, Guillera [28–33] discovered some beautiful series for $1/\pi$ as well as for $1/\pi^2$. He proved some of his series with the help of the WZ-method, and “experimentally” discovered several other series for $1/\pi^2$. For example, using the WZ-method, he proved that

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} (20k^2 + 8k + 1) \left(-\frac{1}{4}\right)^k,$$

$$\frac{128}{\pi^2} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^5}{k!^5} (820k^2 + 180k + 13) \left(-\frac{1}{1024}\right)^k.$$

Using a hypergeometric series transformation arising from the quadratic transformation $z \mapsto -4z/(1-z)^2$ and the two series of Guillera above, Zudilin [42] proved the two new series

$$\frac{10\sqrt{5}}{\pi^2} = \sum_{k=0}^{\infty} U_k \frac{(4k)!}{k!^2(2k)!} \frac{18k^2 - 10k - 3}{(2^8 5^2)^k},$$

$$\frac{5^4 41 \sqrt{41}}{\pi^2} = \sum_{k=0}^{\infty} U_k \frac{(4k)!}{k!^2(2k)!} \frac{1046529k^2 + 227104k + 16032}{(5^4 41^2)^k},$$

where the sequence of integers

$$U_k = \sum_{n=0}^k \binom{2n}{n}^3 \binom{2k-2n}{k-n} 2^{4(k-n)}, \quad k = 0, 1, 2, \dots,$$

satisfies the recurrence relation

$$(k+1)^3 U_{k+1} - 8(2k+1)(8k^2 + 8k + 5)U_k + 4096k^3 U_{k-1} = 0, \quad n = 1, 2, \dots$$

Also see [41–44] for further work accomplished by Zudilin.

In this paper, we again follow Ramanujan's lead from [37] in that we crucially use Eisenstein series, in contrast to most other authors. We employ certain representations for Eisenstein series, identities for hypergeometric series, and the values of singular moduli, to derive many new series for $1/\pi^2$.

2. Preliminary definitions and results

We use the standard shifted or rising factorial notation

$$(a)_0 := 1, \quad (a)_n := a(a+1)(a+2) \cdots (a+n-1), \quad n \geq 1.$$

The hypergeometric functions ${}_pF_{p-1}$, $p \geq 1$, are defined by

$${}_pF_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1}; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \frac{x^n}{n!}, \quad |x| < 1.$$

If

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right),$$

then one of the fundamental results in the theory of elliptic functions [7, p. 101, Entry 6] is given by

$$\phi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right), \quad (2.1)$$

where here, and for the sequel,

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q| < 1. \quad (2.2)$$

We also need Ramanujan's function

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}, \quad (2.3)$$

where the latter identity is Euler's pentagonal number theorem, which is easily derived from Jacobi's triple product identity. Following Ramanujan, define

$$z := z(q) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \phi^2(q), \quad (2.4)$$

where we used (2.1). In the sequel, we often emphasize that x is also a function of q when writing $x = x(q)$.

We end this section by defining a modular equation as understood by Ramanujan. Suppose that the equality

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-k^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\ell^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \ell^2\right)} \quad (2.5)$$

holds for some positive integer n . Then a modular equation of degree n is a relation between the moduli k and ℓ that is implied by (2.5). Ramanujan recorded his modular equations in terms of α and β , where $\alpha = k^2$ and $\beta = \ell^2$. We say that β has degree n over α . The corresponding multiplier m is defined by

$$m := m(\alpha) := \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \ell^2\right)}. \quad (2.6)$$

3. The development of our ideas

Ramanujan’s series representations for $1/\pi$ depend upon Clausen’s product formulas for hypergeometric series and Ramanujan’s Eisenstein series

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}, \quad |q| < 1. \tag{3.1}$$

In our paper [3,4], by combining two different relations between $P(q^2)$ and $P(q^{2n})$, for certain positive integers n , along with a Clausen formula, we obtained series representations for $1/\pi$. Here our main idea is to combine two different representations for $P(q^2)P(q^{2n})$, (3.38) and another one from (3.47) and (3.49), with $q = e^{-\pi/\sqrt{n}}$ for certain positive integers n , along with hypergeometric series identities (3.7)–(3.13). Then we obtain our series for $1/\pi^2$ by appealing to various singular moduli. In the remainder of this section, we explain our method.

By collecting coefficients of x^k , we arrive at the following lemma.

Lemma 3.1. For $|x| < 1$,

$${}_3F_2^2(\alpha, \beta, \gamma; 1, 1; x) = \sum_{k=0}^{\infty} D_k x^k, \tag{3.2}$$

where

$$D_k = \sum_{n=0}^k \frac{(\alpha)_n (\beta)_n (\gamma)_n (\alpha)_{k-n} (\beta)_{k-n} (\gamma)_{k-n}}{(n!)^3 ((k-n)!)^3}.$$

Noting that, for $k \geq n$,

$$\frac{(-k)_n (a)_k}{(1-k-a)_n k!} = \frac{(a)_{k-n}}{(k-n)!},$$

we can also write D_k as

$$\begin{aligned} D_k &= \sum_{n=0}^k \frac{(\alpha)_n (\beta)_n (\gamma)_n (\alpha)_{k-n} (\beta)_{k-n} (\gamma)_{k-n}}{(n!)^3 ((k-n)!)^3} \\ &= \frac{(\alpha)_k (\beta)_k (\gamma)_k}{(k!)^3} \sum_{n=0}^k \frac{(\alpha)_n (\beta)_n (\gamma)_n (-k)_n^3}{(1-k-\alpha)_n (1-k-\beta)_n (1-k-\gamma)_n (n!)^3} \\ &= \frac{(\alpha)_k (\beta)_k (\gamma)_k}{(k!)^3} {}_6F_5(\alpha, \beta, \gamma, -k, -k, -k; 1-k-\alpha, 1-k-\beta, 1-k-\gamma, 1, 1; 1). \end{aligned}$$

We also note that, when $\alpha = \frac{1}{6}$, $\beta = \frac{1}{2}$, and $\gamma = \frac{5}{6}$, Zudilin [42, Eq. (10)] expressed (3.2) in the form

$${}_3F_2^2\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; x\right) = \sum_{k=0}^{\infty} u_k \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{k!^2} x^k, \tag{3.3}$$

where

$$u_k = \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}\right)_{k-n}}{(n!)^3 (k-n)!}.$$

Now, let

$$A_k := \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}\right)_{k-n}^3}{(n!)^3 ((k-n)!)^3}, \tag{3.4}$$

$$B_k := \sum_{n=0}^k \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{4}\right)_{k-n} \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{k-n} \left(\frac{3}{4}\right)_n \left(\frac{3}{4}\right)_{k-n}}{(n!)^3 ((k-n)!)^3}, \tag{3.5}$$

$$C_k := u_k \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{k!^2}, \tag{3.6}$$

and

$$X := 4x(1-x), \quad Y := \frac{4x}{(1-x)^2}, \quad U := \frac{x^2}{4(1-x)}, \quad V := \frac{16x(1-x)^2}{(1+x)^4},$$

$$W := \frac{4X}{(1-X)^2}, \quad L := \frac{27X^2}{(4-X)^3} \quad \text{and} \quad M := \frac{27X}{(1-4X)^3}.$$

If z is defined as in (2.4), then from Theorems 5.7(i)–(vi) and Formula (5.5.9) in [15, pp. 180–181], (3.2), and (3.3) above, we arrive at

$$z^4 = {}_3F_2^2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; X \right) = \sum_{k=0}^{\infty} A_k X^k, \quad 0 \leq x \leq \frac{1}{2}, \tag{3.7}$$

$$= \frac{1}{(1-x)^2} {}_3F_2^2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; -Y \right) = \frac{1}{(1-x)^2} \sum_{k=0}^{\infty} (-1)^k A_k Y^k, \tag{3.8}$$

$$0 \leq x \leq 3 - 2\sqrt{2},$$

$$= \frac{1}{1-x} {}_3F_2^2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; -U \right) = \frac{1}{1-x} \sum_{k=0}^{\infty} (-1)^k A_k U^k, \tag{3.9}$$

$$0 \leq x \leq 2\sqrt{2} - 2,$$

$$= \frac{1}{(1+x)^2} {}_3F_2^2 \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; V \right) = \frac{1}{(1+x)^2} \sum_{k=0}^{\infty} B_k V^k, \tag{3.10}$$

$$0 \leq x \leq 3 - 2\sqrt{2},$$

$$= \frac{1}{1-X} {}_3F_2^2 \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; -W \right) = \frac{1}{1-X} \sum_{k=0}^{\infty} (-1)^k B_k W^k, \tag{3.11}$$

$$0 \leq x \leq \frac{1}{2} \left(1 - 2^{1/4} \sqrt{2 - \sqrt{2}} \right),$$

$$= \frac{4}{4-X} {}_3F_2^2 \left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; L \right) = \frac{4}{4-X} \sum_{k=0}^{\infty} C_k L^k, \quad 0 \leq x \leq \frac{1}{2}, \tag{3.12}$$

$$= \frac{1}{1-4X} {}_3F_2 \left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1, 1; -M \right) = \frac{1}{1-4X} \sum_{k=0}^{\infty} (-1)^k C_k M^k, \\ 0 \leq x \leq \frac{1}{2}. \quad (3.13)$$

In the sequel we also need series representations for $z^3 \frac{dz}{dx}$ and $3z^2 \left(\frac{dz}{dx} \right)^2$. We derive these representations for the form (3.12).

We rewrite (3.12) in the form

$$(1-x+x^2)z^4 = \sum_{k=0}^{\infty} C_k L^k. \quad (3.14)$$

Differentiating (3.14) with respect to x , we find that

$$4(1-x+x^2)z^3 \frac{dz}{dx} = (1-2x)z^4 + \sum_{k=0}^{\infty} C_k k L^{k-1} \cdot \frac{4(8+X)(1-2x)}{X(4-X)} L, \quad (3.15)$$

where $X = 4x(1-x)$. From (3.12) and (3.15), we deduce that

$$z^3 \frac{dz}{dx} = \frac{4(1-2x)}{(4-X)^2} \sum_{k=0}^{\infty} C_k L^k + \frac{4(8+X)(1-2x)}{X(4-X)^2} \sum_{k=0}^{\infty} C_k k L^k. \quad (3.16)$$

Now,

$$\frac{d}{dx} \left(z^3 \frac{dz}{dx} \right) = 3z^2 \left(\frac{dz}{dx} \right)^2 + z^3 \frac{d^2z}{dx^2}. \quad (3.17)$$

But, z satisfies the hypergeometric differential equation

$$x(1-x) \frac{d^2z}{dx^2} + (1-2x) \frac{dz}{dx} - \frac{z}{4} = 0, \quad (3.18)$$

from which we easily arrive at

$$z^3 \frac{d^2z}{dx^2} = \frac{z^4}{X} - \frac{4(1-2x)}{X} z^3 \frac{dz}{dx}. \quad (3.19)$$

Employing (3.12) and (3.16) in (3.19), we find that

$$z^3 \frac{d^2z}{dx^2} = \frac{12}{(4-X)^2} \sum_{k=0}^{\infty} C_k L^k - \frac{16(8+X)(1-X)}{X^2(4-X)^2} \sum_{k=0}^{\infty} C_k k L^k. \quad (3.20)$$

Using (3.20) in (3.17), we obtain

$$\frac{d}{dx} \left(z^3 \frac{dz}{dx} \right) = 3z^2 \left(\frac{dz}{dx} \right)^2 + \frac{12}{(4-X)^2} \sum_{k=0}^{\infty} C_k L^k - \frac{16(8+X)(1-X)}{X^2(4-X)^2} \sum_{k=0}^{\infty} C_k k L^k. \quad (3.21)$$

Now, differentiating the right-hand side of (3.16), equating it with (3.21), and then simplifying, we deduce that

$$3z^2 \left(\frac{dz}{dx} \right)^2 = -\frac{12(4+X)}{(4-X)^3} \sum_{k=0}^{\infty} C_k L^k - \frac{24X(X^2+16X-8)}{X^2(4-X)^3} \sum_{k=0}^{\infty} C_k k L^k + \frac{16(8+X)^2(1-X)}{X^2(4-X)^3} \sum_{k=0}^{\infty} C_k k^2 L^k. \tag{3.22}$$

Next, recall the representation for $P(q^2)$ given in [7, p. 120, Entry 9(iv)], namely,

$$P(q^2) = (1-2\alpha)z^2 + 6\alpha(1-\alpha)z \frac{dz}{d\alpha}, \tag{3.23}$$

where $\alpha = x(q)$. If β has degree n over α , then

$$P(q^{2n}) = (1-2\beta)z^2(q^n) + 6\beta(1-\beta)z(q^n) \frac{dz(q^n)}{d\beta}, \tag{3.24}$$

where $\beta = x(q^n)$.

Our derivations of series for $1/\pi^2$ depend on two different representations for $P(q^2)P(q^{2n})$ with $q = e^{-\pi/\sqrt{n}}$. To this end, we first transform the expression for $P(q^2)$ into an expression involving β and $z(q^n)$.

If m is the multiplier connecting α and β , then, by (2.6),

$$z = mz(q^n). \tag{3.25}$$

Thus,

$$\frac{dz}{d\alpha} = \frac{dz}{d\beta} \cdot \frac{d\beta}{d\alpha} = \frac{d\beta}{d\alpha} \left\{ \frac{dm}{d\beta} z(q^n) + m \frac{dz(q^n)}{d\beta} \right\}. \tag{3.26}$$

From Entry 24(vi) in [7, p. 217], we recall that

$$\frac{d\beta}{d\alpha} = \frac{n\beta(1-\beta)}{m^2\alpha(1-\alpha)}. \tag{3.27}$$

Employing (3.27) in (3.26), we find that

$$\frac{dz}{d\alpha} = \frac{n\beta(1-\beta)}{m^2\alpha(1-\alpha)} z(q^n) \frac{dm}{d\beta} + \frac{n\beta(1-\beta)}{m\alpha(1-\alpha)} \frac{dz(q^n)}{d\beta}. \tag{3.28}$$

Invoking (3.28) and (3.25) in (3.23), we deduce that

$$P(q^2) = \left\{ (1-2\alpha)m^2 + \frac{6n\beta(1-\beta)}{m} \cdot \frac{dm}{d\beta} \right\} z^2(q^n) + 6n\beta(1-\beta)z(q^n) \frac{dz(q^n)}{d\beta} = Dz^2(q^n) + 6n\beta(1-\beta)z(q^n) \frac{dz(q^n)}{d\beta}, \tag{3.29}$$

where

$$D = (1-2\alpha)m^2 + \frac{6n\beta(1-\beta)}{m} \cdot \frac{dm}{d\beta}. \tag{3.30}$$

Multiplying (3.24) and (3.29), we arrive at

$$P(q^2)P(q^{2n}) = (1 - 2\beta)Dz^4(q^n) + 36n\beta^2(1 - \beta)^2z^2(q^n) \left(\frac{dz(q^n)}{d\beta} \right)^2 \\ + (n(1 - 2\beta) + D)6\beta(1 - \beta)z^3(q^n) \frac{dz(q^n)}{d\beta}. \quad (3.31)$$

Now, replacing q by q^n in (3.12), (3.16), and (3.22), and then using these expressions in (3.31), we find that

$$P(q^2)P(q^{2n}) = \sum_{k=0}^{\infty} \{A(x(q^n)) + B(x(q^n))k + C(x(q^n))k^2\} C_k L^{k+1}, \quad (3.32)$$

where

$$A(x(q^n)) = \frac{1}{27X^2} \{4D(1 - 2\beta)(4 - X)^2 + 6nX(1 - X)(4 - X) \\ + 6DX(4 - X)(1 - 2\beta) - 9nX^2(4 + X)\}, \quad (3.33)$$

$$B(x(q^n)) = \frac{2}{9X^2} \{3nX(8 - 16X - X^2) + (8 + X)(4 - X)\{n(1 - X) \\ + D(1 - 2\beta)\}\}, \quad (3.34)$$

$$C(x(q^n)) = \frac{4n(8 + X)^2(1 - X)}{9X^2}, \quad (3.35)$$

with $X = 4x(q^n)(1 - x(q^n)) = 4\beta(1 - \beta)$ and $L = 27X^2/(4 - X)^3$.

Now set

$$x_n := x(e^{-\pi\sqrt{n}}) \quad \text{and} \quad z_n := z(e^{-\pi\sqrt{n}}). \quad (3.36)$$

The numbers x_n are *singular moduli*. In his notebooks [39], Ramanujan calculated the values of many singular moduli, and in the sequel, we frequently appeal to Ramanujan's values, as recorded and proved in [9]. It also can be easily shown that [1, Chapter 15]

$$1 - x_n = x_{1/n}, \quad z_{1/n} = \sqrt{n}z_n \quad \text{and} \quad m(x_{1/n}) = \sqrt{n}. \quad (3.37)$$

Setting $q = e^{-\pi/\sqrt{n}}$ in (3.32), so that $\beta = x(e^{-\pi\sqrt{n}}) = x_n$, $\alpha = x(e^{-\pi/\sqrt{n}}) = 1 - x_n$, and $m = \sqrt{n}$, we deduce that

$$P(e^{-2\pi/\sqrt{n}})P(e^{-2\pi\sqrt{n}}) = \sum_{k=0}^{\infty} \{A(x_n) + B(x_n)k + C(x_n)k^2\} C_k L_n^{k+1}, \quad (3.38)$$

where

$$A(x_n) = \frac{1}{27X_n^2} \{4D_n(1 - 2x_n)(4 - X_n)^2 + 6nX_n(1 - X_n)(4 - X_n) \\ + 6D_nX_n(4 - X_n)(1 - 2x_n) - 9nX_n^2(4 + X_n)\}, \quad (3.39)$$

$$B(x_n) = \frac{2}{9X_n^2} \{3nX_n(8 - 16X_n - X_n^2) + (8 + X_n)(4 - X_n)\{n(1 - X_n) + D_n(1 - 2x_n)\}\}, \tag{3.40}$$

$$C(x_n) = \frac{4n(8 + X_n)^2(1 - X_n)}{9X_n^2}, \tag{3.41}$$

$X_n = 4x_n(1 - x_n)$, $L_n = 27X_n^2/(4 - X_n)^3$, and, from (3.30),

$$D_n = n(2x_n - 1) + \frac{3\sqrt{n}X_n}{2} \left[\frac{dm}{d\beta} \right]_{q=e^{-\pi/\sqrt{n}}}. \tag{3.42}$$

By similar arguments, series expressions analogous to (3.38) can also be obtained from (3.7)–(3.11) and (3.13).

We note that (3.38) gives one representation for $P(e^{-2\pi/\sqrt{n}})P(e^{-2\pi\sqrt{n}})$. To obtain another representation we rely on a transformation formula for $P(q)$ and Ramanujan’s representations for $f_n(q) := nP(q^{2n}) - P(q^2)$. At first, we determine a transformation formula for $P(q)$. Recall the transformation formula for Ramanujan’s function $f(-q)$ [7, p. 43, Entry 27(iii)]. If $a, b > 0$ with $ab = \pi^2$, then

$$e^{-a/12} a^{1/4} f(-e^{-2a}) = e^{-b/12} b^{1/4} f(-e^{-2b}). \tag{3.43}$$

Taking the logarithm of both sides of (3.43), we see that

$$-\frac{a}{12} + \frac{1}{4} \log a + \sum_{k=1}^{\infty} \log(1 - e^{-2ka}) = -\frac{b}{12} + \frac{1}{4} \log b + \sum_{k=1}^{\infty} \log(1 - e^{-2kb}). \tag{3.44}$$

Differentiating both sides of (3.44) with respect to a , we find that

$$-\frac{1}{12} + \frac{1}{4a} + \sum_{k=1}^{\infty} \frac{2ke^{-2ka}}{1 - e^{-2ka}} = \frac{b}{12a} - \frac{1}{4a} - \sum_{k=1}^{\infty} \frac{(2kb/a)e^{-2kb}}{1 - e^{-2kb}}. \tag{3.45}$$

Multiplying both sides of (3.45) by $12a$, rearranging, and then employing the definition of $P(q)$ from (3.1), we deduce that

$$6 - aP(e^{-2a}) = bP(e^{-2b}). \tag{3.46}$$

Setting $a = \pi/\sqrt{n}$, so that $b = \pi\sqrt{n}$, in (3.46), we arrive at

$$nP(e^{-2\pi\sqrt{n}}) + P(e^{-2\pi/\sqrt{n}}) = \frac{6\sqrt{n}}{\pi}. \tag{3.47}$$

If we set $n = 2$ in (3.47), we deduce that

$$P(e^{-2\pi}) = \frac{3}{\pi}, \tag{3.48}$$

which has been established many, many times in the literature, including by Ramanujan in Section 8 of Chapter 14 of his second notebook [39]. See [6, p. 256] for references to several proofs.

In his paper [37], Ramanujan recorded 12 representations for

$$f_n(q) := nP(q^{2n}) - P(q^2), \tag{3.49}$$

corresponding to 12 values of n , namely, $n = 2, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31$, and 35 . He also recorded the representations for $n = 2$ and 4 in Chapter 17 and for the remaining 10 values and for $n = 9$ and $n = 25$ in Chapter 21 of his second notebook [39]. Such a representation for $q = e^{-\pi/\sqrt{n}}$ combined with (3.47) gives a second representation for $P(e^{-2\pi/\sqrt{n}})P(e^{-2\pi\sqrt{n}})$. Equating this representation with that of (3.38) provides a series representation for $1/\pi^2$. In the next section, we demonstrate this with a series for $1/\pi^2$ corresponding to the case $n = 7$. In our last section, we record several other new series for $1/\pi^2$ obtained by this method.

4. Example: $n = 7$

Theorem 4.1. *If*

$$u_k = \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{k-n}}{n!^3 (k-n)!} \quad \text{and} \quad C_k = u_k \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{k!^2},$$

then

$$\frac{1728}{\pi^2} = \sum_{k=0}^{\infty} (154743372k^2 + 27915552k + 1677376)C_k \left(\frac{64}{614125}\right)^{k+1}. \quad (4.1)$$

Remark. The series above adds roughly five digits a term. In other words, for each succeeding term, we gain about five digits of $1728/\pi^2$.

Proof. Setting $n = 7$ in (3.38), we find that

$$P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}}) = \sum_{k=0}^{\infty} \{A(x_7) + B(x_7)k + C(x_7)k^2\}L_7^{k+1}, \quad (4.2)$$

where

$$A(x_7) = \frac{1}{27X_7^2} \{4D_7(1 - 2x_7)(4 - X_7)^2 + 42X_7(1 - X_7)(4 - X_7) + 6D_7X_7(4 - X_7)(1 - 2x_7) - 63X_7^2(4 + X_7)\}, \quad (4.3)$$

$$B(x_7) = \frac{2}{9X_7^2} \{21X_7(8 - 16X_7 - X_7^2) + (8 + X_7)(4 - X_7)\{7(1 - X_7) + D_7(1 - 2x_7)\}\}, \quad (4.4)$$

$$C(x_7) = \frac{28(8 + X_7)^2(1 - X_7)}{9X_7^2}, \quad (4.5)$$

$X_7 = 4x_7(1 - x_7)$, $L_7 = 27X_7^2/(4 - X_7)^3$, and

$$D_7 = 7(2x_7 - 1) + \frac{3\sqrt{7}X_7}{2} \left[\frac{dm}{d\beta} \right]_{q=e^{-\pi/\sqrt{7}}}. \quad (4.6)$$

To find the explicit values of $A(x_7)$, $B(x_7)$, and $C(x_7)$, we need to evaluate D_7 , and hence to evaluate $\left[\frac{dm}{d\beta}\right]_{q=e^{-\pi/\sqrt{7}}}$. To this end, we recall the following modular equation of degree 7 from Entry 19(v) in [7, p. 314]. If β has degree 7 over α , then

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}. \tag{4.7}$$

Differentiating (4.7) with respect to β , we find that

$$\begin{aligned} -\frac{98}{m^3} \frac{dm}{d\beta} &= \frac{1}{\beta} \left\{ -\frac{\sqrt{\alpha}}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2\sqrt{\alpha}} \frac{d\alpha}{d\beta} \right\} + \frac{1}{1-\beta} \left\{ \frac{\sqrt{1-\alpha}}{2\sqrt{1-\beta}} - \frac{\sqrt{1-\beta}}{2\sqrt{1-\alpha}} \frac{d\alpha}{d\beta} \right\} - \frac{1}{\beta(1-\beta)} \\ &\quad \times \left\{ -\frac{\sqrt{\alpha(1-\alpha)}}{2\sqrt{\beta(1-\beta)}}(1-2\beta) + \frac{\sqrt{\beta(1-\beta)}}{2\sqrt{\alpha(1-\alpha)}}(1-2\alpha) \frac{d\alpha}{d\beta} \right\} - \frac{8}{(\beta(1-\beta))^{2/3}} \\ &\quad \times \left\{ \frac{-(\alpha(1-\alpha))^{1/3}}{3(\beta(1-\beta))^{2/3}}(1-2\beta) + \frac{(\beta(1-\beta))^{1/3}}{3(\alpha(1-\alpha))^{2/3}}(1-2\alpha) \frac{d\alpha}{d\beta} \right\}. \end{aligned} \tag{4.8}$$

Setting $q = e^{-\pi/\sqrt{7}}$ in (4.8), so that, by (3.37), $\beta = x_7$, $\alpha = 1 - x_7$, and $m = \sqrt{7}$, and by (3.27), $\left[\frac{d\alpha}{d\beta}\right]_{q=e^{-\pi/\sqrt{7}}} = 1$, we arrive at

$$-2\sqrt{7} \left[\frac{dm}{d\beta}\right]_{q=e^{-\pi/\sqrt{7}}} = \frac{2x_7 - 1}{2(x_7(1-x_7))^{3/2}} + \frac{19(1-2x_7)}{3x_7(1-x_7)}. \tag{4.9}$$

Now, from [9, p. 290], we note that

$$x_7 = \frac{8 - 3\sqrt{7}}{16}, \tag{4.10}$$

so that

$$X_7 = 4x_7(1-x_7) = \frac{1}{64} \quad \text{and} \quad L_7 = \frac{27X_7^2}{(4-X_7)^3} = \frac{64}{614125}. \tag{4.11}$$

Employing (4.10) in (4.9), we deduce that

$$\left[\frac{dm}{d\beta}\right]_{q=e^{-\pi/\sqrt{7}}} = 80. \tag{4.12}$$

Using (4.10)–(4.12) in (4.6), we conclude that

$$D_7 = -\frac{3\sqrt{7}}{4}. \tag{4.13}$$

Invoking (4.10), (4.11), and (4.13) in (4.3)–(4.5), we arrive at

$$A(x_7) = -\frac{3589649}{192}, \quad B(x_7) = \frac{2326296}{16} \quad \text{and} \quad C(x_7) = \frac{12895281}{16}. \tag{4.14}$$

Using (4.14) and (4.11) in (4.2), we find that

$$\begin{aligned} &P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}}) \\ &= \sum_{k=0}^{\infty} \left\{ -\frac{3589649}{192} + \frac{2326296}{16}k + \frac{12895281}{16}k^2 \right\} C_k \left(\frac{64}{614125} \right)^{k+1}. \end{aligned} \tag{4.15}$$

Next, we find a second expression for $P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}})$. From either [37, 38, p. 33] or [7, p. 468, Entry 5(iii)], we know that

$$7P(q^{14}) - P(q^2) = 3z(q)z(q^7) \left(1 + \sqrt{x(q)x(q^7)} + \sqrt{(1-x(q))(1-x(q^7))} \right). \quad (4.16)$$

Setting $q = e^{-\pi/\sqrt{7}}$ in (4.16) and employing (3.37) with $n = 7$ and (4.10), we deduce that

$$7P(e^{-2\pi\sqrt{7}}) - P(e^{-2\pi/\sqrt{7}}) = 3\sqrt{7} \left(1 + 2\sqrt{x_7(1-x_7)} \right) = \frac{27\sqrt{7}}{8} z_7^2. \quad (4.17)$$

Squaring both sides of (4.17), we see that

$$49P^2(e^{-2\pi\sqrt{7}}) + P^2(e^{-2\pi/\sqrt{7}}) - 14P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}}) = \frac{5103}{64} z_7^4. \quad (4.18)$$

But, setting $q = e^{-\pi/\sqrt{7}}$ in (3.12) and then using (4.11), we find that

$$z_7^4 = \frac{4}{(4-X_7)} \sum_{k=0}^{\infty} C_k L_7^k. \quad (4.19)$$

Employing (4.19) and (4.11) in (4.18), we deduce that

$$\begin{aligned} & 49P^2(e^{-2\pi\sqrt{7}}) + P^2(e^{-2\pi/\sqrt{7}}) - 14P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}}) \\ &= \frac{5103}{64} \cdot \frac{4}{(4-X_7)L_7} \sum_{k=0}^{\infty} C_k L_7^{k+1} \\ &= \frac{12289725}{16} \sum_{k=0}^{\infty} C_k \left(\frac{64}{614125} \right)^{k+1}. \end{aligned} \quad (4.20)$$

Setting $n = 7$ in (3.47) yields

$$7P(e^{-2\pi\sqrt{7}}) + P(e^{-2\pi/\sqrt{7}}) = \frac{6\sqrt{7}}{\pi}. \quad (4.21)$$

Squaring both sides of (4.21), we obtain

$$49P^2(e^{-2\pi\sqrt{7}}) + P^2(e^{-2\pi/\sqrt{7}}) + 14P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}}) = \frac{252}{\pi^2}. \quad (4.22)$$

From (4.20) and (4.22), we deduce that

$$P(e^{-2\pi/\sqrt{7}})P(e^{-2\pi\sqrt{7}}) = \frac{9}{\pi^2} - \frac{1755675}{64} \sum_{k=0}^{\infty} C_k \left(\frac{64}{614125} \right)^{k+1}. \quad (4.23)$$

Now, equating (4.15) and (4.23), we conclude that

$$\begin{aligned} \frac{9}{\pi^2} &= \sum_{k=0}^{\infty} \left\{ \frac{1755675}{64} - \frac{3589649}{192} + \frac{2326296}{16}k + \frac{12895281}{16}k^2 \right\} C_k \left(\frac{64}{614125} \right)^{k+1} \\ &= \sum_{k=0}^{\infty} \left\{ \frac{1677376}{192} + \frac{2326296}{16}k + \frac{12895281}{16}k^2 \right\} C_k \left(\frac{64}{614125} \right)^{k+1}, \end{aligned} \quad (4.24)$$

which is equivalent to (4.1). Thus, we complete the proof. \square

5. Other series for $1/\pi^2$

We present here several new series for $1/\pi^2$. Because the proofs follow along the same lines as that in the previous section, we omit the detailed proofs. Also note that we have used several representations of $f_n(e^{-\pi/\sqrt{n}}) = nP(e^{-2\pi/\sqrt{n}}) - P(e^{-2\pi/\sqrt{n}})$ calculated in [15,1,3]. It is worthwhile to note that many more series can be derived in the same way. We have not listed certain other series that we think are not so elegant.

Theorem 5.1. *If*

$$u_k = \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}\right)_{k-n}}{n!^3(k-n)!} \quad \text{and} \quad C_k = u_k \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{k!^2},$$

then

$$\frac{81}{\pi^2} = \sum_{k=0}^{\infty} (784k^2 + 144k - 3)C_k \left(\frac{27}{125}\right)^{k+1}, \tag{5.1}$$

$$\frac{9}{\pi^2} = \sum_{k=0}^{\infty} (1089k^2 + 279k + 22)C_k \left(\frac{4}{125}\right)^{k+1}, \tag{5.2}$$

$$\frac{27}{\pi^2} = \sum_{k=0}^{\infty} (23814k^2 + 5598k + 430)C_k \left(\frac{8}{1331}\right)^{k+1}, \tag{5.3}$$

$$\begin{aligned} \frac{243}{\pi^2} = & \sum_{k=0}^{\infty} \{(590880 + 265200\sqrt{5})k^2 + (125010 + 56700\sqrt{5})k + 8685 + 4050\sqrt{5}\} \\ & \times C_k \left(\frac{27(884\sqrt{5} - 1975)}{33275}\right)^{k+1}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \frac{9}{\pi^2} = & \sum_{k=0}^{\infty} \{4704(589\sqrt{3} + 1020)k^2 + 18(24500\sqrt{3} + 42417)k + 23400\sqrt{3} + 40485\} \\ & \times C_k \left(\frac{3(230888\sqrt{3} - 399849)}{16194277}\right)^{k+1}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \frac{27}{\pi^2} = & \sum_{k=0}^{\infty} \{(89318908159154573403268800\sqrt{5} - 199723150319930230972608000)k^2 \\ & + (17073215957979902289013200\sqrt{5} - 38176871476577254538418450)k \\ & + 1089591204946673091627000\sqrt{5} - 2436400001946666150405125\} \\ & \times C_k \left(\frac{5697769392\sqrt{5} - 12740595841}{97838353751039}\right)^{k+1}, \end{aligned} \tag{5.6}$$

$$\frac{1728}{\pi^2} = \sum_{k=0}^{\infty} (11907k^2 + 6552k + 1136)(-1)^k C_k \left(\frac{64}{125}\right)^{k+1}. \tag{5.7}$$

Proof. The series (5.1)–(5.6) correspond to the cases $n = 2, 3, 4, 5, 9,$ and $25,$ respectively. Proofs of these follow along the same lines as that in the previous section. As in the previous section, the last series also corresponds to $n = 7,$ but we used (3.13) instead of (3.12). \square

Remark. The series (5.1) adds less than a digit per term. The series (5.2)–(5.6) add roughly 1, 2, 3, 5 and 10 digits per term, respectively. The series (5.7) is very slowly convergent.

Theorem 5.2. *If*

$$B_k = \sum_{n=0}^k \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{4}\right)_{k-n} \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{k-n} \left(\frac{3}{4}\right)_n \left(\frac{3}{4}\right)_{k-n}}{n!^3 (k-n)!^3},$$

then

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} (49k^2 + 5k - 1)B_k \left(\frac{32}{81}\right)^{k+1}, \tag{5.8}$$

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty} (128k^2 + 40k + 3)B_k \left(\frac{1}{9}\right)^{k+1}, \tag{5.9}$$

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} (6400k^2 + 1880k + 177)B_k \left(\frac{1}{81}\right)^{k+1}, \tag{5.10}$$

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty} (115200k^2 + 25896k + 1935)B_k \left(\frac{1}{2401}\right)^{k+1}, \tag{5.11}$$

$$\frac{24}{11\pi^2} = \sum_{k=0}^{\infty} (156800k^2 + 31912k + 2163)B_k \left(\frac{1}{9801}\right)^k, \tag{5.12}$$

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} (44571654400k^2 + 5588768408k + 233588841)B_k \left(\frac{1}{99^4}\right)^{k+1}. \tag{5.13}$$

Proof. The six series above correspond to the singular moduli $x_4, x_6, x_{10}, x_{18}, x_{22},$ and $x_{58},$ respectively. To derive these series we use (3.10) instead of (3.12), which we used for (4.1). \square

Remark. The series (5.8) adds less than a digit per term. The series (5.9)–(5.13) add roughly 1, 2, 3, 4 and 8 digits per term, respectively. The series (5.13) is an analogue of Ramanujan’s famous series

$$\frac{9801}{2\sqrt{2}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3} (26390k + 1103) \left(\frac{1}{99}\right)^{4k}, \tag{5.14}$$

which also adds roughly 8 digits per term.

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Theorem 5.3. *If*

$$B_k = \sum_{n=0}^k \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{4}\right)_{k-n} \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{k-n} \left(\frac{3}{4}\right)_n \left(\frac{3}{4}\right)_{k-n}}{n!^3(k-n)!^3},$$

then

$$\frac{96}{\pi^2} = \sum_{k=0}^{\infty} \{(50787\sqrt{2} - 71476)k^2 + 10(1215\sqrt{2} - 1694)k + 12(81\sqrt{2} - 110)\} \times (-1)^k B_k \left(\frac{1}{4}\right)^{k+1}, \tag{5.15}$$

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} (200k^2 + 110k + 21)(-1)^k B_k \left(\frac{1}{4}\right)^{k+1}, \tag{5.16}$$

$$\frac{768}{\pi^2} = \sum_{k=0}^{\infty} (29575k^2 + 11816k + 1680)(-1)^k B_k \left(\frac{16}{63}\right)^{2k+2}, \tag{5.17}$$

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty} (1176k^2 + 390k + 45)(-1)^k B_k \left(\frac{1}{48}\right)^{k+1}, \tag{5.18}$$

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} (33800k^2 + 9022k + 813)(-1)^k B_k \left(\frac{1}{324}\right)^{k+1}, \tag{5.19}$$

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} (5184200k^2 + 990250k + 63075)(-1)^k B_k \left(\frac{1}{25920}\right)^{k+1}. \tag{5.20}$$

Proof. The series above correspond to the singular moduli $x_4, x_5, x_7, x_9, x_{13}$, and x_{25} , respectively, and we have employed (3.11) instead of (3.12), which we used for (4.1). □

Remark. The first two series add less than a digit per term. The series (5.17) adds roughly 1 digit per term and the series (5.18)–(5.20) give more than 1, 2, and 4 digits per term, respectively.

Theorem 5.4. *If*

$$A_k = \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}\right)_{k-n}^3}{n!^3(k-n)!^3},$$

then

$$\frac{3}{\pi^2} = \sum_{k=0}^{\infty} \{(114 - 80\sqrt{2})k^2 + (188 - 133\sqrt{2})k + 79 - 56\sqrt{2}\} A_k \left(2\sqrt{2} - 2\right)^{3k}, \tag{5.21}$$

$$\frac{1}{\pi^2} = \sum_{k=0}^{\infty} (3k^2 + k) A_k \left(\frac{1}{4}\right)^{k+1}, \tag{5.22}$$

$$\frac{9}{\pi^2} = \sum_{k=0}^{\infty} \{(26892 - 19008\sqrt{2})k^2 + (47352 - 33480\sqrt{2})k + 20976 - 14832\sqrt{2}\} \\ \times A_k(1584\sqrt{2} - 2240)^k, \quad (5.23)$$

$$\frac{12}{\pi^2} = \sum_{k=0}^{\infty} \{80(\sqrt{5} - 2)k^2 + (124\sqrt{5} - 270)k + 50\sqrt{5} - 111\} A_k (9 - 4\sqrt{5})^k, \quad (5.24)$$

$$\frac{6}{\pi^2} = \sum_{k=0}^{\infty} (882k^2 + 308k + 34) A_k \left(\frac{1}{64}\right)^{k+1}. \quad (5.25)$$

Proof. These series correspond to the singular moduli x_2 , x_3 , x_4 , x_5 , and x_7 , respectively. For these we use (3.7) instead of (3.12). \square

Remark. The first three series above give less than one digit per term and the later two series add roughly one digit per term.

Theorem 5.5. *If*

$$A_k = \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}\right)_{k-n}^3}{n!^3 (k-n)!^3},$$

then

$$\frac{12}{\pi^2} = \sum_{k=0}^{\infty} (16k^2 + 16k + 5)(-1)^k A_k, \quad (5.26)$$

$$\frac{1}{\pi^2} = \sum_{k=0}^{\infty} \{(417 - 240\sqrt{3})k^2 + (718 - 414\sqrt{3})k + 312 - 180\sqrt{3}\} \\ \times (-1)^k A_k (416 - 240\sqrt{3})^k, \quad (5.27)$$

$$\frac{6}{\pi^2} = \sum_{k=0}^{\infty} (9k^2 + 5k + 1)(-1)^k A_k \left(\frac{1}{8}\right)^k, \quad (5.28)$$

$$\frac{4}{\pi^2} = \sum_{k=0}^{\infty} \{48(3 - 2\sqrt{2})k^2 + 4(56 - 39\sqrt{2})k + 91 - 64\sqrt{2}\} (-1)^k A_k (\sqrt{2} - 1)^{4k}, \quad (5.29)$$

$$\frac{3}{\pi^2} = \sum_{k=0}^{\infty} \{63(14393 - 5440\sqrt{7})k^2 + (1649872 - 623592\sqrt{7})k + 16(47060 - 17787\sqrt{7})\} \\ \times (-1)^k A_k (129536 - 48960\sqrt{7})^k, \quad (5.30)$$

$$\frac{12}{\pi^2} = \sum_{k=0}^{\infty} \{720(9 - 4\sqrt{5})k^2 + 16(685 - 306\sqrt{5})k + 4697 - 2100\sqrt{5}\} \\ \times (-1)^k A_k (\sqrt{5} - 2)^{4k}, \quad (5.31)$$

$$\frac{4}{\pi^2} = \sum_{k=0}^{\infty} \{2352(49 - 20\sqrt{6})k^2 + 24(8522 - 3479\sqrt{6})k + 91239 - 37248\sqrt{6}\} \times (-1)^k A_k (\sqrt{3} - \sqrt{2})^{8k}, \tag{5.32}$$

$$\frac{12}{\pi^2} = \sum_{k=0}^{\infty} \{17424(99 - 70\sqrt{2})k^2 + 4(774664 - 547767\sqrt{2})k + 1397561 - 988224\sqrt{2}\} \times (-1)^k A_k (\sqrt{2} - 1)^{12k}, \tag{5.33}$$

$$\frac{12}{\pi^2} = \sum_{k=0}^{\infty} \{4547664(9801 - 1820\sqrt{29})k^2 + 16(5222158789 - 969730542\sqrt{29})k + 39216474833 - 7282316556\sqrt{29}\} (-1)^k A_k (70 - 13\sqrt{29})^{4k}. \tag{5.34}$$

Proof. For these derivations, we employ (3.8) instead of (3.12), which was used for (4.1). The nine series above correspond to the singular moduli x_n for $n = 2, 3, 4, 6, 7, 10, 18, 22,$ and 58 . □

Remark. The first series is very slowly convergent. The series (5.27) and (5.28) add roughly 1 digit per term, the series (5.29), (5.30) and (5.31) add roughly 2 digits per term, and the series (5.32)–(5.34) add roughly 4, 5, and 8 digits per term, respectively.

Theorem 5.6. *If*

$$A_k = \sum_{n=0}^k \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}\right)_{k-n}^3}{n!^3 (k-n)!^3},$$

then

$$\frac{6}{\pi^2} = \sum_{k=0}^{\infty} \{2(5\sqrt{2} + 1)k^2 + (11\sqrt{2} - 10)k + 4\sqrt{2} - 5\} (-1)^k A_k \left(\frac{\sqrt{2} - 1}{2}\right)^{3k}, \tag{5.35}$$

$$\frac{4}{\pi^2} = \sum_{k=0}^{\infty} \{(42 - 15\sqrt{3})k^2 + 2(23 - 12\sqrt{3})k + 16 - 9\sqrt{3}\} (-1)^k A_k \left(\frac{3\sqrt{3} - 5}{4\sqrt{2}}\right)^{2k}, \tag{5.36}$$

$$\frac{6}{\pi^2} = \sum_{k=0}^{\infty} \{(99\sqrt{2} - 108)k^2 + 4(36\sqrt{2} - 49)k + 57\sqrt{2} - 80\} \times (-1)^k A_k \left(\frac{99\sqrt{2} - 140}{32}\right)^k, \tag{5.37}$$

$$\frac{48}{\pi^2} = \sum_{k=0}^{\infty} \{63(232 - 85\sqrt{7})k^2 + (23464 - 8838\sqrt{7})k + 9728 - 3675\sqrt{7}\} \times (-1)^k A_k \left(\frac{8 - 3\sqrt{7}}{4}\right)^{3k}. \tag{5.38}$$

Proof. The four series above correspond to the singular moduli x_n for $n = 2, 3, 4$, and 7 . We employ (3.9) instead of (3.12). \square

Remark. The first series adds 2 digits per term, the next two series yield roughly 3 digits per term, and the last series gives roughly 6 digits per term.

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